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The effect on eigenvalues of connected graphs by adding edges*

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Abstract

By the well-known Perron-Frobenius Theorem [3], for a connected graph G, its largest eigenvalue strictly increases when an edge is added. We are interested in how the other eigenvalues of a connected graph change when edges are added. Examples show that all cases are possible: increased, decreased, unchanged. In this paper, we consider the effect on the eigenvalues by suitably adding edges in particular families, say the family of connected graphs with clusters. By using the result, we also consider the effect on the energy by suitably adding edges to the graphs of the above families.

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1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set E. Let $A(G) = (a_{ij})$ be the adjacency matrix of G, where $a_{ij} = 1$ if v_i is adjacent to v_j; and $a_{ij} = 0$, otherwise. It is easy to see that $A(G)$ is a real symmetric matrix. Denote its eigenvalues by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G),$$

which are always enumerated in non-increasing order and repeated according to their multiplicity.

Let X be a column vector with n entries. It will be convenient to associate with X an assignment of G in which vertex v_i is assigned x_i(or $X(v_i)$). Such assignment is sometimes called vertex valuations of G.

For $A = (a_{ij})$ an $n \times n_1$ matrix and B an $m \times m_1$ matrix we denote by $A \otimes B$ the matrix $(a_{ij}B)$ (in block partitioned form) and call it the tensor product of A with B. It is easy to see that $A \otimes B$ is an $nm \times n_1m_1$ matrix and $I_m \otimes I_n = I_{mn}$, where I_m denotes the identity matrix of order m.

By the well-known Perron-Frobenius Theorem, for a connected graph G, its largest eigenvalue strictly increases when an edge is added. We are interested in how the other eigenvalues of a connected graph change when edges are added. Examples show that all cases are possible: increased, decreased, unchanged. In this paper, we consider the effect on the eigenvalues by suitably adding edges in particular families, say the family of connected graphs with clusters. By using the result, we also consider the effect on the energy by suitably adding edges to the graphs of the above families.

2 Lemmas and results

The following results can be found in ([5] p.408).

Lemma 2.1. *If A and C are $m \times m$ matrices, B and D are $n \times n$ matrices, then*

(1). $(A + C) \otimes B = (A \otimes B) + (C \otimes B)$;

(2). $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$;

(3). There exists a permutation matrix P of order mn such that

$$A \otimes B = P^{-1}(B \otimes A)P;$$

(4). If A is an $m\times m$ matrix and B is an $n\times n$ matrix, then $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$, provided that A^{-1} and B^{-1} exist.

In this section, we use the tensor product of matrices to study how the eigenvalues of connected graphs change by suitably adding edges to the graphs in certain particular manners as described below.

Let $N(v_i) = \{v_j : v_iv_j \in E(G)\}$. e_s denotes the s-dimensional column vector with all entries 1. We now consider the property of the eigenvector of some eigenvalue of graphs.

Lemma 2.2. *Suppose that $H_1 \cong H_2 \cong \cdots \cong H_s$ $(s \geq 2)$ are s disjoint graphs of order t $(t \geq 1)$ and they are copies of H, $V(H_i) = \{v_{i1}, v_{i2}, \cdots, v_{it}\}$ and for any $1 \leq i < j \leq s$ and $1 \leq x, y \leq t$, $v_{ix}v_{iy} \in E(H_i)$ if and only if $v_{jx}v_{jy} \in E(H_j)$. Let G be a graph with vertices $v_1, v_2, \cdots, v_r$, G_s $(s \geq 2)$ be the graph on $n = r + st$ vertices obtained from G and $H_1, H_2, \cdots, H_s$ by adding edges between G and H_i $(i = 1, 2, \cdots, s)$ satisfying:*

$$N(v_{ih}) \cap V(G) = N(v_{jh}) \cap V(G) \quad (1 \leq i < j \leq s; \quad 1 \leq h \leq t).$$

Let G_s^* be the graph obtained from G_s by adding $\frac{s(s-1)}{2}$ edges among vertices $v_{1i}, v_{2i}, \cdots, v_{si}$ for each i $(1 \leq i \leq t)$. Then

