REMARKS ON FLAT AND DIFFERENTIAL K-THEORY

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ABSTRACT. In this note we prove some results in flat and differential K-theory. The first one is a proof of the compatibility of the differential topological index and the flat topological index by a direct computation. The second one is the explicit isomorphisms between Bunke-Schick differential K-theory and Freed-Lott differential K-theory.

CONTENTS

1. INTRODUCTION

In this note we prove some results in flat and differential K-theory. While some of these results are known to the experts, the proofs given here have not appeared in the literature. We first prove the compatibility of the flat topological index $\mathrm{ind}^{\mathrm{t}}_{\mathrm{L}}$ and the differential topological index $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}$ by a direct computation, i.e., the following diagram commutes ([7, Proposition 8.10])

$$\begin{array}{ccc} K^{-1}_{\mathrm{L}}(X;\mathbb{R}/\mathbb{Z}) & \xrightarrow{i} & \widehat{K}_{\mathrm{FL}}(X) \\ \mathrm{ind}^{\mathrm{t}}\downarrow & & \downarrow\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}} \\ K^{-1}_{\mathrm{L}}(B;\mathbb{R}/\mathbb{Z}) & \xrightarrow[i]{} & \widehat{K}_{\mathrm{FL}}(B) \end{array} \tag{1}$$

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where i is the canonical inclusion, $K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$ is the geometric model of K-theory with $\mathbb{R}/\mathbb{Z}$ coefficients and $\widehat{K}_{\mathrm{FL}}(X)$ is Freed-Lott differential K-theory. The commutativity of (1) is a consequence of the compatibility of the differential analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{FL}}$ and the flat analytic index $\mathrm{ind}^{\mathrm{a}}_{\mathrm{L}}$ together with the differential family index theorem [7, Theorem 7.35]. The differential topological index $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}$ is defined to be the composition of an embedding pushforward and a projection pushforward. When defining the embedding pushforward, currential K-theory [7, §2.28] is used instead of differential K-theory due to the Bismut-Zhang current [2, Definition 1.3]. It is not clear whether currential K-theory should be regarded as a differential cohomology or a "differential homology" (see [6, §4.5] for a detailed discussion), so it may be clearer by looking at the direct computation.

Second we construct the unique natural isomorphisms between Bunke-Schick differential K-theory [4] and Freed-Lott differential K-theory by writing down the explicit formulas, which are inspired by [4, Corollary 5.5]. The uniqueness follows from [5, Theorem 3.10]. Together with [10, Theorem 4.34] and [8, Theorem 1] all the explicit isomorphisms between all the existing differential K-groups [9], [4], [7], [12] are known.

The paper is organized as follows: Section 2 contains all the necessary background material, including the Freed-Lott differential K-theory, the differential topological index, the pairing between flat K-theory and K-homology, and Bunke-Schick differential K-theory. In Section 3 we prove the main results.

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2. Background material

2.1. Freed-Lott differential K-theory and the differential topological index.

In this section we review Freed-Lott differential K-theory and the construction of the differential topological index [7, §4, 5]. We refer the readers to [7] for the details.

The Freed–Lott differential K-group $\widehat{K}_{\mathrm{FL}}(X)$ is the abelian group generated by quadruples $\mathcal{E} = (E, h, \nabla, \phi)$, where $(E, h, \nabla) \to X$ is a complex vector bundle with a Hermitian metric h and a unitary connection ∇, and $\phi \in \dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$. The only relation is $\mathcal{E}_1 = \mathcal{E}_2$ if and only if there exists a generator $(F, h^F, \nabla^F, \phi^F)$ of $\widehat{K}_{\mathrm{FL}}(X)$ such that $E_1 \oplus F \cong E_2 \oplus F$ and $\phi_1 - \phi_2 = \mathrm{CS}(\nabla^{E_2} \oplus \nabla^F, \nabla^{E_1} \oplus \nabla^F)$.

