

# CIRCULAR $L(j, k)$ -LABELING NUMBERS OF SQUARE OF PATHS\*

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## Abstract

Let  $j$ ,  $k$  and  $\sigma$  be positive numbers, a circular  $\sigma$ - $L(j, k)$ -labeling of a graph  $G$  is a function  $f : V(G) \rightarrow [0, \sigma)$  such that  $|f(u) - f(v)|_\sigma \geq j$  if  $u$  and  $v$  are adjacent, and  $|f(u) - f(v)|_\sigma \geq k$  if  $u$  and  $v$  are at distance two, where  $|a - b|_\sigma = \min\{|a - b|, \sigma - |a - b|\}$ . The minimum  $\sigma$  such that there exist a circular  $\sigma$ - $L(j, k)$ -labeling of  $G$  is called the circular- $L(j, k)$ -labeling number of  $G$  and is denoted by  $\sigma_{j,k}(G)$ . The  $k$ -th power  $G^k$  of an undirected graph  $G$  is a graph with the same set of vertices and an edge between two vertices when their distance in  $G$  is at most  $k$ . In this paper, the circular  $L(j, k)$ -labeling numbers of  $P_n^2$  are determined.

**2000 Mathematics Subject Classification:** Primary 05C78; Secondary 05C15

**Keywords:** Circular  $L(j, k)$ -labeling, square of path, code assignment.

## 1. Introduction

The rapid growth of computer wireless networks highlighted the scarcity of available codes (such as radio frequencies) for communication with minimum interference. For example, the Packet Radio Network (PRN) is a computer network that using radio frequencies to transmit packet among computers. The two major types of interference in PRN are direct collision (or interference), which caused by the transmission of adjacent stations (computers), and hidden terminal collision (or interference), which caused by distance-two stations that transmit to the same receiving station or receive the data from the same transmitting station.

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\*Received: September 12, 2016; Accepted: October 21, 2016.

This work is supported by Tianjin Research Program of Application Foundation and Advanced Technology, Tianjin Municipal Science and Technology Commission (No. 14JCYBJC43100), National Natural Science Foundation of China (No. 11601391), Faculty Research Grant of Hong Kong Baptist University.

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Bertossi and Bonuccelli [1] introduced an optimal code assignment to avoid hidden terminal interference and Jin and Yeh [10] further generalized it to the  $L(j, k)$ -labeling problem with  $j \leq k$ .

Let  $G$  be a graph and let  $V(G)$  and  $E(G)$  be its vertex set and edge set, respectively. For any two vertices  $u$  and  $v$ , let  $d_G(u, v)$  (or simply  $d(u, v)$ ) denote the distance (length of a shortest path) between  $u$  and  $v$  in  $G$ . All notation not defined in this article can be found in the book [2].

For positive numbers  $j$  and  $k$ , an  $L(j, k)$ -labeling  $f$  of  $G$  is an assignment of numbers to vertices of  $G$  such that  $|f(u) - f(v)| \geq j$  if  $uv \in E(G)$ , and  $|f(u) - f(v)| \geq k$  if  $d(u, v) = 2$ . The *span* of  $f$  is the difference between the maximum and the minimum numbers assigned by  $f$ . The  $L(j, k)$ -labeling number of  $G$ , denoted by  $\lambda_{j,k}(G)$ , is the minimum span over all  $L(j, k)$ -labeling of  $G$ .

The introduction of  $L(j, k)$ -labeling with  $j \geq k$  was motivated from the channel assignment problem [7]. Afterwards, Heuvel, Leese and Shepherd initiated the investigation of circular  $L(j, k)$ -labeling of a graph in [8]. They used the ‘‘circular distance’’ (will be defined below) to instead of the ‘‘linear distance’’.

For a positive real number  $\sigma$ , let  $S(\sigma)$  denote the circle obtained from the closed interval  $[0, \sigma]$  by identifying 0 and  $\sigma$  into a single point.  $S(\sigma)$  is called a  $\sigma$ -cycle. For any  $x \in \mathbb{R}$ ,  $[x]_\sigma \in [0, \sigma)$  denotes the remainder of  $x$  upon division of  $\sigma$ . The *circular distance* of two points  $p$  and  $q$  on  $S(\sigma)$  is defined as  $|p - q|_\sigma = \min\{|p - q|, \sigma - |p - q|\}$ .