(1). *Let X be an eigenvector corresponding to some eigenvalue $\lambda(G_s)$ of G_s. If $\lambda(G_s) \notin spec(H)$, where $spec(H)$ denotes the spectrum of $A(H)$, then*

$$X(v_{1i}) = X(v_{2i}) = \cdots = X(v_{si}),\ (i = 1, 2, \cdots, t).$$

(2). *Let Y be an eigenvector corresponding to some eigenvalue $\lambda(G_s^*)$ of G_s^*. If $\lambda(G_s^*) \notin spec(G_s^* - V(G))$, then*

$$Y(v_{1i}) = Y(v_{2i}) = \cdots = Y(v_{si}),\ (i = 1, 2, \cdots, t).$$

Proof. Let G_0 be the graph obtained from G_s by deleting all the edges among vertices $v_{i1}, v_{i2}, \cdots, v_{it}$ $(i = 1, 2, \cdots, s)$. Giving a suitable ordering for the vertices of G_s, we can assume that $A(G_0)$ has the following form:

$$A(G_0) = \begin{bmatrix} O_{s\times s} & \cdots & O_{s\times s} & \alpha_{11} & \cdots & \alpha_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ O_{s\times s} & \cdots & O_{s\times s} & \alpha_{t1} & \cdots & \alpha_{tr} \\ \alpha_{11}^T & \cdots & \alpha_{t1}^T & & & \\ \vdots & \ddots & \vdots & & A(G) & \\ \alpha_{1r}^T & \cdots & \alpha_{tr}^T & & & \end{bmatrix},$$

where $O_{m\times n}$ denotes the $m \times n$ zero matrix and $\alpha_{pq} = c_{pq}e_s$ (either $c_{pq} = 1$ or $c_{pq} = 0$ depending on whether or not v_{1p} $(1 \le p \le t)$ is adjacent to v_q $(1 \le q \le r))$.

Let $C = (c_{pq})_{t\times r}$. Thus we have from (1) of Lemma 2.1 that

$$\begin{aligned} A(G_s) &= A(G_0) + \begin{bmatrix} A(H)\otimes I_s & O_{ts\times r} \\ O_{r\times ts} & O_{r\times r} \end{bmatrix} \\ &= \begin{bmatrix} A(H)\otimes I_s & C\otimes e_s \\ (C\otimes e_s)^T & A(G) \end{bmatrix}, \end{aligned}$$

where I_s is the unit matrix of order s.

Let $X^T = (x_1^T, x_2^T, \cdots, x_t^T; a_1, a_2, \cdots, a_r)$ be an eigenvector of G_s corresponding to $\lambda(G_s)$, where $x_1, x_2, \cdots, x_t$ are column vectors each of them with s components. Then $A(G_s)X = \lambda(G_s)X$. Equating the first ts coordinates of the both sides of this equation, we have

$$(A(H)\otimes I_s)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} = -(C\otimes e_s)\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix} + \lambda(G_s)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix}.$$

So

$$((A(H)-\lambda(G_s)I_t)\otimes I_s)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} = -(C\otimes e_s)\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix}. \tag{2.1}$$

From (2) of Lemma 2.1, we have

$$-(C\otimes e_s)\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix} = -(C\otimes e_s)(\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix}\otimes e_1) = -(C\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix})\otimes e_s. \tag{2.2}$$

Substituting Eq. (2.2) into Eq. (2.1), we have

$$((A(H)-\lambda(G_s)I_t)\otimes I_s)\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} = -(C\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{bmatrix})\otimes e_s \triangleq \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_t \end{bmatrix}\otimes e_s. \tag{2.3}$$

If $\lambda(G_s) \notin spec(H)$, then $M = A(H) - \lambda(G_s)I_t$ is nonsingular. From Eq. (2.3) and (2), (4) of Lemma 2.1, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_t \end{bmatrix} = (M^{-1} \otimes I_s)(\begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_t \end{bmatrix} \otimes e_s) = (M^{-1} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_t \end{bmatrix}) \otimes e_s.$$

Then $x_i = c_i e_s$ $(i = 1, 2, \cdots, t)$, where $c_1, \cdots, c_t$ are some constants. This completes the proof of (1). Now we prove that (2) holds. By similar reasoning as above, we have

$$A(G_s^*) = \begin{bmatrix} I_t \otimes A(K_s) + A(H) \otimes I_s & C \otimes e_s \\ (C \otimes e_s)^T & A(G) \end{bmatrix},$$

where K_s denotes the complete graph with s vertices.