There is an exact sequence [7, (2.20)]

$$0 \longrightarrow K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z}) \xrightarrow{\ i\ } \widehat{K}_{\mathrm{FL}}(X) \xrightarrow{\mathrm{ch}_{\widehat{K}}} \Omega^{\mathrm{even}}_{\mathrm{BU}}(X) \longrightarrow 0 \qquad (2)$$

where $K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$ is the geometric model of $\mathbb{R}/\mathbb{Z}$ K-theory [11], i is the canonical inclusion map,

$$\Omega^{\mathrm{even}}_{\mathrm{BU}}(X) = \{\omega \in \Omega^{\mathrm{even}}_{d=0}(X) | [\omega] \in \mathrm{Im}(r \circ \mathrm{ch} : K^0(X) \to H^{\mathrm{even}}(X;\mathbb{R}))\},$$

and $\mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(E, h, \nabla, \phi) := \mathrm{ch}(\nabla) + d\phi$. Elements in $K_{\mathrm{L}}^{-1}(X;\mathbb{R}/\mathbb{Z})$ are required to have virtual rank zero. The canonical inclusion map i in (2) is defined by $i(E, h, \nabla, \phi) = (E, h, \nabla, \phi)$.

Let $X \to B$ and $Y \to B$ be fiber bundles of smooth manifolds with X compact. Let $g^{T^V X}$ and $g^{T^V Y}$ be metrics on the vertical bundles $T^V X \to X$ and $T^V Y \to Y$ respectively, and assume there are horizontal distributions $T^H X$ and $T^H Y$. Let $\mathcal{E} = (E, h^E, \nabla^E, \phi) \in \widehat{K}_{\mathrm{FL}}(X)$ and $\iota : X \hookrightarrow Y$ be an embedding of manifolds. We assume the codimension of X in Y is even, and the normal bundle $\nu \to X$ of X in Y carries a spin^c structure. As in [7, §5] we assume for each $b \in B$, the map $\iota_b : X_b \to Y_b$ is an isometric embedding and is compatible with projections to B. Denote by $\mathcal{S}(\nu) \to X$ the spinor bundle associated to the spin^c-structure of $\nu \to X$. We can locally choose a spin stricture for $\nu \to X$ with spinor bundle $\mathcal{S}^{\mathrm{spin}}(\nu)$. Then there exists a locally defined Hermitian line bundle $L^{\frac{1}{2}}(\nu)$ such that $\mathcal{S}(\nu) \cong \mathcal{S}^{\mathrm{spin}}(\nu) \otimes L^{\frac{1}{2}}(\nu)$. Note that the tensor product on the right is globally defined, and so is the Hermitian line bundle $L(\nu) \to X$ defined by $L(\nu) := (L^{\frac{1}{2}}(\nu))^2$. Let ∇^ν be a metric compatible connection on $\nu \to X$. It has a unique lift to a connection on $\mathcal{S}^{\mathrm{spin}}(\nu)$, still denoted by ∇^ν. Choose a unitary connection $\nabla^{L(\nu)}$ on $L(\nu) \to X$, which induces a connection on $L^{\frac{1}{2}}(\nu)$. The tensor product of ∇^ν and the induced connection on $L^{\frac{1}{2}}(\nu)$ is a connection on $\mathcal{S}(\nu) \to X$, denoted by $\widehat{\nabla}^\nu$. Define

$$\mathrm{Todd}(\widehat{\nabla}^\nu) := \widehat{A}(\nabla^\nu) \wedge e^{\frac{1}{2}c_1(\nabla^{L(\nu)})}.$$

The embedding pushforward $\widehat{\iota}_* : \widehat{K}_{\mathrm{FL}}(X) \to {}_\delta\widehat{K}_{\mathrm{FL}}(Y)$ [7, Definition 4.14] is defined to be

$$\widehat{\iota}_*(\mathcal{E}) = \left(F, h^F, \nabla^F, \frac{\phi^E}{\mathrm{Todd}(\widehat{\nabla}^\nu)} \wedge \delta_X - \gamma\right),$$

where ${}_\delta\widehat{K}_{\mathrm{FL}}(Y)$ is the currential K-group, δ_X is the current of integration over X and γ is the Bismut-Zhang current. (F, h^F, ∇^F) is a Hermitian bundle with a Hermitian metric and a unitary connection chosen as in [7, Lemma 4.4]. Note that γ satisfies the following transgression formula [2, Theorem 1.4]

$$d\gamma = \mathrm{ch}(\nabla^F) - \frac{\mathrm{ch}(\nabla^E)}{\mathrm{Todd}(\widehat{\nabla}^\nu)} \wedge \delta_X.$$

