

## DOCTORAL THESIS

# Fractional differential equations for modelling financial processes with jumps

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# ABSTRACT

The standard Black-Scholes model is under the assumption of geometric Brownian motion, and the log-returns for Black-Scholes model are independent and Gaussian. However, most of the recent literature on the statistical properties of the log-returns makes this hypothesis not always consistent. One of the ongoing research topics is to find a better financial pricing model instead of the Black-Scholes model.

In the present work, we concentrate on two typical 1-D option pricing models under the general exponential Lévy processes, namely the finite moment log-stable (FMLS) model and the Carr-Geman-Madan-Yor-eta (CGMYe) model, and we also propose a multivariate CGMYe model. Both the frameworks, and the numerical estimations and simulations are studied in this thesis.

In the future work, we shall continue to study the fractional partial differential equations (FPDEs) of the financial models, and seek for the efficient numerical algorithms of the American pricing problems.

**Keywords:** fractional partial differential equation; option pricing models; exponential Lévy process; approximate solution.

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