A *circular  $\sigma$ - $L(j, k)$ -labeling* of  $G$  is a function  $f : V(G) \rightarrow S(\sigma)$  satisfying  $|f(x) - f(y)|_\sigma \geq j$  if  $d(x, y) = 1$  and  $|f(x) - f(y)|_\sigma \geq k$  if  $d(x, y) = 2$ . The minimum  $\sigma$  such that there exists a circular  $\sigma$ - $L(j, k)$ -labeling of  $G$  is called the *circular  $L(j, k)$ -labeling number* of  $G$  and is denoted by  $\sigma_{j,k}(G)$ .  $L(j, k)$ -labeling numbers or circular  $L(j, k)$ -labeling numbers of graphs were studied in many articles, please refer to [4–6, 11, 12, 14–17, 19, 25] for  $j \geq k$  and [1, 3, 10, 13, 20–24] for  $j \leq k$ .

## 2. Main Results

The  $k$ -th power  $G^k$  of an undirected graph  $G$  is a graph with the same set of vertices and an edge between two vertices when their distance in  $G$  is at most  $k$ .  $G^2$  is called the *square* of  $G$ .

**Lemma 2.1.** *Let  $j$  and  $k$  be two positive numbers with  $j \leq k$ . Suppose  $G$  is a graph and  $H$  is an induced subgraph of  $G$ . Then  $\sigma_{j,k}(G) \geq \sigma_{j,k}(H)$ .*

Note that Lemma 2.1 is not true if  $H$  is not an induced subgraph. Suppose  $f$  is a circular  $\sigma$ - $L(j, k)$ -labeling of  $G$ . For any  $a \in \mathbb{R}$ , define  $g(v) = [f(v) - a]_\sigma$  for  $v \in V(G)$ . Here  $g$  is still a circular  $\sigma$ - $L(j, k)$ -labeling of  $G$ . In this paper, the path is denoted by  $P_n = v_0 v_1 \cdots v_{n-1}$ . Without loss of generality, we may always assume  $v_0$  is labeled by 0 when we consider circular  $L(j, k)$ -labelings of  $P_n^2$ . Also, in this paper we shall use  $[a, b]$  to denote the close interval from  $a$  to  $b$ . Similar definition are applied for semi-open intervals  $(a, b]$  and  $[a, b)$ .

**Lemma 2.2.** *Let  $a, b \in \mathbb{R}$  and let  $\sigma, h$  and  $l$  be positive numbers. Suppose  $|a - b| \geq h$  and  $|a - b|_\sigma \geq l$ . Then  $\sigma \geq h + l$ .*

**Proof.** The lemma follows from  $l \leq |a - b|_\sigma = \min\{|a - b|, \sigma - |a - b|\} \leq \sigma - |a - b| \leq \sigma - h$ .  $\square$

**Theorem 2.1.** For  $j \leq k$ ,  $\sigma_{j,k}(P_4^2) = \max\{2k, 4j\}$ .

**Proof.** Let  $\eta = \max\{2k, 4j\}$ . Define a circular labeling  $f$  for  $P_4^2$  as follows:

If  $k \geq 2j$ , let  $f(v_0) = 0$ ,  $f(v_1) = j + k$ ,  $f(v_2) = j$  and  $f(v_3) = k$ .

If  $j \leq k < 2j$ , let  $f(v_0) = 0$ ,  $f(v_1) = 3j$ ,  $f(v_2) = j$  and  $f(v_3) = 2j$ .

It is clearly that  $f$  is a circular  $\eta$ - $L(j, k)$ -labeling of  $P_4^2$  for  $j \leq k$ . Then  $\sigma_{j,k}(P_4^2) \leq \max\{2k, 4j\}$ .