Let $Y^T = (y_1^T, y_2^T, \cdots, y_t^T; b_1, b_2, \cdots, b_r)$ be an eigenvector of G_s^* corresponding to $\lambda(G_s^*)$, where $y_1, y_2, \cdots, y_t$ are column vectors each of them with s components. Then $A(G_s^*)Y = \lambda(G_s^*)Y$. Equating the first ts coordinates of the both sides of this equation, we have

$$(I_t \otimes A(K_s) + A(H) \otimes I_s) \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{bmatrix} = -(C \otimes e_s) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix} + \lambda(G_s^*) \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{bmatrix}.$$

So

$$(I_t \otimes A(K_s) + A(H) \otimes I_s - \lambda(G_s^*)I_{st}) \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{bmatrix} = -(C \otimes e_s) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}. \quad (2.4)$$

From (2) of Lemma 2.1, we have

$$-(C \otimes e_s) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix} = -(C \otimes e_s)(\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix} \otimes e_1) = -(C \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}) \otimes e_s. \quad (2.5)$$

Substituting Eq. (2.5) into Eq. (2.4), we have

$$(I_t \otimes A(K_s) + A(H) \otimes I_s - \lambda(G_s^*)I_{st}) \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{bmatrix}$$

$$= -(C \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix}) \otimes e_s \stackrel{\triangle}{=} \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_t \end{bmatrix} \otimes e_s. \tag{2.6}$$

Note that $A(G_s^* - V(G)) = I_t \otimes A(K_s) + A(H) \otimes I_s$. Thus we have if $\lambda(G_s^*) \notin spec(G_s^* - V(G))$, then $N = I_t \otimes A(K_s) + A(H) \otimes I_s - \lambda(G_s^*)I_{st}$ is nonsingular. From Eq. (2.6) and (2) of Lemma 2.1, we have

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_t \end{bmatrix} = N^{-1}(\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_t \end{bmatrix} \otimes e_s) = (N^{-1} \otimes e_1)(\begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_t \end{bmatrix} \otimes e_s) = (N^{-1} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_t \end{bmatrix}) \otimes e_s.$$

Then $y_i = d_i e_s$ $(i = 1, 2, \cdots, t)$, where $d_1, \cdots, d_t$ are some constants. This completes the proof of (2). $\square$

Now we give an example to illustrate the graphs of Lemma 2.2.

Example 1. Let G_s be the graph of Fig. 1, $H_i : v_{i1}v_{i2}v_{i3}$, $(i = 1, 2)$ be a path with length 2. Then G_s $(s = 2)$ is the graph on $n = r + st = 5 + 2 \times 3 = 11$ vertices obtained from G and H_1, H_2 by adding edges $v_5v_{13}, v_4v_{13}; v_5v_{23}, v_4v_{23}$ between G and H_i $(i = 1, 2)$, where G is the induced subgraph of G_s induced by vertices v_1, v_2, v_3, v_4 and v_5. Furthermore $G_s^* = G_s + v_{11}v_{21} + v_{12}v_{22} + v_{13}v_{23}$.

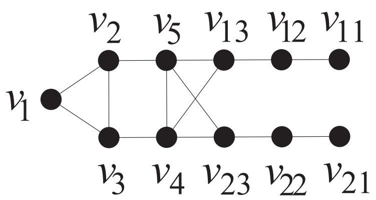

Fig. 1 $G_s : s = 2, t = 3, H \cong P_3$

Let C^n be the complex vector space of complex $n-$vectors. By the well known Courant-Fischer theorem, we have the following.

Lemma 2.3. *([4]) Let A be a Hermitian matrix of order n, the eigenvalues of A be arranged in decreasing order, that is, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and let S_k be a given $k-$dimensional subspace of C^n $(1 \leq k \leq n)$. If there exists a constant c_1 such that $x^*Ax \geq c_1 x^*x$ for all $x \in S_k$, then $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq c_1$. If there exists a constant c_2 such that $x^*Ax \leq c_2 x^*x$ for all $x \in S_k$, then $c_2 \geq \lambda_{n-k+1} \geq \lambda_{n-k+2} \geq \cdots \geq \lambda_n$.*

Now, we give the main result of this paper.

Theorem 2.1. *Let G_s and G_s^* be the two graphs defined in Lemma 2.2. Let G' be a graph obtained from G_s by adding t_i $(0 \leq t_i \leq \frac{s(s-1)}{2})$ edges among vertices $v_{1i}, v_{2i}, \cdots, v_{si}$ for each i $(1 \leq i \leq t)$. Then*

(1). If $\{\lambda_1(G_s), \lambda_2(G_s), \cdots, \lambda_k(G_s)\} \cap Spec(H) = \emptyset$, then $\lambda_i(G') \geq \lambda_i(G_s)$, $i = 1, 2, \cdots, k$.

