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Theoretical Advances on Scattering Theory, Fractional Operators and Their Inverse Problems

XIAO Jingni

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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July 2018

Declaration

I hereby declare that this thesis represents my own work which has been done after registration for the degree of PhD at Hong Kong Baptist University, and has not been previously included in a thesis or dissertation submitted to this or any other institution for a degree, diploma or other qualifications.

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Abstract

Inverse problems arise in numerous fields of science and engineering where one tries to find out the desired information of an unknown object or the cause of an observed effect. They are of fundamental importance in many areas including radar and sonar applications, nondestructive testing, image processing, medical imaging, remote sensing, geophysics and astronomy among others. This study is concerned with three issues in scattering theory, fractional operators, as well as some of their inverse problems.

The first topic is scattering problems for electromagnetic waves governed by Maxwell equations. It will be proved in the current study that an inhomogeneous EM medium with a corner on its support always scatters by assuming certain regularity and admissible conditions. This result implies that one cannot achieve invisibility for such materials. In order to verify the result, an integral of solutions to certain interior transmission problem is to be analyzed, and complex geometry optics solutions to corresponding Maxwell equations with higher order estimate for the residual will be constructed.

The second problem involves the linearized elastic or seismic wave scattering described by the Lamé system. We will consider the elastic or seismic body wave which is composed of two different type of sub-waves, that is, the compressional or primary (P-) and the shear or secondary (S-) waves. We shall prove that the P- and the S-components of the total wave can be completely decoupled under certain geometric and boundary conditions. This is a surprising finding since it is known that the P- and the S-components of the elastic or seismic body wave are coupled in general. Results for decoupling around local boundary pieces, for boundary value problems, and for scattering problems are to be established. This decoupling property will be further applied to derive uniqueness and stability for the associated inverse problem of identifying polyhedral elastic obstacles by an optimal number of scattering measurements.

Lastly, we consider a type of fractional (and nonlocal) elliptic operators and the associated Calderón problem. The well-posedness for a kind of forward problems concerning the fractional operator will be established. As a consequence, the corresponding Dirichlet to Neumann map with certain mapping property is to be defined. As for the inverse problem, it will be shown that a potential can be uniquely identified by local Cauchy data of the associated nonlocal operator, in dimensions larger than or equal to two.

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Chapter 1

Introduction

Inverse problems ask to find out information of a medium that cannot be observed directly. They have wide application in science and engineering, for instance, X-ray tomography, oil detection, seismic inversion, image recovering, just to name a few. In this chapter, we will give a general background of inverse problems, as well as a brief introduction of the three specific topics which will be focused in the current study.

1.1 Inverse Problems

Inverse problems arise in various areas of science and engineering where one tries to find out the desired information of an unknown object or the cause of an observed effect. A general way to do so is to send probing waves, measure the resulting perturbation produced by the presence of the object, and deduce the targeted information from these measurements. For a collection of various topics in inverse problems, one may refer to [121].

As opposed to forward or direct problems, where the general model is known and one tries to find out the output information for given input, inverse problems are based on the setting that some information of the original model is missing and the goal is to find out this targeted information by collecting input and output pairs. Abstractly, suppose that F is a model with the information set Λ. Then forward problems aim to find out the response of the model corresponding to certain input x, namely,

$$x, F[\Lambda] \Longrightarrow F[\Lambda](x).$$

As for inverse problems, by giving input x and measuring the corresponding output $F[\Lambda](x)$, one tries to recover (part of) the unknown information Λ of F, that is,

$$\big(x, F[\Lambda](x)\big) \Longrightarrow \Lambda.$$

A large number of inverse problems are non-linear and ill-posed, including the ones to be considered in the current study.

Inverse scattering is an important area of inverse problems. In forward scattering, incident wave fields will be perturbed due to the presence of a penetrable or impenetrable inhomogeneity in the free space, which is called a scatterer. A scattered field will be generated owning to the interaction between the incident wave and the scatterer. The scattered wave obeys certain decaying property at infinity, namely, the radiation condition. As a consequence, the scattered wave possesses certain asymptotic behavior at infinity, which reveals a one-to-one correspondence to a function called the far-field pattern or scattering amplitude. The scatterer usually appears in two different ways. One is impenetrable or impedance obstacles, in which cases the wave propagation is governed by the corresponding partial differential equation (PDE) in the exterior domain with certain boundary condition, as well as another coupled PDE in the interior if it is impedance. The other type of scatterers is inhomogeneous medium, where the PDE with variable coefficients, reflecting the inhomogeneity, is satisfied in the whole domain. In forward scattering, the scatterer is known, and one aims to find the propagating waves and their properties. Inverse scattering problems are concerned with the opposite direction of forward scattering. It asks to recover targeted information of the scatterer by measuring its corresponding scattering waves. For a comprehensive study of inverse scattering problems, we refer to [30, 17, 19].

The *Calderón problem*, which is also called electrical impedance tomography (EIT), is another type of inverse problems which has received considerable attention since Calderón's paper [22] in 1980. Calderón asked and provided a partial solution to the question of whether the electrical conductivity of a medium can be determined by voltage and current data at the boundary. Mathematically speaking, the Calderón problem concerns whether one can recover (information of) the coefficients of differential operators from the corresponding Dirichlet to Neumann map. There have been many works on the Calderón problems under various settings in the last three decades (cf., the surveys [25, 11, 131]).

There are three main issues in inverse problems. *Uniqueness* concerns whether there is only one object which corresponds to the measurements, or there are several different materials react the same to the measurements. *Stability* asks how similar two objects are to each other, provided that the difference between the corresponding measurements are sufficiently small. *Reconstruction* aims at designing analytical and numerical methods to reconstruct the unknown object from measured data in practice.

1.2 Three Topics to be Explored

Three focused topics in this thesis will be briefly introduced in this section. They are, the electromagnetic scattering in the case when the inhomogeneous medium has a corner at its support, elastic/seismic wave decoupling under certain geometric and boundary conditions, and a type of fractional anisotropic operators and the associated Calderón problem.

1.2.1 Corner Scattering for Electromagnetic Fields

Considering scattering problems for electromagnetic (EM) waves, we are interested in the problem of whether singularities of the scatterer will necessarily affect the far-field behavior of the scattered waves. In fact, we will prove that, if a penetrable EM medium with regular parameters has a right corner at its support, it will *always scatter* incident waves, under certain admissibility conditions.

Besides its own interest, one of the motivations of our study on corner scattering is the possibility of invisibility cloaking. Instead of trying to uniquely determine an inhomogeneity in inverse problems, the goal of cloaking is to make the scatterer undetectable. It has been shown both in mathematics and in engineering that one can make an object invisible by constructing an appropriate cloak (cf. [62, 88, 112, 122, 16, 83, 61, 60, 82, 70, 107, 38, 37]).

Materials needed for perfect cloaking are known to be singular, which is in agreement with the known uniqueness results in corresponding inverse problems for regular coefficients (cf. [127]). Therefore, it is natural to ask whether partial cloaking, invisibility under certain incident waves for instance, is achievable when the cloaking device is of regular parameters. According to our corner scattering results, the answer is no for any regular EM medium which has a corner at its support, unless (perhaps) it is probed by a class of incident waves with special forms.

Our EM corner scattering result is verified by establishing a stronger one, via the analysis of *interior transmission problems (ITPs)* (see, e.g., [33, 111, 18, 66, 21, 19]) associated to the Maxwell system. More precisely, we will prove that if an EM field satisfies certain ITP in a domain with corners, then it cannot be extended across the corner in the way that the Maxwell equations are still satisfied in a neighborhood of this corner.

The corner scattering result concerning acoustic medium scatterers was first derived by Blåsten, Päivärinta and Sylvester [10] in 2014. They proved that, in any dimension n, acoustic medium scatterers with a rectangular corner will always scatter any incident field. Elschner and Hu then showed in [45], by assuming real-analyticity for the refractive index, that any corner in dimension two or any edge in dimension three scatterers every acoustic incident waves. Later on, Päivärinta, Salo and Vesalainen [110] proved similar corner scattering results in dimension two, but with less restriction on the coefficients of the medium. They also showed for dimension three that any convex conic corner, excluding those of angles in a countable set, will scatter acoustic incident fields nontrivially. Results for more general corners and edges can be found in [46].

We shall generalize in this thesis, non-trivially though, the result in [10], from acoustic wave propagation to the more challenging and practical EM case. Due to the complication of non-elliptic and vectorial Maxwell systems compared to the case of elliptic and scalar Helmholtz equations, substantial challenges will occur when generalizing the results. Firstly, by assuming the triviality of the far-field amplitude, an integral identity for (vectorial) solutions to the corresponding ITP for EM waves has to be derived. Secondly, it is to be verified that the Laplace transform for the dot product of certain two vectorial fields cannot vanish identically in any open set. Lastly and crucially, special complex solutions, which obeys certain exponentially decaying property, to corresponding variable coefficients Maxwell equations have to be constructed. Those kind of solutions are called complex geometric optics (CGO) solutions which have also found applications in other kinds of inverse problems; see for instance, the survey article [129].