As noted in [7, p.926] the horizontal distributions of the fiber bundles $X \to B$ and $Y \to B$ need not be compatible. An odd form $\widetilde{C} \in \dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ is defined to correct this non-compatibility, and it satisfies the following transgression formula [7, (5.6)]

$$d\widetilde{C} = \iota^* \operatorname{Todd}(\widehat{\nabla}^{T^V Y}) - \operatorname{Todd}(\widehat{\nabla}^{T^V X}) \wedge \operatorname{Todd}(\widehat{\nabla}^{\nu}).$$

The modified embedding pushforward $\widehat{\iota}_*^{\mathrm{mod}} : \widehat{K}_{\mathrm{FL}}(X) \to {}_{\mathrm{WF}}\widehat{K}_{\mathrm{FL}}(Y)$ [7, Definition 5.8] is defined to be

$$\widehat{\iota}_*^{\mathrm{mod}}(\mathcal{E}) := \widehat{\iota}_*(\mathcal{E}) - j\bigg(\frac{\widetilde{C}}{\iota^* \operatorname{Todd}(\widehat{\nabla}^{T^V Y}) \wedge \operatorname{Todd}(\widehat{\nabla}^{\nu})} \wedge \operatorname{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathcal{E}) \wedge \delta_X\bigg). \quad (3)$$

See [7, §3.1] for the definition of ${}_{\mathrm{WF}}\widehat{K}_{\mathrm{FL}}(X)$.

The differential topological index $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{FL}}(B)$ [7, Definition 5.34] is defined by taking $Y = \mathbb{S}^N \times B$ for some even N and composing the embedding pushforward with the submersion pushforward $\widehat{\pi}_*^{\mathrm{prod}}$ defined in [7, Lemma 5.13], i.e., $\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}} := \widehat{\pi}_*^{\mathrm{prod}} \circ \widehat{\iota}_*^{\mathrm{mod}}$.

2.2. **Pairing between flat K-theory and topological K-homology.** Let X be an odd-dimensional closed spin^c manifold. Let $\mathcal{E} = (E, h^E, \nabla^E, \phi) \in {}_{\delta}\widehat{K}_{\mathrm{FL}}(X)$, and $\mathsf{D}^{X,E}$ be the twisted Dirac operator on $\mathcal{S}(X) \otimes E \to X$. A modified reduced eta-invariant $\bar{\eta}(X, \mathcal{E}) \in \mathbb{R}/\mathbb{Z}$ [7, Definition 2.33] is defined by

$$\bar{\eta}(X, \mathcal{E}) := \bar{\eta}(\mathsf{D}^{X,E}) + \int_X \operatorname{Todd}(\widehat{\nabla}^{TX}) \wedge \phi \mod \mathbb{Z}.$$

$\bar{\eta} : {}_{\delta}\widehat{K}_{\mathrm{FL}}(X) \to \mathbb{R}/\mathbb{Z}$ is a well defined homomorphism [7, Prop 2.25]. If $\mathcal{E}$ is a generator of $K^{-1}_{\mathrm{L}}(X; \mathbb{R}/\mathbb{Z})$, by [7, (2.37)] we have

$$\bar{\eta}(X, i(\mathcal{E})) = \langle [X], \mathcal{E} \rangle, \quad (4)$$

where $[X] \in K_{-1}(X)$ is the fundamental K-homology class. Here $\langle [X], \mathcal{E} \rangle$ is the perfect pairing between flat K-theory and topological K-homology [11, Prop 3]

$$K^{-1}_{\mathrm{L}}(X; \mathbb{R}/\mathbb{Z}) \times K_{-1}(X) \to \mathbb{R}/\mathbb{Z}. \quad (5)$$

2.3. **Bunke-Schick differential K-theory.** In this subsection we briefly recall Bunke-Schick differential K-theory $\widehat{K}_{\mathrm{BS}}$, and refer to [4] for the details.

