

CENTRAL LIMIT THEOREM FOR GERMINATION–GROWTH MODELS IN \mathbb{R}^d WITH NON-POISSON LOCATIONS

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Abstract

Seeds are randomly scattered in \mathbb{R}^d according to an m -dependent point process. Each seed has its own potential germination time. From each seed that succeeds in germinating, a spherical inhibited region grows to prohibit germination of any seed with later potential germination time. We show that under certain conditions on the distribution of the potential germination time, the number of germinated seeds in a large region has an asymptotic normal distribution.

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1 Introduction

The model considered in this paper was motivated by applications in many diverse fields, such as modelling crystal growth (Johnson and Mehl, 1939; Kolmogorov, 1937) and various biological processes (Bennett and Robinson, 1990; Cowan *et al.*, 1995; Quine and Robinson, 1990, 1992; Vanderbei and Shepp, 1988; Wolk, 1975). The most commonly used name for it is the Johnson–Mehl model (see Okabe *et al.*, 2000, for more references), which can be described by the following spatio-temporal germination and growth process. Seeds are randomly scattered in \mathbb{R}^d according to a point process Ψ at time $t = 0$. The seed at location x_i will be stimulated by an internal or external stimulus after a time t_i that is an independent and identically distributed realisation of a random variable with distribution function F . A seed, once stimulated, immediately tries to germinate and at the same time to prohibit other seeds from germination by generating a spherical inhibited region the radius of which grows at a speed $v > 0$. A seed stimulated at time t_i fails to germinate if and only if its location

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has been inhibited on or before t_i . Let Φ denote the spatial point process of germinated seeds.

In the literature the lion's share of attention was devoted to the case that Ψ is a stationary Poisson process. A major advantage of using the Poisson process is that the numbers of points in disjoint subsets of \mathbb{R}^d are independent. Under the Poisson assumption, Chiu and Quine (1997) showed the asymptotic normality of the total number of germinated seeds. Note that there is a terminological change from Chiu and Quine (1997); our seeds and germinated seeds here were called points and seeds, respectively, in that paper. Quine and Szczotka (2000) considered the model generated by non-Poisson processes in the case $d = 1$ and showed that, under certain conditions, the asymptotic normality of the number of germinated seeds still holds.

Such a germination–growth process can be regarded as a germ-grain model in the spatio-temporal space: The locations and potential germination times of seeds are the germs in the spatio-temporal space and the typical grain is an infinite cone rising along the time axis from the origin. The germ process is time-inhomogeneous. The germinated seeds are the exposed germs whilst the ungerminated seeds are germs lying in the interior of other grains. Heinrich and Molchanov (1999) proved a central limit theorem for a class of random measures, including the counting measure of the exposed germs, of germ-grain models with β -mixing germs. However, they considered only stationary germ processes and assumed that the mean grain size is bounded. Thus, their result cannot be applied to germination–growth processes.

In this paper we consider the case that d is arbitrary and Ψ is a stationary m -dependent point process, which means the following. Let B_1 and B_2 be two subsets of \mathbb{R}^d , and denote by $\|B_1 - B_2\|$ the infimum of the Euclidean distances between elements in B_1 and B_2 . Define the strong mixing coefficient of Ψ to be

$$\alpha^\Psi(k) = \sup\{|\Pr(A_1 \cap A_2) - \Pr(A_1)\Pr(A_2)| : A_u \in \sigma(\Psi \cap B_u), u = 1, 2, \\ \|B_1 - B_2\| \geq k\},$$

where $\sigma(\Psi \cap B_u)$ is the σ -algebra generated by the points of Ψ within B_u . The point process Ψ is m -dependent if $\alpha^\Psi(k) = 0$ for $k > m$. However, since Φ is a dependently thinned version of Ψ , it is not m -dependent. If Ψ is Poisson, then $m = 0$.

2 Central Limit Theorem

We show that the distribution of the total number of germinated seeds in a large subset of \mathbb{R}^d , after suitable normalisation, converges weakly to the standard normal distribution. We follow the framework used in Chiu and Quine (1997). Let ξ_z be the number of germinated seeds within the unit cube $\{z + [0, 1)^d\} = \{z + x : x \in [0, 1)^d\}$, where $z \in \mathbb{Z}^d$. Then $\{\xi_z : z \in \mathbb{Z}^d\}$ is a stationary random field.

We first consider the moments of ξ_z . Since all germinated seeds are seeds, $\mathbf{E}(\xi_z^n)$ is bounded above by the n th order moment of the number of seeds in a unit cube, but the exact formula will be a messy integral depending on the reduced n -point Palm distribution

of Ψ ; we do not bother to express it here. In the case that Ψ is a Poisson process, the first moment is given in Chiu and Quine (1997, equation (1.1)).