The construction of CGO solutions with sufficiently sharp estimates for residual terms is of crucial importance in our proof of the main result. CGO solutions for Maxwell equations were first constructed in [32], where authors considered EM scatterers of variable electric permittivity and conductivity but constant magnetic permeability. It was then generalized in [109] (see also [108]) for the construction of CGO solutions to the fully variable Maxwell system. The main idea in [109] is to introduce two extra scalar fields in addition to the six-dimensional EM field, and then convert the Maxwell system to a system of eight Schrödinger equations, for which there are many more known results on CGO solutions. However, most of the known CGO solutions to Maxwell systems are constructed with an L^2 estimate for the residual term, which is not sufficient for our current study. Therefore, CGO solutions with L^p residuals of decaying order larger than $3/p$, for any $p \geq 2$, will also be constructed.

1.2.2 Decoupling of Elastic Fields and its Applications in Scattering and Inverse Scattering

Elastic, or seismic, body waves are mainly composed of primary (P-) and secondary (S-) waves, alias, compressional and shear waves. P-waves propagate longitudinally and faster than S-waves which travel transversally. These two kinds of waves are known to be coupled in general. For example, P-waves can convert to S-waves after striking an interface, and vice versa. However, we shall show in Chapter 3 that the P- and the S-waves can be completely decoupled in certain circumstances.

Let $\mathbf{U} = (U_j)_{j=1}^3$ denote the displacement field of a time-harmonic elastic or seismic wave field. It satisfies the Lamé system, or the reduced Navier equation,

$$(\Delta^* + \omega^2)\mathbf{U} = 0,$$

with the Lamé operator Δ^* given by

$$\Delta^*\mathbf{U} = \mu\Delta\mathbf{U} + (\lambda + \mu)\,\nabla\,(\nabla \cdot \mathbf{U})\,.$$

Denote the Helmholtz decomposition of $\mathbf{U}$ as

$$\mathbf{U} = -\,\nabla\,\phi + \operatorname{curl}\mathbf{F}.$$

It is known (and can be verified straightforwardly) that the scalar field ϕ, or $v := -\,\nabla \cdot \mathbf{U}$, which is closely related to the *P-wave*, satisfies the Helmholtz equation, and the vector field $\mathbf{F}$, or $\mathbf{E} := \operatorname{curl}\mathbf{U}$, corresponding to the *S-wave*, is a solution to the Maxwell equations.

We shall prove that, under certain geometric and boundary conditions, elastic/seismic displacement fields have a special correspondence to solutions of Helmholtz equations and those of Maxwell systems with the Dirichlet or Neumann conditions. More precisely, in the case of scattering on a fourth kind obstacle D, we will show that the scalar field ϕ, or v, satisfies the sound-soft acoustic scattering problem, and $\mathbf{F}$, or $\mathbf{E}$ applies to the electromagnetic one with a perfect electric conductor, provided that ∂D satisfies a piecewise geometric condition (see, Section 3.2). Otherwise if the obstacle D is polyhedral and of the third kind, then $\mathbf{F}$ or $\mathbf{E}$ satisfies the electromagnetic scattering model for a perfect magnetic conductor, and ϕ and v solves the sound-hard acoustic one. Decoupling results in domains connected to a third or fourth boundary piece, as well as decoupling for homogeneous third or fourth boundary value problems will also be established.

With the help of these decoupling properties, we are able to further obtain *uniqueness* as well as *stability* for corresponding linearized inverse elastic or seismic scattering problems in the identification of a third or fourth kind polyhedral obstacle. from an optimal number of scattering (near-field or far-field) measurements. In particular, we shall show that if a polyhedral obstacle without flat components is either a third or a fourth kind scatterer, then it can be identified from only one single total scattering measurement, or a single P-part (resp., S-part) one by assuming that the incident plane wave has nontrivial P-part (resp., S-part). In the case that the polyhedral scatter might contain two-dimensional pieces, we will verify the uniqueness of determining a fourth kind scatter from a single (total or P-) scattering measurement whose incident field has nontrivial P-part, or by two (total or S-) scattering measurements whose incident fields has nontrivial S-part. Otherwise if it is a third kind scatter with two-dimensional pieces, it can be identified by two (total or S-) measurements with incident fields of nonzero S-part, or by three (total or P-) measurements for incident waves with nonzero P-part.

It is important in practice to reduce the number of needed measurements in detecting a scatterer. Results for identifying an acoustic or electromagnet obstacle with particular property by minimal (and finite) number of measurement can be found in references, for instance, [26, 3, 47, 96, 94, 95, 67, 71, 73], and corresponding stability estimates are contained in [114, 123, 91, 90, etc.]. As for counterparts in inverse elastic/seismic scattering, much less has been found.

The first result in inverse elastic scattering is given by Hähner and Hsiao [65]. They showed the uniqueness of determination of an impenetrable elastic obstacle with the Dirichlet (first kind) boundary condition by infinitely number of far-field measurements. Gintides and Sini [58] then proved similar results but need only to use pure P- (or S-) far-field measurements generated by pure P- (or S-) type of incident plane waves.

As for detection of elastic obstacles by a few number of measurements, authors in [68] showed that one can detect a ball or a convex polyhedron by one measurement, which is also in accordance with counterpart results in inverse acoustic or EM scattering problems. Later on, the authors in [49] proved that a fourth kind polyhedral obstacle is uniquely identifiable by a single P-far-field measurement, and a third kind one can be detected by two S-far-field measurement. It is worth mentioning that the last reference [49] is the most relevant to our current study on uniqueness in inverse elastic scattering. But our uniqueness results apply to more general settings of scatterers and measurements. In addition, the arguments we shall use to prove our uniqueness results are completely different to those in [49]. Moreover, based one the uniqueness results, we will consider the stability for corresponding inverse elastic scattering of linearized elastic or seismic waves. The stability estimates are of logarithmic type, which is known to be optimal for inverse scattering problems at fixed energy.

The main idea in proving our uniqueness and stability results is to convert the elastic scattering into the acoustic scattering for the P-wave and the EM one for the S-wave, by applying the aforementioned decoupling properties. Then the uniqueness and stability in recovering third or fourth kind elastic/seismic obstacles can be obtained as a consequence of the known uniqueness and stability results for corresponding acoustic and EM inverse problems with polyhedral scatterers (see, [26, 3, 47, 97, 94, 114, 90, 91]).

1.2.3 A Fractional Operator and an Associated Calderón Problem

The Calderón problem, which is also called *Electrical Impedance Tomography (EIT)*, is an important type of inverse problems which has numerous applications in geophysics, medical imaging, non-destructive testing and many other fields. There has been considerable literature on Calderón problems under various settings (see, the survey papers [25, 11, 131]), since Calderón's work [22] in 1980. The Calderón problem can be formulated in a mathematical way as, to recover the conductivity γ inside a given domain Ω, from information of the Dirichlet to Neumann (DtN) map This is closely related to another inverse problem which asks to determine the potential q from the DtN map associated with the operator $\mathcal{L}_\gamma + q$.

Instead of the classical Calderón problem concerning (local) elliptic partial differential operators, we shall consider the one for the *(nonlocal) fractional anisotropic* operator $\mathcal{L}_\gamma^s$ with $s \in (0,1)$. The Calderón problem for fractional operators was first introduced in [55], where the authors considered to recover the potential q in a bounded domain Ω, from the corresponding DtN map of the nonlocal operator $(-\Delta)^s + q$ in Ω. It was proved in [55] that the potential q can be uniquely recovered from partial DtN data. Several related work on inverse problems concerning the fractional Laplacian has been done afterwards, see the review paper [120].

The current study is concerned with a more general fractional operator $\mathcal{L}_\gamma^s$. Unlike the case for the usual fractional Laplacian $(-\Delta)^s$, which is well defined and understood via the Fourier transform, the definition of $\mathcal{L}_\gamma^s$ is less clear. In fact, only implicit form of $\mathcal{L}_\gamma^s$ is available by far. The fractional operator $\mathcal{L}_\gamma^s$ considered in the current study is a bounded map from $H^s(\mathbb{R}^n)$ to $H^{-s}(\mathbb{R}^n)$, which has the following weak form

$$\left\langle \mathcal{L}_\gamma^s f, g \right\rangle := \int_{\mathbb{R}^n \times \mathbb{R}^n} \big(f(x) - f(z)\big)\big(g(x) - g(z)\big)\mathcal{K}_s(x,z)\,dxdz, \tag{1.1}$$

where $\mathcal{K}_s = \mathcal{K}_{\gamma,s}$ is defined via the heat kernel of the elliptic operator $\mathcal{L}_\gamma$. The definition (1.1) originates from the function $f(t) = t^s$ for operators via the spectral theorem for densely defined self-adjoint operators, but is more general in mapping properties. In addition, the weak form (1.1) coincides with the definition of $(-\Delta)^s$ with $\mathcal{K}_s$ defined by the heat kernel of $-\Delta$, see for instance, [86].

Nonlocal operators with the form (1.1) have found applications in various fields such as random walk and Lévy flights [100, 101], anomalous diffusion

[124], image processing [79, 57, 12], machine learning [118], shape optimization [34], population dynamics [98]. Mathematical theory for the fractional Laplacian or more general fractional/nonlocal operators as in the form (1.1) can be found in, just to name a few, [42, 36, 13, 51, 52, 116, 117, 14, 44, 40].