A generator of $\widehat{K}_{\mathrm{BS}}(B)$ is of the form $(\mathscr{E}, \phi)$, where $\mathscr{E}$ is an even-dimensional geometric family [4, Definition 2.2] over a compact manifold B and $\phi \in \dfrac{\Omega^{\mathrm{odd}}(B)}{\mathrm{Im}(d)}$. Roughly speaking a geometric family over B is the geometric data needed to construct the index bundle. There is a well defined notion of isomorphic and sum of generators [4, Definition 2.5, 2.6]. Two geometric families $(\mathscr{E}_0, \phi_0)$ and $(\mathscr{E}_1, \phi_1)$ are equivalent if there exists a geometric family

$(\mathscr{E}', \phi')$ such that $(\mathscr{E}_0, \rho_0) + (\mathscr{E}', \phi')$ is paired with $(\mathscr{E}_1, \rho_1) + (\mathscr{E}', \phi')$ [4, Definition 2.10, Lemma 2.13]. Two generators $(\mathscr{E}_0, \phi_0)$ and $(\mathscr{E}_1, \phi_1)$ are paired if

$$\rho_1 - \rho_0 = \eta^{\mathrm{B}}((\mathscr{E}_0 \sqcup_B (\mathscr{E}_1)^{\mathrm{op}})_t),$$ [1]

where $(\mathscr{E} \sqcup_B (\mathscr{E}')^{\mathrm{op}})_t$ is a certain tamed geometric family [4, Definition 2.7], and η^{B} is the Bunke eta form [3].

As noted in [4, 2.14] and [3, 4.2.1], a complex vector bundle $E \to B$ with a Hermitian metric h^E and a unitary connection ∇^E can be naturally considered as a zero-dimensional geometric family over B, denoted by $\mathbb{E}$.

3. Main results

3.1. **Compatibility of the topological indices.** Note that every element $\mathcal{E} - \mathcal{F} \in \widehat{K}_{\mathrm{FL}}(X)$ can be written in the form

$$\widetilde{\mathcal{E}} - [n].$$

Here $\widetilde{\mathcal{E}} = (E \oplus G, h^E \oplus h^G, \nabla^E \oplus \nabla^G, \phi^E + \phi^G)$, where $(G, h^G, \nabla^G, \phi^G)$ is a generator of $\widehat{K}_{\mathrm{FL}}(X)$ such that

$$(F \oplus G, h^F \oplus h^G, \nabla^F \oplus \nabla^G, \phi^F + \phi^G) = (\mathbb{C}^n, h, d, 0) =: [n].$$

The existence of the connection ∇^G such that $\mathrm{CS}(\nabla^F \oplus \nabla^G, d) = 0$, where d is the trivial connection on the trivial bundle $X \times \mathbb{C}^n \to X$, follows from [12, Theorem 1.8]. Here $\phi^G := -\phi^F$. Henceforth we assume an element of $\widehat{K}_{\mathrm{FL}}(X; \mathbb{R}/\mathbb{Z})$ is of the form $\mathcal{E} - [n]$. These arguments also apply to elements in $K_{\mathrm{L}}^{-1}(X; \mathbb{R}/\mathbb{Z})$.

