

## DOCTORAL THESIS

### Intrinsic meshless methods for PDEs on manifolds and applications

Chen, Meng

*Date of Award:*  
2018

[Link to publication](#)

#### General rights

Copyright and intellectual property rights for the publications made accessible in HKBU Scholars are retained by the authors and/or other copyright owners. In addition to the restrictions prescribed by the Copyright Ordinance of Hong Kong, all users and readers must also observe the following terms of use:

- Users may download and print one copy of any publication from HKBU Scholars for the purpose of private study or research
- Users cannot further distribute the material or use it for any profit-making activity or commercial gain
- To share publications in HKBU Scholars with others, users are welcome to freely distribute the permanent URL assigned to the publication

# Abstract

Radial basis function (RBF) methods for partial differential equations (PDEs), either in bulk domains, on surfaces, or in a combination of the formers, arise in a wide range of practical applications. This thesis proposes numerical approaches of RBF-based meshless techniques to solve these three kinds of PDEs on stationary and nonstationary surfaces and domains.

In Chapter 1, we introduce the background of RBF methods, some basic concepts, and error estimates for RBF interpolation. We then provide some preliminaries for manifolds, restricted RBFs on manifolds, and some convergence properties of RBF interpolation. Finally, implicit-explicit time stepping schemes are briefly presented.

In Chapter 2, we propose methods to implement meshless collocation approaches intrinsically to solve elliptic PDEs on smooth, closed, connected, and complete Riemannian manifolds with arbitrary codimensions. Our methods are based on strong-form collocations with oversampling and least-squares minimizations, which can be implemented either analytically or approximately. By restricting global kernels to the manifold, our methods resemble their easy-to-implement domain-type analogies, that is, Kansa methods. Our main theoretical contribution is a robust convergence analysis under some standard smoothness assumptions for high-order convergence. We simulate reaction-diffusion equations to generate Turing patterns and solve shallow water problems on manifolds.

In Chapter 3, we consider convective-diffusion problems that model surfactants or heat transport along moving surfaces. We propose two time-space algorithms by combining the methods of lines and kernel-based meshless collocation techniques intrinsic to surfaces. We use a low-order time discretization for fair comparison, and higher-order schemes in time are possible. The proposed methods can achieve second-order convergence. They use either analytic or approximated spatial discretization of the surface operators, which do not require regeneration of point clouds at each temporal iteration. Thus, they are alternatively applied to handle models on two types of evolving surfaces, which are defined as prescribed motions and governed by

geometric evolution laws, respectively. We present numerical examples on various evolving surfaces for the performance of our algorithms and apply the approximated one to merging surfaces.

In Chapter 4, a kernel-based meshless method is developed to solve coupled second-order elliptic PDEs in bulk domains and on surfaces, subject to Robin boundary conditions. It combines a least-squares kernel-based collocation method with a surface-type intrinsic approach. We can thus use each pair for discrete point sets, RBF kernels (globally and restrictedly), trial spaces, and some essential assumptions, to search for least-squares solutions in bulks and on surfaces, respectively. We first analyze error estimates for a domain-type Robin-boundary problem. Based on this analysis and the existing results for surface PDEs, we discuss the theoretical requirements for the Sobolev kernels used. We then select the orders of smoothness for the kernels in bulks and on surfaces. Finally, several numerical experiments are demonstrated to test the robustness of the coupled method in terms of accuracy and convergence rates under different settings.

**Keywords:** intrinsic kernel-based meshless collocation methods, radial basis functions, convergence analysis, least-squares minimizations, evolving and merging surfaces, parabolic PDEs, coupled bulk-surface PDEs.