An interesting property for the fractional Laplacian $(-\Delta)^s$ or the more general one $\mathcal{L}_\gamma^s$ is the extension problem in one more dimension. In particular, it is proved in [15] that u and $(-\Delta)^s u$ defined in $\mathbb{R}^n$ are exactly the Dirichlet and the Neumann data of a $(n+1)$-dimensional degenerate operator $\nabla \cdot (x_{n+1}^{1-2s} \nabla U)$ in the half space. The result was then extended by [126] into the case for the more general fractional operator $\mathcal{L}_\gamma^s$, and the corresponding degenerate problem in one more dimension turns to $\nabla \cdot (x_{n+1}^{1-2s} \gamma \nabla U)$.

Based on the extension problem, an important unique continuation property for the fractional operator $\mathcal{L}_\gamma^s$ can be derived, that is, if both u and $\mathcal{L}_\gamma^s u$ vanish in an arbitrary open set, then u has to be zero in the whole space. This is completely different to the case for ordinary (local) differential operators. This property was first introduced in [119] as a unique continuation property of the fractional Laplacian $(-\Delta)^s$, and a complete version was proved in [55]. It was then generated in [54] for $s \in [1/4, 1)$ that, as long as u satisfies $(-\Delta)^s u + qu = 0$ in an open set and vanishes in a subset with positive measure (not necessarily containing any open set), then u must be identically zero in the whole space. In comparison with *complex geometrical optics (CGO)* solutions which plays a crucial role in Calderón problems for ordinary (local) differential operators, the unique continuation property for $\mathcal{L}_\gamma^s$ is crucial in obtaining uniqueness and other result for fractional inverse problems. It is noticed that the unique continuation property for $(-\Delta)^s$ is equivalent to the Runge approximation which states that any L^2 function in a bounded open set can be approximated by solutions in a larger domain. For the Runge approximation properties of the fractional operator, please see [55, 39, 53, 54, etc.] for references.

We shall generalize the results in [55] by showing that the potential q in any dimension $n \geq 2$ can be uniquely determined by *partial data* of the DtN map $\Lambda_q^s = \Lambda_{\gamma,q}^s$ associated with the nonlocal operator $\mathcal{L}_\gamma^s + q$. The study for

$\mathcal{L}^s$ is much more complicated than the one for $(-\Delta)^s$. The main challenges are twofold. Firstly, the fractional Laplacian $(-\Delta)^s$ has been well defined and understood via the Fourier transform, which is not available for $\mathcal{L}^s$ with γ a general positive definite matrix. Therefore, one has to obtain a proper definition of the fractional operator $\mathcal{L}^s_\gamma$ which satisfies desired mapping properties and coincides with the fractional Laplacian when the conductivity is the identical matrix. In addition, it is also natural to expect that a well-defined operator $\mathcal{L}^s_\gamma$ should be compatible with the definition for fractional operators by the spectral theorem. Another challenge is to prove the unique continuation property for the fractional Laplacian $(-\Delta)^s$ into the one for general fractional elliptic operators as $\mathcal{L}^s_\gamma$. The main idea to prove such a unique continuation property is to look into the corresponding extension problem.

Our uniqueness result on recovering q from the DtN map of the nonlocal operator $\mathcal{L}^s_\gamma + q$ is "stronger" than the counterpart for ordinary Calderón problems concerned with the local differential operator $\mathcal{L}_\gamma + q$ in three aspects. One is our results and arguments are the same for any dimension n, while those for classical Calderón problems are different in dimension two and higher dimensions. The second is that the results in our study are valid for general isotropic conductivity, but the three dimensional case for classical Calderón problems has not been solved completely yet. Lastly, our recovery of q will be set up with a very general partial data setting, in the sense that the Dirichlet and the Neumann data can be taken on two arbitrary open subsets of the full data set. However for classical Calderón problems, the best result for dimension two gives a partial data recovery with the Dirichlet and the Neumann data measured in arbitrary but the same data set. Even less is known for three dimensions classical Calderón problems with partial data. For more details of research concerning the classical Calderón problem, one can refer to the survey articles [130, 77, 131]

In order for the inverse problem to be accessed, the *well-posedness* of the (forward) "boundary" value problem for the fractional operator $\mathcal{L}^s_\gamma + q$ will be first established. Based on that, a proper definition of the *DtN map* Λ^s_q will be further provided. Owing to the nonlocal property of $\mathcal{L}^s_\gamma$, the "boundary" data is no longer measurements taken merely on the boundary of the domain, but instead the ones measured in (an open subset of) the exterior domain.

Chapter 2

Corner Scattering for Electromagnetic Fields

The time-harmonic electromagnetic scattering problem will be considered in this chapter. We are interested in the phenomenon where no perturbation of a probing wave can be observed outside or far from a scatterer. We will show that an electromagnetic medium with regular coefficients but a cubic corner at its support, will *always scatter* incident waves, under certain admissibility conditions. This implies that one cannot achieve *invisibility* for a regular electromagnetic medium with corners, except *perhaps* for a class of incident waves with special forms. In fact, we will prove a stronger result, which states that electromagnetic fields satisfying corresponding interior transmission problems cannot be extended across the corner in a way that the Maxwell equations are still satisfied. The result will verified by analyzing an integral of solutions to an associated interior transmission problem in the support of the medium. Complex geometric optics (CGO) solutions with sufficient sharp estimate for the residual terms shall play an important role in the proof of the main result.

2.1 Introduction

In this chapter, the case when a electromagnetic (EM) scatterer does not scatter an incident field is to be investigated. This is closely related to two questions in scattering theory and inverse problems; they are, the *non-scattering* *energy* as well as the *interior transmission problem*, and the *invisibility cloaking*.

The notion of non-scattering energies originates from the question that whether there is a case in scattering where the far-field pattern is zero while the incident field is nontrivial. If this is the case, then the corresponding wavenumber is called a *non-scattering energy* (cf. [132]). A relevant field of research is the *interior transmission problem (ITP)* [19]. In fact, a non-scattering energy must be an *interior transmission eigenvalue (ITEV)* in a domain, as long as the unique continuation principle applies to the scattered wave from the far-field up to the boundary of the domain. However, whether an ITEV can be a non-scattering energy depends on the possibility of the corresponding *interior transmission eigenfunction (ITEF)* extending as a solution to the scattering problem.

It is known that some important reconstruction techniques in inverse scattering such as the linear sampling method [29], which assumes injectivity of the far-field operator, may fail at the non-scattering energy. Therefore, non-scattering energies or ITEVs had been trying to avoid in inverse scattering, until researchers found that ITEVs might imply some physical properties of the medium and could be brought out by far-field measurements; see for instance, the survey articles [18, 21].

Unlike the existence of ITEVs, which has been almost solved completely (cf. [20]), there are very few results for the existence of non-scattering energies. A typical case is the spherically stratified medium, where non-scattering energies are exactly the same as ITEVs. Another type of known results is the nonexistence of the non-scattering energy. In particular, authors in [10] showed that acoustic non-scattering energies do not exist when the scattering potential has a cubic corner at its support. This result was then generated in different aspects by authors in [45, 72, 110, 46, etc.].

From another point of view, a scatterer would be invisible to observers from far away when it is probed by an incident wave which produces no far-field pattern. This leads to an interesting question of whether one can cloak an object such that it can be invisible, namely, the possibility of *invisibility* *cloaking.*

Invisibility cloaking appeals to both scientific and engineering community for its important and interesting applications in practice. It has been showed both theoretically and practically that invisibility is achievable by applying extremely irregular materials, which are called *metamaterials*; cf., [62, 88, 112, 122, 16, 83, 61, 60, 82, 70, 107, 38, 37].

On the other hand, metamaterials are not easy nor cheap to construct owning to the extreme properties they are supposed to have. Therefore, it is natural to ask whether invisibility is available for materials with regular parameters. Unfortunately, according to uniqueness results in inverse problems (cf. [30, 131]), this can only be done in an approximate way, or as a partial invisibility which is valid under particular circumstances. The results in [10] implies that invisibility is not even partially achievable when the inhomogeneous medium as a rectangular corner at its support, by showing the nonexistence of non-scattering energies.

We shall extend the results in [10] from acoustic scattering to the more challenging and practical case of EM corners scattering. We will prove that an EM medium with regular coefficients but a right corner, will always scatter incident waves, under certain regularity and admissibility conditions. This result implies that one cannot achieve invisibility for EM waves to a regular medium with corners, except perhaps for a class of incident waves.

The EM corner scattering result will be proved by establishing a stronger result, via the analysis for interior transmission eigenvalue problems (cf. [111, 80, 18]) associated with the Maxwell system. More specifically, we shall show that EM fields satisfying certain interior transmission eigenvalue problems cannot be extended across the corner, in such a way that the Maxwell equations remain to be satisfied in the extended neighborhood. This will be done by analyzing an integral identity for (vectorial) solutions to the corresponding ITP, establishing a non-vanishing property for Laplace transforms of the dot product for certain two EM fields, and constructing complex geometric optics (CGO) solutions with residuals decaying in a order larger than $3/p$ in L^p norm for any $p > 6$.

2.2 Electromagnetic Scattering from a Penetrable Corner

In this section, the main corner scattering result will be presented. We first introduce some backgrounds for EM scattering problems. The result for corner scattering as well as the stronger version concerning the eigenfunctions of an interior transmission problem is to be given and proved. Auxiliary results to be applied in the proof of the main results will also be stated.