Proposition 1. Let $\pi : X \to B$ be a fiber bundle with X compact and such that the fibers are of even dimension. The following diagram commutes.

$$\begin{array}{ccc} K_{\mathrm{L}}^{-1}(X; \mathbb{R}/\mathbb{Z}) & \xrightarrow{i} & \widehat{K}_{\mathrm{FL}}(X) \\ {\scriptstyle \mathrm{ind}^{\mathrm{t}}} \downarrow & & \downarrow {\scriptstyle \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}} \\ K_{\mathrm{L}}^{-1}(B; \mathbb{R}/\mathbb{Z}) & \xrightarrow[i]{} & \widehat{K}_{\mathrm{FL}}(B) \end{array}$$

Proof. Let $\mathcal{E}' - [n]' \in K_{\mathrm{L}}^{-1}(X)$ and write $\mathcal{E} - [n] = i(\mathcal{E}' - [n]')$, where i is given in (2). Consider the difference

$$h := \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n]) - i(\mathrm{ind}^{\mathrm{t}}(\mathcal{E}' - [n]')).$$

[1] It differs by a sign in [4].

We prove that $h = 0$. By [7, Lemma 5.36] and the fact that $\mathrm{ch}_{\widehat{K}_{\mathrm{FL}}} \circ i = 0$ (see (2)), we have

$$\begin{aligned} &\mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) - \mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(i(\mathrm{ind}^{\mathrm{t}}(\mathcal{E}' - [n]'))) \\ &= \mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) \\ &= \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^V X}) \wedge (\mathrm{ch}(\nabla^E) - \mathrm{rank}(E) + d\phi^E) \\ &= 0. \end{aligned}$$

By (2), there exists an element $a \in K^{-1}(B; \mathbb{R}/\mathbb{Z})$ such that $i(a) = h$. To prove $a = 0 \in K^{-1}_{\mathrm{L}}(B; \mathbb{R}/\mathbb{Z})$, it follows from (5) that it is sufficient to show that for all $\alpha \in K_{-1}(B; \mathbb{Z})$,

$$\langle \alpha, a \rangle = 0 \in \mathbb{R}/\mathbb{Z}. \tag{6}$$

Using the geometric picture of K-homology [1], we may, without loss of generality, let $\alpha = f_*[M]$ for some smooth map $f : M \to B$, where M is a closed odd-dimensional spin^c manifold, and $[M]$ is the fundamental K-homology in $K_{-1}(M)$. Since $\langle \alpha, a \rangle = \langle [M], f^* a \rangle$, we pull everything back to M and we may assume B is an arbitrary closed odd-dimensional spin^c manifold. Thus proving (6) is equivalent to proving

$$\langle [B], a \rangle = 0 \in \mathbb{R}/\mathbb{Z}. \tag{7}$$

Since

$$\langle [B], a \rangle = \bar{\eta}(B, \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) - \bar{\eta}(B, i(\mathrm{ind}^{\mathrm{t}}(\mathcal{E}' - [n]'))) \mod \mathbb{Z},$$

proving (7) is equivalent to proving

$$\bar{\eta}(B, \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) = \bar{\eta}(B, i(\mathrm{ind}^{\mathrm{t}}(\mathcal{E}' - [n]'))) \mod \mathbb{Z}. \tag{8}$$

In the following, we write $a \equiv b$ as $a = b \mod \mathbb{Z}$. By [7, (6.7)], we have

$$\begin{aligned} \bar{\eta}(B, \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E} - [n])) &\equiv \bar{\eta}(\mathsf{D}^{X, E-n}) + \int_X \frac{\iota^* \mathrm{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N \times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi^E \\ &\quad - \int_X \frac{\pi^* \mathrm{Todd}(\widehat{\nabla}^{TB})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \widetilde{C} \wedge \mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathcal{E} - [n]) \qquad (9) \\ &\equiv \bar{\eta}(\mathsf{D}^{X, E-n}) + \int_X \frac{\iota^* \mathrm{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N \times B)})}{\mathrm{Todd}(\widehat{\nabla}^{\nu})} \wedge \phi^E \end{aligned}$$

as $\mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathcal{E} - [n]) = \mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(i(\mathcal{E}' - [n]')) = 0$. On the other hand, by [11, (49)], we have

$$\begin{aligned} \bar{\eta}(B, i(\mathrm{ind}^{\mathrm{t}}(\mathcal{E}' - [n]'))) &\equiv \langle [B], \mathrm{ind}^{\mathrm{t}}(\mathcal{E}' - [n]') \rangle \\ &= \langle \pi^! [B], \mathcal{E} - [n] \rangle = \langle [X], \mathcal{E} - [n] \rangle = \bar{\eta}(X, \mathcal{E} - [n]) \\ &\equiv \bar{\eta}(\mathsf{D}^{X, E-n}) + \int_X \mathrm{Todd}(\widehat{\nabla}^{TX}) \wedge \phi^E. \end{aligned} \tag{10}$$