On the other hand, let  $f$  be a circular  $\sigma$ - $L(j, k)$ -labeling for  $P_4^2$ . Note that by our assumption,  $f(v_0) = 0$ . Since  $d(v_0, v_3) = 2$ ,  $f(v_3) \geq k$ . By Lemma 2.2, this implies that  $\sigma \geq 2k$ . Moreover, since any two vertices of  $P_4^2$  are adjacent or of distance two,  $\max_{1 \leq i \leq 3} f(v_i) \geq 3j$ . Since  $|f(v_s) - f(v_t)|_\sigma \geq j$  and  $\max\{|f(v_s) - f(v_t)|\} \geq 3j$  for  $0 \leq s < t \leq 3$ , by Lemma 2.2 we have  $\sigma \geq 4j$ . Thus,  $\sigma \geq \max\{2k, 4j\}$ .

Hence,  $\sigma_{j,k}(P_4^2) = \max\{2k, 4j\}$ .  $\square$

**Theorem 2.2.** For  $j \leq k$ ,  $\sigma_{j,k}(P_5^2) = \max\{j + 2k, 5j\}$ .

**Proof.** Let  $\eta = \max\{j + 2k, 5j\}$ . Given a circular labeling  $f$  for  $P_5^2$  as follows:

If  $k \geq 2j$ , let  $f(v_0) = 0$ ,  $f(v_1) = 2j + k$ ,  $f(v_2) = j$ ,  $f(v_3) = j + k$  and  $f(v_4) = k$ .

If  $j \leq k < 2j$ , let  $f(v_0) = 0$ ,  $f(v_1) = 4j$ ,  $f(v_2) = j$ ,  $f(v_3) = 3j$  and  $f(v_4) = 2j$ .

It is easy to verify that  $f$  is a circular  $\eta$ - $L(j, k)$ -labeling of  $P_5^2$  for  $j \leq k$ . Then  $\sigma_{j,k}(P_5^2) \leq \max\{j + 2k, 5j\}$ .

On the other hand, let  $f$  be a circular  $\sigma$ - $L(j, k)$ -labeling for  $P_5^2$ . Since  $d(v_0, v_3) = d(v_0, v_4) = 2$  and  $v_3$  and  $v_4$  are adjacent,  $\max_{i=3,4} f(v_i) \geq j + k$ . Again since  $d(v_0, v_i) = 2$  for  $i = 3, 4$ , by Lemma 2.2 we have  $\sigma \geq j + 2k$ . Meantime, since any two vertices of  $P_5^2$  are adjacent or of distance two,  $\max_{1 \leq i \leq 4} f(v_i) \geq 4j$ . Since  $|f(v_s) - f(v_t)|_\sigma \geq j$  and  $\max\{|f(v_s) - f(v_t)|\} \geq 4j$  for  $0 \leq s < t \leq 4$ , by Lemma 2.2 we have  $\sigma \geq 5j$ . Thus,  $\sigma \geq \max\{j + 2k, 5j\}$ .

Hence,  $\sigma_{j,k}(P_5^2) = \max\{j + 2k, 5j\}$  for  $j \leq k$ .  $\square$

**Theorem 2.3.** Let  $j$  and  $k$  be two positive numbers with  $j \leq k < 3j$ . For  $n \geq 6$ ,  $\sigma_{j,k}(P_n^2) = 4j + k$ .

**Proof.** Let  $\sigma = 4j + k$ . Define a circular labeling  $f$  of  $P_n^2$  as follows.

$f(v_i) = [ij]_\sigma$ , for  $0 \leq i \leq n - 1$ . It is easy to verify that  $f$  is a circular  $\sigma$ - $L(j, k)$ -labeling of  $P_n^2$  for  $j \leq k < 3j$  and  $n \geq 6$ . Then  $\sigma_{j,k}(P_n^2) \leq 4j + k$  for  $j \leq k < 3j$  and  $n \geq 6$ .

On the other hand, since  $P_6^2$  is an induced subgraph of  $P_n^2$  for  $n \geq 6$ , by Lemma 2.1, we have  $\sigma_{j,k}(P_n^2) \geq \sigma_{j,k}(P_6^2)$ . It suffices to find a lower bound of  $\sigma = \sigma_{j,k}(P_6^2)$ .