2.2.1 EM Forward Scattering

The time-harmonic EM wave propagation at the frequency $\omega > 0$ is characterized by the following Maxwell system,

$$\operatorname{curl} \mathbf{E} - \mathtt{i}\omega\mu\mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} + \mathtt{i}\omega\gamma\mathbf{E} = 0, \tag{2.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ denote, respectively, the electric and the magnetic fields. The real coefficients $\mu = \mu(x) > 0$ in (2.1) stands for the magnetic permeability, γ is defined by

$$\gamma := \varepsilon + \mathtt{i}\sigma/\omega,$$

where $\varepsilon = \varepsilon(x)$ and $\sigma = \sigma(x)$ are, respectively, the electric permittivity and the electric conductivity.

Let $\Sigma \subset \mathbb{R}^3$ be an inhomogeneous EM medium with the EM parameters ε, μ and σ. Suppose that Σ is open and bounded in $\mathbb{R}^3$, and that

$$\varepsilon(x) = \varepsilon_0,\ \mu(x) = \mu_0,\ \sigma(x) = 0, \quad \text{for } x \in \mathbb{R}^3 \backslash \overline{\Sigma},$$

where ε_0 and μ_0 are two positive constants denoting the permittivity and the permeability of the free space. We shall fix the constants ε_0 and μ_0 through out this chapter. Let $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$ be a pair of incident fields satisfying in $\mathbb{R}^3$

$$\operatorname{curl} \mathbf{E}^{\mathrm{inc}} - \mathtt{i}\omega\mu_0\mathbf{H}^{\mathrm{inc}} = 0, \quad \operatorname{curl} \mathbf{H}^{\mathrm{inc}} + \mathtt{i}\omega\varepsilon_0\mathbf{E}^{\mathrm{inc}} = 0. \tag{2.2}$$

The scattering problem associating the inhomogeneous medium $(\Sigma; \varepsilon, \gamma)$ and the incident field $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$ is described by

$$\begin{cases} \operatorname{curl} \mathbf{E} - \mathtt{i}\omega\mu\mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} + \mathtt{i}\omega\gamma\mathbf{E} = 0, \qquad \text{in } \mathbb{R}^3, \\ \lim\limits_{|x|\to\infty} (\mathbf{H}^{\mathrm{sca}} \times x - |x|\mathbf{E}^{\mathrm{sca}}) = 0, \quad \text{uniformly for all } \hat{x} := x/|x| \in \mathbb{S}^2, \end{cases} \tag{2.3}$$

where $(\mathbf{E},\mathbf{H})$ represents the total EM field, and $(\mathbf{E}^{\mathrm{sca}},\mathbf{H}^{\mathrm{sca}}) := (\mathbf{E},\mathbf{H}) - (\mathbf{E}^{\mathrm{inc}},\mathbf{H}^{\mathrm{inc}})$ is the scattered field. The limit in the second line of (2.3) is the Silver-Müller radiation condition, which guarantees that the wave propagation is outgoing. As a consequence, the following asymptotic behavior at infinity applies,

$$\mathbf{E}^{\mathrm{sca}}(x) = \frac{e^{\mathrm{i}\omega|x|}}{|x|}\mathbf{E}^{\infty}(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad \mathbf{H}^{\mathrm{sca}}(x) = \frac{e^{\mathrm{i}\omega|x|}}{|x|}\mathbf{H}^{\infty}(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right).$$

The functions $\mathbf{E}^{\infty}$ and $\mathbf{H}^{\infty}$ defined on $\mathbb{S}^2$ are known as the far-field pattern, or the scattering amplitude, of the scattered electronic and magnetic fields respectively. It is known that there is one-to-one correspondence between $\mathbf{E}^{\mathrm{sca}}$ and $\mathbf{E}^{\infty}$, as well as $\mathbf{H}^{\mathrm{sca}}$ and $\mathbf{H}^{\infty}$. Moreover, one has that

$$\mathbf{H}^{\infty} = \hat{x} \times \mathbf{E}^{\infty} \qquad \text{for all } \hat{x} \in \mathbb{S}^2.$$

Since the scattered field as well as the far-field pattern is determined by both the incident waves and the scatterer, we sometimes write $\mathbf{E}^{\infty}$ as $\mathbf{E}^{\infty}(\Sigma; \mathbf{E}^{\mathrm{inc}}, \mathbf{E}^{\mathrm{inc}})$. Analogous notations also apply to $\mathbf{E}^{\mathrm{sca}}$, $\mathbf{H}^{\mathrm{sca}}$ and $\mathbf{H}^{\infty}$. The EM scattering problem (2.3) has been thoroughly studied among numerous references. For the well-posedness and other properties of the EM scattering system, readers may refer to [103, 106, 30, 91, etc.]

We are concerned with the possibility of the case $\mathbf{E}^{\infty}(\Sigma; \mathbf{E}^{\mathrm{inc}}, \mathbf{E}^{\mathrm{inc}}) \equiv 0$, or equivalently $\mathbf{H}^{\infty}(\Sigma; \mathbf{E}^{\mathrm{inc}}, \mathbf{E}^{\mathrm{inc}}) \equiv 0$. If it happens, then the scatterer induces no perturbation at the outside under the probing wave $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$, and hence is invisible to this probing field for any outside observer.

2.2.2 The Main Results

We shall show that an EM corner will always scatter incident waves except for a specific class of waves we have no assertion on it yet. Before presenting our main result for EM corners scattering in this section, we first specify mathematically what a cornered scatterer is.

Definition 2.1. An EM medium $(\Sigma; \varepsilon, \gamma)$ as described in Section 2.2.1 is called *cornered* if the following conditions are satisfied.

(1) There exists a sufficiently small ball B_{ϵ_0} in $\mathbb{R}^3$ such that

$$\gamma(x) \neq \varepsilon_0 \quad \text{and} \quad \mu(x) \neq \mu_0, \qquad \text{for any } x \in B_{\epsilon_0} \cap \mathcal{K},$$

where $\mathcal{K}$ denotes the *closed* positive orthant of $\mathbb{R}^3$.

(2) There is a sufficiently large ball B_{R_0} in $\mathbb{R}^3$ such that

$$\operatorname{supp}(\varepsilon_0 - \gamma) \subset \overline{B_{R_0} \cap \mathcal{K}} =: D \quad \text{and} \quad \operatorname{supp}(\mu_0 - \mu) \subset D.$$

In Definition 2.1, the definite position of the corner is constrained at the origin 0. However, this is not necessary. Therefore, we given the following a generalized definition of a cornered medium.

Definition 2.1'. An EM medium $(\Sigma; \varepsilon, \gamma)$ is called *cornered* if it satisfies the conditions in Definition 2.1 after a rigid change of coordinates.

Our main result concerning EM corners scattering is stated as the following.

Theorem 2.1. *Let $(\Sigma; \varepsilon, \gamma)$ be a cornered scatterer as defined in Definition 2.1. Assume more that $\varepsilon, \gamma \in C^4(D)$, and they do not vanish in D. Then Σ scatterers every incident field contained in the class $\mathscr{A}_{\mathrm{EM}}$, which will be defined later in Definition 2.3.*

Proof. Assume the opposite that there exists an incident field $(\mathbf{E}^{\mathrm{inc}}, \mathbf{E}^{\mathrm{inc}})$ such that the corresponding far-field pattern $\mathbf{E}^{\infty}(\Sigma; \mathbf{E}^{\mathrm{inc}}, \mathbf{E}^{\mathrm{inc}})$ to the scattering system (2.3) is identically zero. Then as a result of Rellich's lemma (cf. [30]) or the unique continuation property, one has that

$$(\mathbf{E}^{\mathrm{sca}}, \mathbf{H}^{\mathrm{sca}}) \equiv 0 \quad \text{in } \mathbb{R}^3 \backslash D,$$

where D is the bounded and closed set specified in Definition 2.1. By recalling the Maxwell equations that the incident field $(\mathbf{E}^{\mathrm{inc}}, \mathbf{E}^{\mathrm{inc}})$ and the total field $(\mathbf{E}, \mathbf{H})$ are subjected to, one has

$$\begin{cases} \operatorname{curl} \mathbf{E} - \mathtt{i}\omega\mu\mathbf{H} = 0, & \operatorname{curl} \mathbf{H} + \mathtt{i}\omega\gamma\mathbf{E} = 0, \quad \text{in } D, \\ \operatorname{curl} \mathbf{E}^{\mathrm{inc}} - \mathtt{i}\omega\mu_0\mathbf{H}^{\mathrm{inc}} = 0, & \operatorname{curl} \mathbf{H}^{\mathrm{inc}} + \mathtt{i}\omega\varepsilon_0\mathbf{H}^{\mathrm{inc}} = 0, \quad \text{in } D, \\ \nu \times \mathbf{E} = \nu \times \mathbf{E}^{\mathrm{inc}}, & \nu \times \mathbf{H} = \nu \times \mathbf{H}^{\mathrm{inc}}, \quad \text{on } \partial D, \end{cases} \tag{2.4}$$

with ν the unit normal of ∂D pointing outwards. Therefore, the two pairs $(\mathbf{E}^{\rm inc}, \mathbf{E}^{\rm inc})$ and $(\mathbf{E}, \mathbf{H})$ solve the problem (2.4). Moreover, $(\mathbf{E}^{\rm inc}, \mathbf{E}^{\rm inc})$ satisfies the Maxwell equation (2.2) in the whole space $\mathbb{R}^3$. This is a contradiction to Theorem 2.2 which will be presented soon afterwards. $\square$

The system (2.4) is actually an interior transmission (eigenvalue) problem (ITP) for Maxwell equations. It is noticed from the proof that, if the EM scattering system (2.3) produces trivial scattered field outside the scatterer, then the corresponding wave fields satisfy the ITP (2.5). Moreover, if $\big(\mathbf{E}^{\rm inc}, \mathbf{E}, \mathbf{H}^{\rm inc}, \mathbf{H}\big)$ is a solution to the ITP (2.4), and $\big(\mathbf{E}^{\rm inc}, \mathbf{H}^{\rm inc}\big)$ can be extended into the whole space $\mathbb{R}^3$ as a solution to the Maxwell system (2.2) in $\mathbb{R}^3$, then as an incident field, $\big(\mathbf{E}^{\rm inc}, \mathbf{H}^{\rm inc}\big)$ will not be scattered by the EM medium Σ.