(2). If $\{\lambda_{n-k+1}(G_s^*), \cdots, \lambda_n(G_s^*)\} \cap Spec(G_s^* - V(G)) = \emptyset$, then $\lambda_i(G') \leq \lambda_i(G_s^*)$, $i = n-k+1, \cdots, n$.

Proof. Let X_i be a unit eigenvector corresponding to $\lambda_i(G_s)$ $(i = 1, 2, \cdots, k)$ and $X_1, \cdots, X_k$ are linearly independent and orthogonal to each other. Let $S_k = \{X_1, \cdots, X_k\}$ which is a $k-$dimensional subspace of C^n $(1 \leq k \leq n)$. If $x \in S_k$, then there exist real numbers $a_1, \cdots, a_k$ such that $x = a_1X_1 + \cdots + a_kX_k$. It is easy to see that $x^*x = a_1^2 + \cdots + a_k^2$, where x^* is the transpose of x. From Lemma 2.2, we have

$$\begin{aligned}
x^*A(G')x &= (a_1X_1^* + \cdots + a_kX_k^*)A(G')(a_1X_1 + \cdots + a_kX_k) \\
&= (a_1X_1^* + \cdots + a_kX_k^*)A(G_s)(a_1X_1 + \cdots + a_kX_k) \\
&+ 2\sum_{i=1}^{t} t_i[a_1X_1(v_{1i}) + a_2X_2(v_{1i}) + \cdots + a_kX_k(v_{1i})]^2 \\
&\geq (a_1X_1^* + \cdots + a_kX_k^*)(\lambda_1(G_s)a_1X_1 + \cdots + \lambda_k(G_s)a_kX_k) \\
&= \lambda_1(G_s)a_1^2X_1^*X_1 + \cdots + \lambda_k(G_s)a_k^2X_k^*X_k \\
&\geq \lambda_k(G_s)a_1^2X_1^*X_1 + \cdots + \lambda_k(G_s)a_k^2X_k^*X_k \\
&= \lambda_k(G_s)(a_1^2 + \cdots + a_k^2) \\
&= \lambda_k(G_s)x^*x
\end{aligned}$$

From Lemma 2.3, we have $\lambda_k(G') \geq \lambda_k(G_s)$. This completes the proof of (1).

We now prove that (2) holds. Let Y_i be a unit eigenvector corresponding to $\lambda_i(G_s^*)$ $(i = n-k+1, n-k+2, \cdots, n)$ and $Y_{n-k+1}, \cdots, Y_n$ are linearly independent and orthogonal to each other. Let $S_k = \{Y_{n-k+1}, \cdots, Y_n\}$ which is a $k-$dimensional subspace of C^n $(1 \leq k \leq n)$. If $y \in S_k$, then there exist real numbers $b_{n-k+1}, \cdots, b_n$ such that $y = b_{n-k+1}Y_{n-k+1} + \cdots + b_nY_n$. It is easy to see that $y^*y = b_{n-k+1}^2 + \cdots + b_n^2$. From Lemma 2.2, we have

$$\begin{aligned}
y^*A(G')y &= (b_{n-k+1}Y_{n-k+1}^* + \cdots + b_nY_n^*)A(G')(b_{n-k+1}Y_{n-k+1} + \cdots + b_nY_n) \\
&= (b_{n-k+1}Y_{n-k+1}^* + \cdots + b_nY_n^*)A(G_s^*)(b_{n-k+1}Y_{n-k+1} + \cdots + b_nY_n) \\
&- 2\sum_{i=1}^{t}(\frac{s(s-1)}{2} - t_i)[b_{n-k+1}Y_{n-k+1}(v_{1i}) + \cdots + b_nY_n(v_{1i})]^2 \\
&\leq (b_{n-k+1}Y_{n-k+1}^* + \cdots + b_nY_n^*)\lambda_{n-k+1}(G_s^*)(b_{n-k+1}Y_{n-k+1} + \cdots \\
&+ \lambda_n(G_s^*)b_nY_n) \\
&= \lambda_{n-k+1}(G_s^*)b_{n-k+1}^2Y_{n-k+1}^*Y_{n-k+1} + \cdots + \lambda_n(G_s^*)b_n^2Y_n^*Y_n \\
&\leq \lambda_{n-k+1}(G_s^*)b_{n-k+1}^2Y_{n-k+1}^*Y_{n-k+1} + \cdots + \lambda_{n-k+1}(G^*)b_n^2Y_n^*Y_n \\
&= \lambda_{n-k+1}(G_s^*)(b_{n-k+1}^2 + \cdots + b_n^2) \\
&= \lambda_{n-k+1}(G_s^*)y^*y
\end{aligned}$$