Let  $f$  be a circular  $\sigma$ - $L(j, k)$ -labeling for  $P_6^2$ . Note that  $\sigma \leq 4j + k$ . As mentioned before  $f(v_0) = 0$ . By considering the distances of  $v_i$  from  $v_0$ , we have  $f(v_1), f(v_2) \in [j, \sigma - j]$  and  $f(v_3), f(v_4) \in [k, \sigma - k]$ . Without loss of generality, we may assume that  $f(v_1) < f(v_4)$ , otherwise consider the new labeling  $[\sigma - f]_\sigma$ . Here we have  $f(v_1) \in [j, \sigma - 2k]$  and  $f(v_4) \in [2j, \sigma - k]$ . Since  $d(v_1, v_5) = 2$ , we have  $|f(v_1) - f(v_5)|_\sigma \geq k$ , that is,  $k \leq |f(v_1) - f(v_5)| \leq \sigma - k$ . So we have  $f(v_5) \in [j + k, \sigma)$  when  $f(v_1) < f(v_5)$  or  $f(v_5) \in [0, \sigma - 3k]$  when  $f(v_1) > f(v_5)$ .

**Case 1.** Suppose  $2j < k < 3j$ . Here  $\sigma - 3k < 0$ . Under this condition, by the same argument above, we also have  $f(v_2) < f(v_5)$ . Thus  $f(v_5) \in [j+k, \sigma)$ . This implies that  $f(v_2) \in [j, \sigma - k)$ . Since the length of  $[j, \sigma - 2k]$  is  $\sigma - j - 2k \leq 3j - k < j$ ,  $f(v_2), f(v_3) \in [2j, \sigma - k]$ . Now  $f(v_2), f(v_3), f(v_4) \in [2j, \sigma - k]$ . Since  $v_2, v_3, v_4$  induce a  $K_3$ ,  $\sigma - k - 2j \geq 2j$ . Hence  $\sigma \geq 4j + k$ .

**Case 2.** Suppose  $j \leq k \leq 2j$ . We have two cases:

- a. Suppose  $f(v_5) \in [0, \sigma - 3k]$ . Similarly, we obtain that  $f(v_1), f(v_2) \in [k, \sigma - j]$ , and  $f(v_3), f(v_4) \in [k, \sigma - k]$ . We can conclude that  $f(v_i) \in [k, \sigma - j]$  for  $1 \leq i \leq 4$ . Since the distances between any of two of  $v_1, v_2, v_3, v_4$  are at most 2, it means that  $\sigma - j - k \geq 3j$ . That is,  $\sigma \geq 4j + k$ .
- b. Suppose  $f(v_5) \in [j+k, \sigma)$ .  
 If  $f(v_2) < f(v_5)$ , then  $f(v_2) \in [j, \sigma - k)$ . We can conclude that  $f(v_i) \in [j, \sigma - k]$  for  $1 \leq i \leq 4$ . Similar to the above case, we have  $\sigma \geq 4j + k$ .  
 If  $f(v_2) > f(v_5)$ , then  $f(v_2) \in [j+2k, \sigma - j]$  and  $f(v_5) \in [j+k, \sigma - j - k]$ . Combining with  $f(v_3), f(v_4) \in [k, \sigma - k]$ , we can conclude that  $f(v_i) \in [k, \sigma - j]$  for  $2 \leq i \leq 5$ . Similar to the above case, we have  $\sigma \geq 4j + k$ .

This completes the proof.  $\square$

**Theorem 2.4.** Let  $j$  and  $k$  be two positive numbers with  $k \geq 3j$ . We have  $\sigma_{j,k}(P_6^2) = j + 2k$ .

**Proof.** Define a circular labeling  $f$  for  $P_n^2$  as follows:

$f(v_0) = 0, f(v_1) = j, f(v_2) = 2j, f(v_3) = k, f(v_4) = j + k$  and  $f(v_5) = 2j + k$ .

It is clearly that  $f$  is a circular  $(j+2k)$ - $L(j,k)$ -labeling for  $P_6^2$ . Then  $\sigma_{j,k}(P_6^2) \leq j + 2k$ .