Therefore, the essential gap between the non-scattering energy and the interior transmission eigenvalue (ITEV), is whether the eigenfunction can be extended to the exterior as a solution to the Maxwell system. A relevant result in [31] shows that when the medium scatterer is *spherically stratified*, then any associated ITEF can be extended as such an incident field which will not be scattered by the scatterer. In this case the two problems, non-scattering and ITEF, can be regarded as equivalent.

However, for *cornered* EM medium scatterers, the following theorem denies the equivalence between the non-scattering energies and the ITEVs. In other words, interior transmission eigenfunctions of Maxwell equations in a cornered domain D cannot be extended outside D in a way that the extended functions can be fields of a EM scattering problem. In fact, they are not even able to be extended across the corner in such a way.

Theorem 2.2. *Let $(\Sigma; \varepsilon, \gamma)$ be a cornered EM medium satisfying the same conditions as in Theorem 2.1. Let D be a bounded and closed set specified in Definition 2.1. Consider the following interior transmission eigenvalue* *problem*

$$\begin{cases} \operatorname{curl}\mathbf{E}^- - \mathrm{i}\omega\mu\mathbf{H}^- = 0, & \operatorname{curl}\mathbf{H}^- + \mathrm{i}\omega\gamma\mathbf{E}^- = 0, \quad \text{in } D, \\ \operatorname{curl}\mathbf{E}^{\mathrm{int}} - \mathrm{i}\omega\mu_0\mathbf{H}^{\mathrm{int}} = 0, & \operatorname{curl}\mathbf{H}^{\mathrm{int}} + \mathrm{i}\omega\varepsilon_0\mathbf{E}^{\mathrm{int}} = 0, \quad \text{in } D, \\ \nu\times\mathbf{E}^- = \nu\times\mathbf{E}^{\mathrm{int}}, & \nu\times\mathbf{H}^- = \nu\times\mathbf{H}^{\mathrm{int}}, \quad \text{on } \partial D, \end{cases} \tag{2.5}$$

with ν *the unit normal of* ∂D *pointing outwards. Assume that* (2.5) *admits a solution* $\left(\mathbf{E}^-, \mathbf{E}^{\mathrm{int}}, \mathbf{H}^-, \mathbf{H}^{\mathrm{int}}\right)$*, with* $\left(\mathbf{E}^{\mathrm{int}}, \mathbf{H}^{\mathrm{int}}\right) \in \mathscr{A}_{\mathrm{EM}}$ *(see, Definition 2.3). Then for any open neighborhood* $\mathcal{N}_\epsilon$ *of the corner at the support of the medium* Σ*, there is no solution* $(\mathbf{E}^0, \mathbf{H}^0)$ *to the Maxwell equations*

$$\operatorname{curl}\mathbf{E}^0 - \mathrm{i}\omega\mu_0\mathbf{H}^0 = 0, \quad \operatorname{curl}\mathbf{H}^0 + \mathrm{i}\omega\varepsilon_0\mathbf{E}^0 = 0, \qquad \text{in } \mathcal{N}_\epsilon, \tag{2.6}$$

such that

$$\left(\mathbf{E}^0, \mathbf{H}^0\right)\Big|_D = \left(\mathbf{E}^{\mathrm{int}}, \mathbf{H}^{\mathrm{int}}\right). \tag{2.7}$$

2.2.3 The Class of Incident Fields

In this section, the class of EM waves fields stated in Theorems 2.1 and 2.2 will be specified.

Definition 2.2. A three-dimensional vectorial field $(\mathbf{E}^{\mathrm{int}}, \mathbf{H}^{\mathrm{int}})$ is said to be *inadmissible* if, after a rigid change of coordinate, it satisfies the following conditions (I) and (IIa) or (IIb).

Let $\mathbf{P}_{\mathbf{E}^{\mathrm{int}}}$ (resp. $\mathbf{P}_{\mathbf{H}^{\mathrm{int}}}$) be the first nontrivial term of the Taylor expansion for $\mathbf{E}^{\mathrm{int}}$ (resp. $\mathbf{H}^{\mathrm{int}}$) at 0. Let N be the smaller one within the order of $\mathbf{P}_{\mathbf{E}^{\mathrm{int}}}$ and $\mathbf{P}_{\mathbf{H}^{\mathrm{int}}}$. Denote

$$\mathcal{S} := \begin{cases} \{\mathbf{E}^{\mathrm{int}}\}, & \text{if } \ N = N_{\mathbf{E}^{\mathrm{int}}} < N_{\mathbf{H}^{\mathrm{int}}}, \\ \{\mathbf{H}^{\mathrm{int}}\}, & \text{if } \ N = N_{\mathbf{H}^{\mathrm{int}}} < N_{\mathbf{E}^{\mathrm{int}}}, \\ \{\mathbf{E}^{\mathrm{int}}, \mathbf{H}^{\mathrm{int}}\}, & \text{if } \ N = N_{\mathbf{E}^{\mathrm{int}}} = N_{\mathbf{H}^{\mathrm{int}}}. \end{cases}$$

(I) $N \geq 1$.

(IIa) N is odd, and for any $\mathbf{S} \in \mathcal{S}$, the lowest-order homogeneous polynomial $\mathbf{P}_{\mathbf{S}}$ can be represented as

$$\mathbf{P}_{\mathbf{S}} = \left(x^{(j)}\, P^{(j)}_{N-1}(x)\right)_{j=1}^{3},$$

with homogeneous polynomials $P^{(j)}_{N-1}$, $j = 1, 2, 3$, of order $N - 1$;

(IIb) N is even, and for any $\mathbf{S} \in \mathcal{S}$, the lowest-order homogeneous polynomial $\mathbf{P_S}$ has the following representation

$$\mathbf{P_S} = \left(P^{(j)}_{N-2}(x) \prod_{\substack{l=1 \\ l \neq j}}^{3} x^{(l)} \right)_{j=1}^{3}.$$

Definition 2.3. A pair of real-analytic three-dimensional vectorial functions $(\mathbf{E}^{\mathrm{int}}, \mathbf{H}^{\mathrm{int}})$ belongs to the class $\mathscr{A}_{\mathrm{EM}}$ if and only if it is not inadmissible.

Remark 2.1. The conditions of admissibility in Definition 2.3 are equivalent to the property that (see also Theorem 2.5) there always exists a complex vector $\boldsymbol{\rho}$ satisfying $\boldsymbol{\rho} \cdot \boldsymbol{\rho}$ and

$$\int_{\mathcal{K}} \boldsymbol{\rho} \cdot \mathbf{P_S}(x)\, e^{-x \cdot \boldsymbol{\rho}} dx \neq 0. \tag{2.8}$$

Remark 2.2. Beside Definition 2.2, the admissible class $\mathscr{A}_{\mathrm{EM}}$ has an equivalent specification by using the vectorial spherical harmonics. It is presented and verified in a joint paper [93]. For more information on vectorial spherical harmonics, readers can refer to references such as [30].

Before proceeding to the next part, we would like to point out that the two common types of incident waves, i.e., plane waves and point sources, are not inadmissible; that is, they belong to the admissible class $\mathscr{A}_{\mathrm{EM}}$ and will be scattered by any cornered EM medium scatterer.

Lemma 2.1 ([93])**.** *Let* $(\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}})$ *be either a pair of plane waves of the form*

$$\mathbf{E}^{\mathrm{inc}} = e^{\mathrm{i}kx\cdot\mathbf{d}}\mathbf{d}^{\perp}, \quad \mathbf{H}^{\mathrm{inc}} = \sqrt{\varepsilon_0/\mu_0}\, e^{\mathrm{i}kx\cdot\mathbf{d}}\mathbf{d} \times \mathbf{d}^{\perp},$$

with $k := \omega\sqrt{\varepsilon_0\mu_0}$*, and* $\mathbf{d}, \mathbf{d}^{\perp} \in \mathbb{S}^2$ *being perpendicular to each other, or a point source*

$$\mathbf{E}^{\mathrm{inc}} = \mathrm{curl}\,(\mathbf{a}\Phi_y)\,, \quad \mathbf{H}^{\mathrm{inc}} = \frac{1}{\mathrm{i}\omega\mu_0}\,\mathrm{curl}\,\mathrm{curl}\,(\mathbf{a}\Phi_y)\,,$$

with $\mathbf{a} \in \mathbb{R}^3$ *a constant vector,* $y \notin \mathcal{N}_\epsilon$*, and*

$$\Phi_y(x) := \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|}, \quad x \neq y.$$

Then a cornered scatterer $(\Sigma; \varepsilon, \gamma)$ *always scatters* $(\mathbf{E}^{\rm inc}, \mathbf{H}^{\rm inc})$*, namely, the corresponding far-field pattern of the scattering problem* (2.3) *can not be identically zero.*

2.3 Proof of Theorem 2.2

We will prove Theorem 2.2 in this section, which is concerned with the extension of solutions to (2.5).