# Table of Contents

Declaration	i
Abstract	ii
Acknowledgements	iv
Table of Contents	v
Chapter 1 Background and preliminaries	1
1.1 Radial basis function methods . . . . .	1
1.1.1 RBFs . . . . .	2
1.1.2 Native spaces . . . . .	4
1.1.3 Error estimates for RBF interpolation . . . . .	6
1.2 RBFs restricted on manifolds . . . . .	7
1.2.1 Manifolds . . . . .	7
1.2.2 Sobolev spaces on manifolds . . . . .	8
1.2.3 Restricted RBFs and error estimates . . . . .	10
1.3 Time stepping . . . . .	11
1.4 Outline of Thesis . . . . .	13
Chapter 2 Intrinsic Kansa-type meshless methods for PDEs on manifolds	15
2.1 Introduction . . . . .	15
2.2 Sobolev spaces on manifolds and embedding domains . . . . .	17
2.3 Partial differential equations on surfaces . . . . .	19
2.3.1 Kernels and discrete settings . . . . .	20

2.3.2	Stability and consistency . . . . .	22
2.4	Intrinsic meshless collocation methods and implementation . . . . .	26
2.4.1	Kansa-type projection method . . . . .	26
2.4.2	Approximated Kansa method . . . . .	28
2.5	Numerical demonstrations . . . . .	29
2.5.1	Effects of oversampling . . . . .	30
2.5.2	Smoothness of kernels . . . . .	32
2.5.3	Pattern formations and robustness . . . . .	33
2.5.4	Shallow water problems on surfaces . . . . .	36
2.6	Conclusions . . . . .	39
2.7	Future works . . . . .	39
2.8	Appendix . . . . .	42
2.8.1	Parametric equations for all tested manifolds . . . . .	42
2.8.2	Initial conditions of shallow water equations on surfaces . . . . .	43
Chapter 3	Intrinsic meshless collocation methods for solving PDEs on evolving and merging surfaces	44
3.1	Introduction . . . . .	44
3.2	Parameterized surfaces with prescribed evolution and an algorithm for convection-diffusion equations . . . . .	47
3.2.1	Semi-discretization in time . . . . .	48
3.2.2	Algorithm for solving PDEs on prescribed moving surfaces . . . . .	50
3.3	Moving surfaces based on point clouds . . . . .	52
3.3.1	Geometric evolution of surfaces . . . . .	53
3.3.2	Numerical solver based on point clouds . . . . .	54
3.4	Numerical examples . . . . .	57
3.4.1	Prescribed evolving surfaces . . . . .	57
3.4.2	Mass conservation on prescribed surfaces and simulations on geometric evolving surfaces . . . . .	63
3.5	Conclusions . . . . .	67

3.6	Future works . . . . .	68
Chapter 4 Kernel-based meshless collocation methods for solving coupled bulk-		
	surface PDEs . . . . .	73
4.1	Introduction . . . . .	73
4.2	Coupled bulk-surface elliptic PDEs . . . . .	74
4.3	Discrete settings, kernels and trial spaces . . . . .	75
4.3.1	Discrete settings . . . . .	75
4.3.2	Kernels . . . . .	75
4.3.3	Bulk and surface trial spaces . . . . .	76
4.4	Convergence analysis for bulk PDEs with Robin boundary conditions . . . . .	77
4.4.1	Essential inequalities . . . . .	78
4.4.2	Convergence results for bulk PDEs . . . . .	81
4.5	Meshless collocation methods for coupled bulk-surface PDEs . . . . .	83
4.5.1	Implementation . . . . .	83
4.5.2	Some discussions on convergence . . . . .	84
4.6	Numerical experiments . . . . .	87
4.6.1	Smoothness orders of kernels for accuracy and convergence . . . . .	88
4.6.2	Point settings for bulk and surface in a torus . . . . .	93
4.6.3	Simulations on other surfaces . . . . .	95
4.7	Conclusions . . . . .	97
4.8	Future works . . . . .	98
4.8.1	Error analysis . . . . .	98
4.8.2	Other problems . . . . .	99
	Bibliography . . . . .	101
	Curriculum Vitae . . . . .	111