From (9) and (10) we have

$$\begin{aligned}
&\bar{\eta}(B, \mathrm{ind}^{\mathrm{t}}_{\mathrm{FL}}(\mathcal{E}-[n])) - \bar{\eta}(B, i(\mathrm{ind}^{\mathrm{t}}(\mathcal{E}'-[n]'))) \\
\equiv &\int_X \left(\frac{\iota^* \operatorname{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N\times B)})}{\operatorname{Todd}(\widehat{\nabla}^{\nu})} - \operatorname{Todd}(\widehat{\nabla}^{TX}) \right) \wedge \phi^E \\
\equiv &\int_X \left(\frac{\iota^* \operatorname{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N\times B)}) - \operatorname{Todd}(\widehat{\nabla}^{TX}) \wedge \operatorname{Todd}(\widehat{\nabla}^{\nu})}{\operatorname{Todd}(\widehat{\nabla}^{\nu})} \right) \wedge \phi^E.
\end{aligned} \tag{11}$$

Since $\mathrm{ch}_{\widehat{K}_{\mathrm{FL}}}(\mathcal{E}-[n]) = 0$, it follows from (3) that

$$\widehat{\iota}_*^{\mathrm{mod}}(\mathcal{E}-[n]) = \widehat{\iota}_*(\mathcal{E}-[n]). \tag{12}$$

Recall that the purpose of the modified embedding pushforward $\widehat{\iota}_*^{\mathrm{mod}}$ is to correct the non-compatibility of the horizontal distributions $T^H(\mathbb{S}^N \times B)$ and $T^H X$. By (12) we may assume that the horizontal distributions $T^H(\mathbb{S}^N \times B)$ and $T^H X$ are compatible by changing the one for X to be the restriction of the one for $\mathbb{S}^N \times B$. Thus

$$\iota^* \operatorname{Todd}(\widehat{\nabla}^{T(\mathbb{S}^N\times B)}) = \operatorname{Todd}(\widehat{\nabla}^{TX}) \wedge \operatorname{Todd}(\widehat{\nabla}^{\nu}),$$

which implies that (11) is zero, and therefore $h = 0$. □

3.2. **Explicit isomorphisms between $\widehat{K}_{\mathrm{BS}}$ and $\widehat{K}_{\mathrm{FL}}$.** In this subsection we construct the explicit isomorphisms between Bunke-Schick differential K-group and the Freed-Lott differential K-group.

Proposition 2. Let B be a compact manifold. Define two maps $f : \widehat{K}_{\mathrm{FL}}(B) \to \widehat{K}_{\mathrm{BS}}(B)$ and $g : \widehat{K}_{\mathrm{BS}}(B) \to \widehat{K}_{\mathrm{FL}}(B)$ by

$$\begin{aligned}
f(E, h, \nabla, \phi) &= [\mathbb{E}, \phi], \\
g([\mathscr{E}, \phi]) &= (\mathrm{ind}^{\mathrm{a}}(\mathscr{E}), h^{\mathrm{ind}^{\mathrm{a}}(\mathscr{E})}, \nabla^{\mathrm{ind}^{\mathrm{a}}(\mathscr{E})}, \phi),
\end{aligned}$$

where, in the definition of f, $\mathbb{E}$ is the zero-dimensional geometric family associated to (E, h, ∇). Then f and g are well defined ring isomorphisms and are inverses to each other.

Proof. Note that it suffices to prove the statement under the assumption that $\mathrm{ind}^a(\mathscr{E}) \to B$ is actually given by a kernel bundle $\ker(\mathsf{D}^E) \to B$ in the definition of g. Indeed, by a standard perturbation argument every class in $\widehat{K}_{\mathrm{BS}}$ has a representative where this is satisfied.