2.3.1 Auxiliary Results

In this section, results which will be used in the proof of Theorem 2.2 will be presented.

We first state the result on a special class of solutions to the Maxwell equations which are referred to as CGO solutions. We present here a simplified version which suffices for the proof of Theorem 2.2, while the full version will be given as Theorem 2.6 and be presented and proved in Section 2.4.

Theorem 2.3. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^3$*. Let* γ *and* μ *be functions in* $C^4(\Omega)$ *which does not vanish in* Ω*. Then for any given* $p > 6$*,* $c \in \mathbb{R}$ *and any* $\boldsymbol{\rho} \in \mathbb{C}^3 \backslash \{0\}$ *with* $|\boldsymbol{\rho}|$ *sufficiently large, the Maxwell system*

$$\operatorname{curl} \mathbf{E} - \mathtt{i}\omega\mu\mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} + \mathtt{i}\omega\gamma\mathbf{E} = 0 \qquad in\ \Omega, \tag{2.9}$$

has a solution $(\mathbf{E}, \mathbf{H})$ *with the following the form*

$$\mathbf{E} = e^{-\boldsymbol{\rho}\cdot x}\left(\gamma^{-1/2}\hat{\boldsymbol{\rho}} + \tilde{\mathbf{E}}_{\boldsymbol{\rho}}\right), \quad \mathbf{H} = e^{-\boldsymbol{\rho}\cdot x}\left(c\mu^{-1/2}\hat{\boldsymbol{\rho}} + \tilde{\mathbf{H}}_{\boldsymbol{\rho}}\right), \tag{2.10}$$

which satisfies

$$\left\|\left(\gamma^{1/2}\tilde{\mathbf{E}}_{\boldsymbol{\rho}}, \mu^{1/2}\tilde{\mathbf{H}}_{\boldsymbol{\rho}}\right)\right\|_{L^p(\Omega)^6} \le \frac{C}{|\boldsymbol{\rho}|^{3/p+\delta}}, \tag{2.11}$$

with two positive constants δ *and* C *independent of* $(\mathbf{E}, \mathbf{H})$ *and* $\boldsymbol{\rho}$*.*

The next result is about the Taylor expansion of solutions to (2.2). Notice that any solution of (2.2) in an open domain is real-analytic (cf. [30]). The following theorem and its corollary can be verified via analyzing the Taylor expansion of the corresponding vector fields. A detailed proof is contained in a joint paper [92], which will be skipped here.

Theorem 2.4. *Let* $\left(\mathbf{E}^{\text{int}}, \mathbf{H}^{\text{int}}\right)$ *be a solution to* (2.2) *in a neighborhood of the origin* 0*. For either* $\mathbf{V} = \mathbf{E}^{\text{int}}$ *or* $\mathbf{V} = \mathbf{H}^{\text{int}}$*, let* $\mathbf{P}_{\mathbf{V}}$ *be the first nonzero term in the Taylor expansion of* $\mathbf{V}$ *at* 0*, with the lowest order* $N_{\mathbf{V}}$*. Then* $\mathbf{V}$ *can be represented in an open neighborhood* $\mathcal{N}_\epsilon$ *of* 0 *as*

$$\mathbf{V} = \mathbf{P}_{\mathbf{V}} + \mathbf{M}_{N_{\mathbf{V}}+1}\mathbf{R}_{\mathbf{V}},$$

which satisfies

(1) $\mathbf{P}_{\mathbf{V}}$ *is harmonic and divergence-free.*

(2) $\mathbf{M}_{N_{\mathbf{V}}+1}$ *is a diagonal matrix whose diagonal entries* $M^{(j)}_{N_{\mathbf{V}}+1}$*,* $j = 1, 2, 3$*, are homogeneous polynomials of order larger than or equal to* $N_{\mathbf{V}} + 1$*;*

(3) $\mathbf{R}_{\mathbf{V}}$ *is a* 3*-dimensional vectorial function which is bounded in* $\mathcal{N}_\epsilon$*;*

(4) $\mathbf{P}_{\mathbf{E}^{\text{int}}}$ *or* $\mathbf{P}_{\mathbf{H}^{\text{int}}}$*, which ever is of higher order (or both of them if they have the same order), is (are) curl free.*

Corollary 2.1. *The vector field* $\mathbf{V}$ *given in Theorem 2.4 has another representation as* $\mathbf{V} = \mathbf{M}_{N_{\mathbf{V}}}\tilde{\mathbf{V}}$*, where* $\tilde{\mathbf{V}}$ *satisfies* (2.4)*, and* $\mathbf{M}_{N_{\mathbf{V}}}$ *is a diagonal* 3×3 *matrix satisfying* (2.4) *but with the integer* $N_{\mathbf{V}} + 1$ *replaced by* $N_{\mathbf{V}}$*.*

The last result in this subsection is also of important use in the proof of Theorem 2.2. Denote $\mathscr{L}$ as the Laplace transform operator defined by

$$\mathscr{L}[f](\boldsymbol{\rho}) := \int_{\mathcal{K}} e^{-x\cdot\boldsymbol{\rho}} f(x)dx. \tag{2.12}$$

Theorem 2.5. *Let* $(\mathbf{E}^{\text{int}}, \mathbf{H}^{\text{int}}) \in \mathscr{A}_{\text{EM}}$ *be a non-trivial pair of three-dimensional vectorial functions. Suppose that* $(\mathbf{E}^{\text{int}}, \mathbf{H}^{\text{int}})$ *solves* (2.2) *in a neighborhood of the origin* 0*. Let* $\mathbf{P} = \mathbf{P}_{\mathbf{S}}$ *be the homogeneous polynomial specified in Theorem 2.4. Then* $\boldsymbol{\rho} \cdot \mathscr{L}[\mathbf{P}](\boldsymbol{\rho})$ *will not vanish identically in any open subset of* $\{\boldsymbol{\rho} \in \mathbb{C}^3;\ \boldsymbol{\rho} \cdot \boldsymbol{\rho} = 0\}$*.*

For the proof of Theorem 2.5, we refer to the joint paper [93].

2.3.2 Proof of Theorem 2.2

Proof of Theorem 2.2. Let $\left(\mathbf{E}, \mathbf{E}^{\text{int}}, \mathbf{H}, \mathbf{H}^{\text{int}}\right)$ be a solution to (2.5). Then one has from integration by parts (see also from [108] or [93]) that

$$I_0 := \int_D \tilde{\gamma}\, \mathbf{E}^{\text{int}} \cdot \mathbf{E} - \tilde{\mu}\, \mathbf{H}^{\text{int}} \cdot \mathbf{H} = 0, \tag{2.13}$$

where $\tilde{\gamma} := \gamma - \varepsilon_0$, $\tilde{\mu} := \mu - \mu_0$, and the pair $\mathbf{E}, \mathbf{H}$ is any solution to

$$\operatorname{curl} \mathbf{E} - \mathtt{i}\omega\mu \mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} + \mathtt{i}\omega\gamma \mathbf{E} = 0, \qquad \text{in } D. \tag{2.14}$$

Assume the opposite of the result in Theorem 2.2 that, there exists an open neighborhood $\mathcal{N}_{\epsilon_0}$ of the corner, and a pair of EM fields $(\mathbf{E}^0, \mathbf{H}^0)$ solving the Maxwell system (2.6) in $\mathcal{N}_{\epsilon_0}$ and satisfying (2.7). We assume with out lose of generality that the corner is at the origin 0.

According to Theorem 2.3, we choose $(\mathbf{E}, \mathbf{H})$ as the following

$$\mathbf{E} = e^{-\boldsymbol{\rho}\cdot x} \left(\gamma^{-1/2} \hat{\boldsymbol{\rho}} + \tilde{\mathbf{E}}\right), \quad \mathbf{H} = e^{-\boldsymbol{\rho}\cdot x} \tilde{\mathbf{H}}, \tag{2.15}$$

where we take $\boldsymbol{\rho}$ as $\boldsymbol{\rho} = |\boldsymbol{\rho}|\hat{\boldsymbol{\rho}}_0$, with $\boldsymbol{\rho}_0$ a fixed complex vector satisfying $\Re \hat{\boldsymbol{\rho}}_0{}^{(j)} > c > 0$ for $j = 1, 2, 3$. The exact value of $\boldsymbol{\rho}_0$ will be specified later.

We define the new integrals

$$I_1 := \int_{D \setminus \mathcal{N}^+} \tilde{\gamma}\, \mathbf{E}^{\text{int}} \cdot \mathbf{E} - \tilde{\mu}\, \mathbf{H}^{\text{int}},$$

and

$$I_2 := I_0 - I_1 = \int_{\mathcal{N}^+} \tilde{\gamma}\, \mathbf{E}^{\text{int}} \cdot \mathbf{E} - \tilde{\mu}\, \mathbf{H}^{\text{int}},$$

where $\mathcal{N}^+ := \mathcal{N}_{\epsilon_0} \cap D$. We claim that

$$|I_1| \le C e^{-c\epsilon_0 |\boldsymbol{\rho}|} \quad \text{and} \quad |I_2| \sim |\boldsymbol{\rho}|^{-N+3}, \tag{2.16}$$

where N is an integer whose value will be specified later in the proof. If this is the case, then it contradicts the fact that $I_0 = I_1 + I_2 = 0$, which completes the proof.