For z_1 and z_2 in \mathbb{Z}^d , let $d(z_1, z_2) = \max_{1 \leq i \leq d} |z_1(i) - z_2(i)|$, where $z(i)$, $1 \leq i \leq d$, are the components of $z \in \mathbb{Z}^d$. For $\Gamma \subset \mathbb{Z}^d$, denote by $\#(\Gamma)$ the number of elements in Γ and by $\partial\Gamma$ the set $\{z \in \Gamma : \text{there exists } z' \notin \Gamma \text{ such that } d(z, z') = 1\}$. Let $\Gamma_n \uparrow \mathbb{Z}^d$ be a fixed sequence of finite subsets of \mathbb{Z}^d satisfying the regularity condition that $\lim_{n \rightarrow \infty} \#(\partial\Gamma_n)/\#(\Gamma_n) = 0$. It implies that the sequence $\{\Gamma_n\}$ does not increase in only one direction, except in the case $d = 1$. Define S_n to be $\sum_{z \in \Gamma_n} \{\xi_z - \mathbf{E}(\xi_z)\}$ for each $n \in \mathbb{N}$. Let $S_0 = 0$.

For $\Gamma^{(1)}, \Gamma^{(2)} \subset \mathbb{Z}^d$, let $d(\Gamma^{(1)}, \Gamma^{(2)}) = \inf\{d(z_1, z_2) : z_u \in \Gamma^{(u)}, u = 1, 2\}$. Define the mixing coefficient for the random field $\{\xi_z : z \in \mathbb{Z}^d\}$ to be

$$\alpha_{a,b}(k) \equiv \sup\{|\mathbf{P}(A_1 \cap A_2) - \mathbf{P}(A_1)\mathbf{P}(A_2)| : A_u \in \sigma(\xi_z : z \in \Gamma^{(u)}), \\ \#(\Gamma^{(1)}) \leq a, \#(\Gamma^{(2)}) \leq b, d(\Gamma^{(1)}, \Gamma^{(2)}) \geq k\}$$

where $k \in \mathbb{N}$, $a, b \in \mathbb{N} \cup \{\infty\}$ and $\sigma(\xi_z : z \in \Gamma)$ is the σ -algebra generated by $\{\xi_z : z \in \Gamma\}$.

Lemma 1. (Bolthausen, 1982) *Suppose $\{\xi_z : z \in \mathbb{Z}^d\}$ is stationary. If $\sum_{k=1}^{\infty} k^{d-1} \alpha_{a,b}(k) < \infty$ for $a + b \leq 4$, $\alpha_{1,\infty}(k) = o(k^{-d})$, and $\mathbf{E}|\xi_z|^{2+\delta} < \infty$ and $\sum_{k=1}^{\infty} k^{d-1} \alpha_{1,1}(k)^{\delta/(2+\delta)} < \infty$ for some $\delta > 0$, then $\sum_{z \in \mathbb{Z}^d} |\mathbf{cov}(\xi_{z_0}, \xi_z)| < \infty$ and if $\sigma^2 = \sum_{z \in \mathbb{Z}^d} \mathbf{cov}(\xi_{z_0}, \xi_z) > 0$, then the distribution of $S_n/\sqrt{\#(\Gamma_n)\sigma^2}$ converges weakly to the standard normal distribution as $n \rightarrow \infty$.*

The following lemmas aim to establish sufficient conditions on the distribution function F of the potential germination time under which the conditions in Lemma 1 will be fulfilled. The approach is similar to that in Chiu and Quine (1997), in which we used a sophisticated argument to give an upper bound for $\alpha_{a,b}(k)$. Here we argue in a much simpler way to obtain a bound that is true not only for Poisson but also m -dependent seeds at the cost that if the seeds form a Poisson process, this general bound leads to stronger conditions on F .

For notational simplicity, hereinafter the index i refers to $z_i \in \Gamma^{(1)}$ and j refers to $z_j \in \Gamma^{(2)}$. Let T_i be zero if there is no seed in $z_i + [0, 1]^d$ and be the first potential germination time of seeds in $z_i + [0, 1]^d$ otherwise. Because of stationarity, T_i has the same distribution as T_j .

Lemma 2.