On the other hand, since  $P_5^2$  is an induced subgraph of  $P_6^2$ , by Lemma 2.1 and Theorem 2.2, we have  $\sigma_{j,k}(P_6^2) \geq \sigma_{j,k}(P_5^2) \geq j + 2k$ . Hence, the proof completes.  $\square$

**Theorem 2.5.** Let  $j$  and  $k$  be two positive numbers with  $k \geq 3j$ . We have  $\sigma_{j,k}(P_n^2) = \frac{7}{3}k$  for  $n \geq 7$ .

**Proof.** Let  $\sigma = \frac{7}{3}k$ . For  $n \geq 7$ , define a circular labeling  $f$  of  $P_n^2$  by  $f(v_i) = [ik/3]_\sigma$  for  $0 \leq i \leq n - 1$ . It is easy to verify that  $f$  is a circular  $\frac{7}{3}k$ - $L(j,k)$ -labeling of  $P_n^2$ . Thus,  $\sigma_{j,k}(P_n^2) \leq \frac{7}{3}k$ .

Since  $P_7^2$  is an induced subgraph of  $P_n^2$  for  $n \geq 7$ , by Lemma 2.1, we have  $\sigma_{j,k}(P_n^2) \geq \sigma_{j,k}(P_7^2)$ . It suffices to find a lower bound of  $\sigma_{j,k}(P_7^2)$ .

Let  $\sigma = \sigma_{j,k}(P_7^2)$ . Note that  $\sigma \leq \frac{7}{3}k$ . Let  $f$  be a circular  $\sigma$ - $L(j,k)$ -labeling of  $P_7^2$ . We define the *traveling distance* from  $v_i$  to  $v_j$ , denoted by  $\mathcal{L}(i, j)$ , to be the length from  $f(v_i)$  to  $f(v_j)$  along the clockwise direction in the  $\sigma$ -cycle. In  $P_7^2$ , if we travel the vertices from  $v_0, v_4, v_1, v_5, v_2, v_6, v_3$  one by one, then the total traveling distance from  $v_0$  to  $v_3$  is  $T = \mathcal{L}(0, 4) + \mathcal{L}(4, 1) + \mathcal{L}(1, 5) + \mathcal{L}(5, 2) + \mathcal{L}(2, 6) + \mathcal{L}(6, 3)$ . As  $k \leq \mathcal{L}(u, v) \leq \sigma - k$  for  $d(u, v) = 2$ , we have

$$6k \leq T \leq 6(\sigma - k). \quad (2.1)$$

Since  $\sigma \leq \frac{7}{3}k$ , we have  $2 < \frac{6k}{\sigma} \leq \frac{T}{\sigma} \leq 6 - \frac{6k}{\sigma} < 4$ . It implies that we arrive  $f(v_3)$  from  $f(v_0)$  after rotating the  $\sigma$ -cycle 2 or 3 times.

Suppose we arrive  $f(v_3)$  from  $f(v_0)$  after rotating the  $\sigma$ -cycle 2 times. Since  $v_3$  and  $v_0$  are at distance 2 and  $f(v_0) = 0$ , it follows that  $|T - 2\sigma|_\sigma \geq k$ , that is  $2\sigma + k \leq T \leq 3\sigma - k$ . From (2.1), we have  $6k \leq 3\sigma - k$ , and thus,  $\sigma \geq \frac{7}{3}k$ .

Suppose we arrive  $f(v_3)$  from  $f(v_0)$  after rotating the  $\sigma$ -cycle 3 times. Since  $v_3$  and  $v_0$  are at distance 2 and  $f(v_0) = 0$ , it follows that  $|T - 3\sigma|_\sigma \geq k$ , that is  $3\sigma + k \leq T \leq 4\sigma - k$ . From (2.1), we have  $3\sigma + k \leq 6(\sigma - k)$ , and hence  $\sigma \geq \frac{7}{3}k$ .

Then, we have  $\sigma_{j,k}(P_n^2) \geq \sigma_{j,k}(P_7^2) = \sigma \geq \frac{7}{3}k$ . The proof completes.  $\square$

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