Proof of Claim (2.16)

Now we verify the claim (2.16). The first part is easier. It is noticed for any $x \in D$ that

$$\Re(x \cdot \boldsymbol{\rho}) \ge |x| \min_j \Re \boldsymbol{\rho}_j > c|\boldsymbol{\rho}||x|.$$

Hence by inserting (2.15) into the integral we have for $|\boldsymbol{\rho}|$ sufficiently large that

$$
\begin{aligned}
\left|\int_{D\setminus\mathcal{N}^+} \tilde{\gamma}\,\mathbf{E}^{\text{int}}\cdot\mathbf{E}\right| =& \left|\int_{D\setminus\mathcal{N}_{\epsilon_0}} \tilde{\gamma} e^{-x\cdot\boldsymbol{\rho}}\mathbf{E}^{\text{int}}\cdot(\gamma^{-1/2}\hat{\boldsymbol{\rho}}+\tilde{\mathbf{E}})\right| \\
\leq& \int_{D\setminus\mathcal{N}_{\epsilon_0}} e^{-\Re(x\cdot\boldsymbol{\rho})}\left|\tilde{\gamma}\gamma^{-1/2}\left(\mathbf{E}^{\text{int}}\cdot\hat{\boldsymbol{\rho}}+\gamma^{1/2}\tilde{\mathbf{E}}\right)\right| \\
\leq& C e^{-c\epsilon_0|\boldsymbol{\rho}|}.
\end{aligned}
$$

The other part of the integral in I_1 concerning $\mathbf{H}$ can be treated in the same manner. Therefore, we have verified the first assertion in (2.16).

We are left to prove the second part of the claim, that is

$$
|I_2| \sim |\boldsymbol{\rho}|^{-N+3}. \tag{2.17}
$$

We shall write $\mathbf{E}^{\text{int}}$ and $\mathbf{H}^{\text{int}}$ in $\mathcal{N}_{\epsilon_0}$, according to Theorem 2.4, as

$$
\mathbf{E}^{\text{int}} = \mathbf{P}_{\mathbf{E}} + \mathbf{M}_{N_{\mathbf{E}}+1}\mathbf{R}_{\mathbf{E}}, \quad \mathbf{H}^{\text{int}} = \mathbf{P}_{\mathbf{H}} + \mathbf{M}_{N_{\mathbf{H}}+1}\mathbf{R}_{\mathbf{H}}. \tag{2.18}
$$

We assume without loss of generality that $N := N_{\mathbf{E}} \leq N_{\mathbf{H}}$, and denote for notational simplicity that $\mathbf{P}_N := \mathbf{P}_{\mathbf{E}}$ and $\mathbf{M}_{N+1} := \mathbf{M}_{N_{\mathbf{E}}+1}$.

Denote c_0 be the value of $\tilde{\gamma}\gamma^{-1/2}$ at the corner 0. By applying the expansion (2.18), we further split the integral I_2 into three parts: II_j, $j = 0, 1, 2$, which are defined as

$$
II_0 := c_0 \int_{\mathcal{N}_{\epsilon_0}^+} e^{-x\cdot\boldsymbol{\rho}}\,\boldsymbol{\rho}\cdot\mathbf{P}_N, \tag{2.19}
$$

$$
II_1 := \int_{\mathcal{N}_{\epsilon_0}^+} \tilde{\gamma} e^{-x\cdot\boldsymbol{\rho}}\tilde{\mathbf{E}}\cdot\mathbf{E}^{\text{int}} - \int_{\mathcal{N}_{\epsilon_0}^+} \tilde{\mu} e^{-x\cdot\boldsymbol{\rho}}\tilde{\mathbf{H}}\cdot\mathbf{H}^{\text{int}}, \tag{2.20}
$$

and

$$
II_2 := \int_{\mathcal{N}_{\epsilon_0}^+} \gamma^{-1/2} e^{-x\cdot\boldsymbol{\rho}}\,\boldsymbol{\rho}\cdot\left((\tilde{\gamma}-c_0\gamma^{1/2})\mathbf{P}_N + \tilde{\gamma}\mathbf{M}_{N+1}\mathbf{R}_{\mathbf{E}}\right). \tag{2.21}
$$

We will show that

$$
|II_0| \sim |\boldsymbol{\rho}|^{-N+3}, \quad |II_1| \leq C|\boldsymbol{\rho}|^{-(N+3+\delta)} \quad \text{and} \quad |II_2| \leq C|\boldsymbol{\rho}|^{-(N+4)}. \tag{2.22}
$$

Estimate of Integral II_0

For a given constant $c \in (0, 1/\sqrt{6})$, denote

$$
\tilde{\mathcal{U}}_c := \{\boldsymbol{\rho} = (\boldsymbol{\rho}^{(j)})_{j=1}^3 \in \mathbb{C}^3;\ \min_j \Re\hat{\boldsymbol{\rho}}^{(j)} > c\},
$$

and

$$\mathcal{U}_c := \tilde{\mathcal{U}}_c \cap \{\boldsymbol{\rho} \in \mathbb{C}^3; |\boldsymbol{\rho}| > 1\} \cap \{\boldsymbol{\rho} \in \mathbb{C}^3;\ \boldsymbol{\rho}\cdot\boldsymbol{\rho} = 0\}.$$

Then $\mathcal{U}_c$ is a non-empty open subset of the variety $\{\boldsymbol{\rho} \in \mathbb{C}^3;\ \boldsymbol{\rho}\cdot\boldsymbol{\rho} = 0\}$ (cf. [10]). Hence, one can obtain from Theorem 2.5 a point $\boldsymbol{\rho}_0 \in \mathcal{U}_c$ satisfying

$$\boldsymbol{\rho}_0 \cdot \mathscr{L}[\boldsymbol{\rho}_0 \cdot \mathbf{P}_N](\boldsymbol{\rho}_0) \neq 0.$$

Therefore, we can derive for any $\boldsymbol{\rho} = |\boldsymbol{\rho}|\hat{\boldsymbol{\rho}}_0$ that

$$\begin{aligned} \mathscr{L}[\boldsymbol{\rho}\cdot\mathbf{P}_N](\boldsymbol{\rho}) &= \int_{\mathcal{K}} e^{-x\cdot\boldsymbol{\rho}}\, \boldsymbol{\rho}\cdot\mathbf{P}_N(x)\,dx \\ &= |\boldsymbol{\rho}|^{-(N+3)} \int_{\mathcal{K}} e^{-y\cdot\hat{\rho}_0}\, \boldsymbol{\rho}\cdot\mathbf{P}_N(y)\,dy \\ &= |\boldsymbol{\rho}|^{-(N+3)} \mathscr{L}[\hat{\boldsymbol{\rho}}_0\cdot\mathbf{P}_N](\hat{\boldsymbol{\rho}}_0) := C|\boldsymbol{\rho}|^{-(N+3)}, \end{aligned} \tag{2.23}$$

where C is independent of $|\boldsymbol{\rho}|$. In addition, one has for sufficiently large $|\boldsymbol{\rho}|$ that [10]

$$\left| \int_{\mathcal{K}\backslash\mathcal{N}_{\epsilon_0}} e^{-x\cdot\boldsymbol{\rho}}\, \boldsymbol{\rho}\cdot\mathbf{P}_N \right| \leq Ce^{-c\epsilon_0|\boldsymbol{\rho}|},$$

which in combination with (2.23) yields

$$|II_0| \sim |\boldsymbol{\rho}|^{-(N+3)}. \tag{2.24}$$

Estimate of Integral II_1

From Corollary 2.1 we obtain the following representation

$$\mathbf{E}^{\mathrm{int}} = \mathbf{M}_{N_{\mathbf{E}}}\tilde{\mathbf{E}}_P, \quad \mathbf{H}^{\mathrm{int}} = \mathbf{M}_{N_{\mathbf{H}}}\tilde{\mathbf{H}}_P.$$

It is observed that

$$\left| \int_{\mathcal{N}_{\epsilon_0}^+} \tilde{\gamma} e^{-x\cdot\boldsymbol{\rho}} \tilde{\mathbf{E}}_P \cdot \mathbf{M}_{N_{\mathbf{E}}}\tilde{\mathbf{E}} \right| \leq \left\| \tilde{\gamma}\tilde{\mathbf{E}}_P \right\|_{L^\infty(\mathcal{N}_{\epsilon_0}^+)^3} \int_{\mathcal{N}_{\epsilon_0}^+} \left| e^{-x\cdot\boldsymbol{\rho}}\,\mathbf{M}_{N_{\mathbf{E}}}\tilde{\mathbf{E}} \right|.$$

Applying similar strategies as in (2.28) and (2.29), we have

$$\begin{aligned} \int_{\mathcal{N}_{\epsilon_0}^+} \left| e^{-x\cdot\boldsymbol{\rho}}\,\mathbf{M}_{N_{\mathbf{E}}}\tilde{\mathbf{E}} \right| &= \frac{1}{|\boldsymbol{\rho}|^{N+3}} \int_{\mathcal{N}_{|\boldsymbol{\rho}|\epsilon}} \left| e^{-y\cdot\hat{\rho}}\,\mathbf{M}_{N_{\mathbf{E}}}(y)\tilde{\mathbf{E}}(y/|\boldsymbol{\rho}|) \right| dy \\ &\leq \frac{\|\mathbf{F}_{\mathbf{E}}\|_{L^{p'}(\mathcal{K})}}{|\boldsymbol{\rho}|^{N+3}} \|\tilde{\mathbf{E}}(\cdot/|\boldsymbol{\rho}|)\|_{L^p(\mathcal{N}_{|\boldsymbol{\rho}|\epsilon})^3} \\ &= \frac{\|\mathbf{F}_{\mathbf{E}}\|_{L^{p'}(\mathcal{K})}}{|\boldsymbol{\rho}|^{N+3-3/p}} \|\tilde{\mathbf{E}}\|_{L^p(\mathcal{N}_{\epsilon_0}^+)^3}, \end{aligned} \tag{2.25}$$

with $\mathbf{F}_{\mathbf{E}}(x) := e^{-x\cdot\hat{\boldsymbol{\rho}}}\mathbf{M}_{N_{\mathbf{E}}}(x)$. The estimate for the rest part of the integral II_1 concerning $\mathbf{H}$ can be derived in a similar way. Then by applying the estimates (2.32) for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ we obtain

$$|II_1| \leq \frac{C}{|\boldsymbol{\rho}|^{N+3+\delta}}. \tag{2.26}$$

Estimate of Integral II_2.