$$\alpha_{a,b}(k) \leq 2ab \Pr\{T_i \geq (k - m - 2\sqrt[d]{d})/(2v)\}.$$

Proof. The locations and the stimulation times of seeds in $B_i \equiv z_i + [0, 1]^d \oplus b(0, vt_i + 1)$, will suffice to determine ξ_{z_i} , where \oplus denotes the Minkowski (elementwise) addition and $b(0, r)$ a closed ball centred at 0 with radius r . This is still true even if there is no seed in $z_i + [0, 1]^d$ because in this case $\xi_{z_i} = 0$ no matter where the other seeds are. Moreover, $\|\cup_i B_i - \cup_j B_j\| \geq \min_{i,j} \{k - v(t_i + t_j) - 2\sqrt[d]{d}\}$. Since Ψ is m -dependent, for $A_u \in \sigma(\xi_z : z \in \Gamma^{(u)})$, where $u = 1$ and 2, $|\Pr(A_1 \cap A_2 | T_i = t_i, T_j = t_j \text{ for all } i \text{ and } j) - \Pr(A_1 | T_i = t_i, T_j = t_j \text{ for all } i \text{ and } j) \Pr(A_2 | T_i = t_i, T_j = t_j \text{ for all } i \text{ and } j)|$ is not zero only if $\min_{i,j} \{k - v(t_i + t_j) - 2\sqrt[d]{d}\} \leq m$ and is always bounded above by 1. Thus, if G is the joint distribution of

all T_i and T_j , we have

$$\begin{aligned}\alpha_{ab}(k) &\leq \int \cdots \int_{\max_{i,j}(t_i+t_j) \geq (k-m-2\sqrt[d]{d})/v} 1 \, dG \\ &\leq \sum_{i:z_i \in \Gamma^{(1)}} \sum_{j:z_j \in \Gamma^{(2)}} \Pr\{(T_i + T_j) \geq (k - m - 2\sqrt[d]{d})/v\} \\ &\leq 2ab \Pr\{T_i \geq (k - m - 2\sqrt[d]{d})/(2v)\}.\end{aligned}$$

Lemma 3. For sufficiently large x ,

$$\Pr(T_i > x) \leq 2\{1 - F(x)\}.$$

Proof. Let N_i denote the number of seeds in $z_i + [0, 1)^d$. For $x \geq 0$ and $F(x) > 0$, we have

$$\Pr(T_i > x) = \sum_{n=1}^{\infty} (1 - F(x))^n \Pr(N_i = n) \leq 1/F(x) - 1.$$

Since for sufficiently large x , $F(x) \geq 1/2$, the result follows.

Lemma 4.

$$\alpha_{1,\infty}(k) \leq 2^{d^2} \sum_{h \geq k} h^{d-1} \Pr(T_i \geq (h - m - 2\sqrt[d]{d})/(2v)).$$

Proof. Let $\Gamma^{(1)} = \{z_1\}$. By the same argument as in the proof of Lemma 2, we have

$$\alpha_{1,\infty}(k) \leq \sum_{h \geq k} \sum_{j:d(z_1, z_j)=h} \Pr\{(T_1 + T_j) \geq (h - m - 2\sqrt[d]{d})/v\}$$

Since the number of z_j in $\Gamma^{(2)}$ such that $d(z_1, z_j) = h$ is at most $(2h+1)^d - (2h-1)^d$, which is less than $2^{d^2-1}h^{d-1}$, the result follows.

Because of Lemma 2, the condition $\sum_{k=1}^{\infty} k^{d-1} \alpha_{1,1}(k)^{\delta/(2+\delta)} < \infty$ for some $\delta > 0$ implies the condition $\sum_{k=1}^{\infty} k^{d-1} \alpha_{a,b}(k) < \infty$ in Lemma 1. Hence, only two conditions on $\alpha_{a,b}(k)$, together with the moment conditions, will suffice to guarantee the asymptotic normality.

Theorem 1. Suppose that $\sigma^2 = \sum_{z \in \mathbb{Z}^d} \mathbf{cov}(\xi_{z_0}, \xi_z) > 0$, and the number of points in Ψ within a bounded subset of \mathbb{R}^d has a finite moment up to $(2+\delta)$ th order for some $\delta > 0$, then the distribution of $S_n/\sqrt{\#(\Gamma_n)\sigma^2}$ converges weakly to the standard normal distribution, if F satisfies the following conditions:

$$\begin{aligned}\sum_{h=k}^{\infty} h^{d-1} \left\{ 1 - F\left(\frac{h - m - 2\sqrt[d]{d}}{2v}\right) \right\} &= o(k^{-d}), \\ \sum_{k=1}^{\infty} k^{d-1} \left\{ 1 - F\left(\frac{k - m - 2\sqrt[d]{d}}{2v}\right) \right\}^{\frac{\delta}{2+\delta}} &< \infty.\end{aligned}$$

These two conditions on the tail of F are satisfied if F is, for example, the Weibull distribution function. Typically, sufficient conditions for the positivity of σ^2 are difficult to formulate for a mixing random sequence or field. Chiu and Lee (2001) showed that if the locations and the potential germination times of seeds form a spatio-temporal Poisson process, then $\sigma^2 > 0$ for $d = 1$. Chiu and Quine (1997) showed numerically that if this spatio-temporal Poisson process is homogeneous, then $\sigma^2 > 0$ for $d = 2, 3$ and 4 . We believe that $\sigma^2 > 0$ under rather mild conditions on the point process of seeds but it remains an open problem.

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