First of all we prove that f is well defined. Suppose

$$(E, h^E, \nabla^E, \phi^E) = (F, h^F, \nabla^F, \phi^F) \in \widehat{K}_{\mathrm{FL}}(B).$$

Then there exists a generator $(G, h^G, \nabla^G, \phi^G)$ of $\widehat{K}_{\mathrm{FL}}(B)$ such that

$$\begin{aligned}
E \oplus G &\cong F \oplus G, \\
\phi^F - \phi^E &= \mathrm{CS}(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G).
\end{aligned} \tag{13}$$

Denote by $\mathbb{F}$ and $\mathbb{G}$ the zero-dimensional geometric families associated to (F, h^F, ∇^F) and (G, h^G, ∇^G), respectively. We prove that $[\mathbb{E}, \phi^E] = [\mathbb{F}, \phi^F] \in$

$\widehat{K}_{\mathrm{BS}}(B)$. Indeed, we prove that $(\mathbb{E}+\mathbb{G},\phi^E+\phi^G)$ is paired with $(\mathbb{F}+\mathbb{G},\phi^F+\phi^G)$. We need to show $\mathbb{E}\sqcup_B\mathbb{G}\cong\mathbb{F}\sqcup_B\mathbb{G}$ and

$$(\phi^F+\phi^G)-(\phi^E+\phi^G)=\eta^{\mathrm{B}}(((\mathbb{E}\sqcup_B\mathbb{G})\sqcup_B(\mathbb{F}\sqcup_B\mathbb{G})^{\mathrm{op}})_t) \tag{14}$$

if such a taming exists. In the case of zero-dimensional geometric family, $\mathbb{E}\sqcup_B\mathbb{G}\cong E\oplus G$ as vector bundles over B. Thus the first equality (13) implies $\mathbb{E}\sqcup_B\mathbb{G}\cong\mathbb{F}\sqcup_B\mathbb{G}$. Since the underlying proper submersion of the trivial geometric family is the identity map, the corresponding kernel bundle is just $E\to B$ by the remark of [6, Definition 4.7]. Thus the taming in (14) exists and the definition of η^{B} shows that

$$\eta^{\mathrm{B}}(((\mathbb{E}\sqcup_B\mathbb{G})\sqcup_B(\mathbb{F}\sqcup_B\mathbb{G})^{\mathrm{op}})_t)=\mathrm{CS}(\nabla^E\oplus\nabla^G,\nabla^F\oplus\nabla^G). \tag{15}$$

From (13) and (14) we see that $(\mathbb{E}+\mathbb{G},\phi^E+\phi^G)$ is paired with $(\mathbb{F}+\mathbb{G},\phi^F+\phi^G)$. Thus f is well defined.

For the map g, note that under our assumption we have $[\mathscr{E},0]=[\mathbb{K},\widetilde{\eta}(\mathscr{E})]$ by [4, Corollary 5.5], where $\mathbb{K}$ is the trivial geometric family associated to $(\ker(\mathsf{D}^E),h^{\ker(\mathsf{D}^E)},\nabla^{\ker(\mathsf{D}^E)})$ and $\widetilde{\eta}(\mathscr{E})$ is the associated Bismut-Cheeger eta form. Since $[\mathscr{E},\phi]=[\mathbb{K},\widetilde{\eta}(\mathscr{E})+\phi]$, g can be written as

$$g([\mathscr{E},\phi])=g([\mathbb{K},\widetilde{\eta}(\mathscr{E})+\phi])=(\ker(\mathsf{D}^E),h^{\ker(\mathsf{D}^E)},\nabla^{\ker(\mathsf{D}^E)},\widetilde{\eta}(\mathscr{E})+\phi).$$