From Lemma 2.3, we have $\lambda_{n-k+1}(G') \leq \lambda_{n-k+1}(G_s^*)$. This completes the proof of (2). □

Remark: Let G_s be the graph of Fig. 1. Then the graph G' in Theorem 2.1 is one of the following graphs: $G_s, G_s + v_{11}v_{21}, G_s + v_{12}v_{22}, G_s + v_{13}v_{23}, G_s + v_{11}v_{21} + v_{12}v_{22}, G_s + v_{11}v_{21} + v_{13}v_{23}, G_s + v_{12}v_{22} + v_{13}v_{23}, G_s^* = G_s + v_{11}v_{21} + v_{12}v_{22} + v_{13}v_{23}$.

Let G be a simple undirected graph of order n. A cluster [2] in G of order k and degree s, is a pair of vertex subsets (C, S), where C is a set of cardinality $|C| = k \geq 2$ of pairwise co-neighbor vertices sharing the same set S of s neighbors.

Corollary 2.1. *Let G be a connected graph on n vertices with a cluster (C, S) of order k and degree s and $V(C) = \{v_1, v_2, \cdots, v_k\}$, G_k^* be the graph obtained from G by adding $\frac{k(k-1)}{2}$ edges among vertices $v_1, v_2, \cdots, v_k$ and G' be the graph obtained from G by adding t $(0 \leq t \leq \frac{k(k-1)}{2})$ edges among vertices $v_1, v_2, \cdots, v_k$.*

(i). If $\lambda_i(G) > 0$, then $\lambda_i(G) \leq \lambda_i(G')$.

(ii). If $\{k-1, -1\} \cap \{\lambda_{n-t+1}(G_k^*), \cdots, \lambda_n(G_k^*)\} = \emptyset$, then $\lambda_i(G') \leq \lambda_i(G_k^*)$, $i = n-t+1, \cdots, n$.

Proof. By using (1) of Theorem 2.1 for H is an isolated vertex, (i) holds. From (2) of Theorem 2.1 and note that the eigenvalues of K_k are $k-1$ and -1 with multiplicity $k-1$, respectively, (ii) follows. □

The energy of G was first defined by Gutman in 1978 as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

Just as Brualdi [1] pointed out that since K_n does not have maximum energy, it is not true in general that $E(G') \leq E(G)$, where G' is a spanning subgraph of G. He proposed the following basic problem: when does this inequality hold?

By the well-known Perron-Frobenius Theorem and Corollary 2.1, we have the following.

Corollary 2.2. *Let G be a connected graph with a cluster (C, S) of order k and degree s and $V(C) = \{v_1, v_2, \cdots, v_k\}$, and G' be the graph obtained from G by adding t $(1 \leq t \leq \frac{k(k-1)}{2})$ edges among vertices $v_1, v_2, \cdots, v_k$. Then*

$$E(G) < E(G').$$

Let $m_G(\lambda)$ denote the multiplicity of λ as an eigenvalue of the graph G. In the following, we give some properties of eigenvalues of the graph G_s^*.

Theorem 2.2. *Let G_s $(s \geq 2)$ be the graph defined in Lemma 2.2 and $\lambda_1(H) \geq \lambda_2(H) \geq \cdots \geq \lambda_t(H)$ be the eigenvalues of the graph H. Then*

$$m_{G_s}(\lambda_k(H)) \geq (s-1), \quad k = 1, 2, \cdots, t.$$

Proof. Let X be a eigenvector of H_1 corresponding to $\lambda_k(H_1)$, $(1 \leq k \leq t)$ and Y_j $(2 \leq j \leq s)$ be a valuation of G_s defined by

$$\begin{cases} Y(v_{1i}) = X(v_{1i}), \ i = 1, 2, \cdots, t; \\ Y(v_{ji}) = -X(v_{1i}), \ i = 1, 2, \cdots, t; \\ Y(u) = 0, \ u \neq v_{11}, v_{12}, \cdots v_{1t}; v_{j1}, v_{j2}, \cdots v_{jt}. \end{cases}$$

Then Y_j $(2 \leq j \leq s)$ is an eigenvector of G_s corresponding to $\lambda_k(H)$, $(1 \leq k \leq t)$. The results follows. □

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