We prove that g is well defined. Suppose $[\mathscr{E}_1,\phi^1]=[\mathscr{E}_2,\phi^2]\in\widehat{K}_{\mathrm{BS}}(B)$. Since $[\mathscr{E}_i,\phi^i]=[\mathbb{K}_i,\widetilde{\eta}(\mathscr{E}_i)+\phi^i]$ for $i=1,2$, to prove $g([\mathscr{E}_1,\phi^1])=g([\mathscr{E}_2,\phi^2])$ it suffices to show

$$\begin{aligned}&(\ker(\mathsf{D}^{E^1}),h^{\ker(\mathsf{D}^{E^1})},\nabla^{\ker(\mathsf{D}^{E^1})},\widetilde{\eta}(\mathscr{E}_1)+\phi^1)\\&=(\ker(\mathsf{D}^{E^2}),h^{\ker(\mathsf{D}^{E^2})},\nabla^{\ker(\mathsf{D}^{E^2})},\widetilde{\eta}(\mathscr{E}_2)+\phi^2).\end{aligned} \tag{16}$$

Since $[\mathscr{E}_1,\phi^1]=[\mathscr{E}_2,\phi^2]$, there exists a taming $(\mathscr{E}_1\sqcup_B(\mathscr{E}_2)^{\mathrm{op}})_t$, and therefore $\ker(\mathsf{D}^{E^1})=\ker(\mathsf{D}^{E^2})\in K(B)$. Thus it suffices to show

$$\mathrm{CS}(\nabla^{\ker(\mathsf{D}^{E^2})},\nabla^{\ker(\mathsf{D}^{E^1})})=\widetilde{\eta}(\mathscr{E}_1)-\widetilde{\eta}(\mathscr{E}_2)+\phi^2-\phi^1\in\frac{\Omega^{\mathrm{odd}}(B)}{\Omega^{\mathrm{odd}}_{\mathrm{BU}}(B)} \tag{17}$$

by the exactness of [7, (2.21)]. Since

$$[\mathbb{K}_1,\widetilde{\eta}(\mathscr{E}_1)+\phi^1]=[\mathscr{E}_1,\phi^1]=[\mathscr{E}_2,\phi^2]=[\mathbb{K}_2,\widetilde{\eta}(\mathscr{E}_2)+\phi^2],$$

it follows that there exists a taming $(\mathbb{K}_2\sqcup_B(\mathbb{K}_1)^{\mathrm{op}})_t$ such that

$$\widetilde{\eta}(\mathscr{E}_1)-\widetilde{\eta}(\mathscr{E}_2)+\phi^1-\phi^2=\eta^{\mathrm{B}}((\mathbb{K}_2\sqcup_B(\mathbb{K}_1)^{\mathrm{op}})_t). \tag{18}$$

By the same reason as in (15) we have

$$\eta^{\mathrm{B}}((\mathbb{K}_2\sqcup_B(\mathbb{K}_1)^{\mathrm{op}})_t)=\mathrm{CS}(\nabla^{\ker(\mathsf{D}^{E^2})},\nabla^{\ker(\mathsf{D}^{E^1})}). \tag{19}$$

(17) follows by comparing (18) and (19). Thus g is well defined.

We prove that f and g are inverses to each other. Let (E,h,∇,ϕ) be a generator of $\widehat{K}_{\mathrm{FL}}(B)$. Then

$$(g\circ f)(E,h,\nabla,\phi)=g[(\mathbb{E},\phi)]=(E,h,\nabla,\phi).$$

On the other hand, for a generator $(\mathscr{E}, \phi)$ of $\widehat{K}_{\mathrm{BS}}(B)$,

$$\begin{aligned}(f \circ g)([\mathscr{E}, \phi]) &= f(\ker(\mathsf{D}^E), h^{\ker(\mathsf{D}^E)}, \nabla^{\ker(\mathsf{D}^E)}, \widetilde{\eta}(\mathscr{E}) + \phi) \\ &= [\mathbb{K}, \widetilde{\eta}(\mathcal{E}) + \phi] \\ &= [\mathscr{E}, \phi]\end{aligned}$$

by [4, Corollary 5.5] again.

Since f is a ring homomorphism, the same is true for g. Thus f and g are ring isomorphisms and are inverses to each other. □

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