By recalling $\tilde{\gamma}(0) - c_0\gamma^{1/2}(0) = 0$ and noticing from the Taylor expansion of $(\tilde{\gamma} - c_0\gamma^{1/2})$ around the corner, the integral II_2 in (2.21) can actually be regarded as

$$II_2 = \int_{\mathcal{N}_{\epsilon_0}^+} \gamma^{-1/2} e^{-x\cdot\boldsymbol{\rho}}\,\hat{\boldsymbol{\rho}}\cdot\left(\tilde{\mathbf{Q}}^{N+1}\mathbf{F}\right), \tag{2.27}$$

with $\mathbf{F}$ bounded in $\mathcal{N}_{\epsilon_0}$, and $\tilde{\mathbf{Q}}^{N+1}$ a 3×3 diagonal matrix homogeneous polynomials of order larger than or equal to $N+1$. Then one has

$$\begin{aligned} II_2 &= \int_{\mathcal{N}_{\epsilon_0}^+} \gamma^{-1/2} e^{-|\boldsymbol{\rho}|x\cdot\hat{\boldsymbol{\rho}}}\,\hat{\boldsymbol{\rho}}\cdot\left(\tilde{\mathbf{Q}}^{N+1}\mathbf{F}\right)\,dx \\ &= \frac{1}{|\boldsymbol{\rho}|^{N+4}}\int_{\mathcal{N}_{|\boldsymbol{\rho}|\epsilon}} e^{-y\cdot\hat{\boldsymbol{\rho}}}\,\gamma^{-1/2}(y/|\boldsymbol{\rho}|)\,\tilde{\mathbf{Q}}^{N+1}(y)\mathbf{F}(y/|\boldsymbol{\rho}|)\,dy. \end{aligned} \tag{2.28}$$

Hence

$$\begin{aligned} |II_2| &\leq \frac{1}{|\boldsymbol{\rho}|^{N+4}}\|\gamma^{-1/2}\mathbf{F}\|_{L^\infty(\mathcal{N}_{\epsilon_0}^+)^3}\int_{\mathcal{K}} e^{-y\cdot\Re\hat{\boldsymbol{\rho}}}\left|\tilde{\mathbf{Q}}^{N+1}(y)\right|\,dy \\ &\leq \frac{C}{|\boldsymbol{\rho}|^{N+4}}. \end{aligned} \tag{2.29}$$

Therefore, we have verified all the estimates in (2.22), and hence (2.17) as well as the complete claim (2.16). $\square$

2.4 CGO Solutions for Maxwell Equations

In this section Theorem 2.3 concerning CGO solutions for Maxwell equations will be proved. In fact, it is a particular case of the following result.

Theorem 2.6. *Let γ, μ and Ω be the same as in Theorem 2.3. Let $\boldsymbol{\rho}, \boldsymbol{\zeta} \in \mathbb{C}^3\backslash\{\mathbf{0}\}$ be such that $\boldsymbol{\rho}\cdot\boldsymbol{\rho} = 0$ and $\boldsymbol{\rho}\cdot\boldsymbol{\zeta} = 0$. Then for any $p > 6$, and any constants c_{ij}, $i,j = 1,2$, the Maxwell equations* (2.14) *has a solution* $(\mathbf{E},\mathbf{H})$ *which is of the following form*

$$\mathbf{E} = e^{-\boldsymbol{\rho}\cdot x}\left(\gamma^{-1/2}\hat{\boldsymbol{\rho}} + \tilde{\mathbf{E}}_{\boldsymbol{\rho},\hat{\boldsymbol{\rho}}}\right), \quad \mathbf{H} = e^{-\boldsymbol{\rho}\cdot x}\left(\mu^{-1/2}\mathbf{H}_0 + \tilde{\mathbf{H}}_{\boldsymbol{\rho},\mathbf{H}_0}\right), \tag{2.30}$$

with

$$\mathbf{E}_0 = c_{11}\hat{\boldsymbol{\rho}} + c_{21}\boldsymbol{\zeta} \times \hat{\boldsymbol{\rho}} \quad \textit{and} \quad \mathbf{H}_0 = c_{12}\hat{\boldsymbol{\rho}} + c_{22}\boldsymbol{\zeta} \times \hat{\boldsymbol{\rho}}. \tag{2.31}$$

Moreover, there exists some constants $C, \delta > 0$*, which are independent of* $(\mathbf{E}, \mathbf{H})$ *and* $\boldsymbol{\rho}$*, such that*

$$\left\| \left(\gamma^{1/2} \tilde{\mathbf{E}}_{\boldsymbol{\rho},\hat{\boldsymbol{\rho}}}, \mu^{1/2} \tilde{\mathbf{H}}_{\boldsymbol{\rho},\mathbf{H}_0} \right) \right\|_{L^p(\Omega)^6} \leq \frac{C}{|\boldsymbol{\rho}|^{3/p+\delta}}. \tag{2.32}$$

The CGO solution (2.30) will be constructed by inducing two auxiliary scalar fields concerning the divergence on the electromagnetic field, following the idea in [109].

2.4.1 From Maxwell Equations to Schrödinger Equations

We briefly introduce the idea from [109] for construction of CGO solutions to Maxwell equations, which was originated from [108]. Most of the arguments and statements in this subsection can be extracted or slightly modified from those in [109, 23, 93].

Notations

We first introduce some notations. Define a operator $\mathscr{P}_{\mp}$ as following:

$$\mathscr{P}_{\mp}(\nabla) := \begin{pmatrix} 0 & \mathscr{P}_{-}(\nabla) \\ \mathscr{P}_{+}(\nabla) & 0 \end{pmatrix},$$

with

$$\mathscr{P}_{+}(\nabla) := \begin{pmatrix} 0 & \nabla\cdot \\ \nabla & \nabla\times \end{pmatrix} \quad \text{and} \quad \mathscr{P}_{-}(\nabla) := \begin{pmatrix} 0 & \nabla\cdot \\ \nabla & -\nabla\times \end{pmatrix}.$$

Notice that for any three-dimensional vector field $\boldsymbol{\rho}$, the 8-by-8 matrix $\mathscr{P}_{\mp}(\boldsymbol{\rho})$ is also well understood by the above definition. The following notations shall apply for vector fields $\boldsymbol{\rho}, \boldsymbol{\zeta} \in \mathbb{C}^3$ as well,

$$\mathscr{P}_{\mp}(\boldsymbol{\rho}, \boldsymbol{\zeta}) := \begin{pmatrix} 0 & \mathscr{P}_{-}(\boldsymbol{\rho}) \\ \mathscr{P}_{+}(\boldsymbol{\zeta}) & 0 \end{pmatrix} \quad \text{and} \quad \mathscr{P}_{\pm}(\boldsymbol{\rho}, \boldsymbol{\zeta}) := \begin{pmatrix} 0 & \mathscr{P}_{+}(\boldsymbol{\rho}) \\ \mathscr{P}_{-}(\boldsymbol{\zeta}) & 0 \end{pmatrix}.$$

Denote

$$\boldsymbol{\alpha} := \gamma^{-1} \nabla \gamma \ \text{ and } \ \boldsymbol{\beta} := \mu^{-1} \nabla \mu.$$

The following 8-by-8 matrix valued functions $\mathcal{V}, \mathcal{W}, \mathcal{W}'$ and $\mathcal{Q}$ which depend on γ and μ shall also be used [93]:

$$\mathcal{V} = \mathcal{V}_{\mu,\gamma} := \begin{pmatrix} \mathtt{i}\omega\mu & 0 & 0 & \boldsymbol{\alpha}\cdot \\ 0 & \mathtt{i}\omega\mu\mathbf{I}_3 & \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{\beta}\cdot & \mathtt{i}\omega\gamma & 0 \\ \boldsymbol{\beta} & 0 & 0 & \mathtt{i}\omega\gamma\mathbf{I}_3 \end{pmatrix},$$

$$\mathcal{W} = \mathcal{W}_{\mu,\gamma} := \mathtt{i}\omega(\gamma\mu)^{1/2}\mathbf{I}_8 + \frac{1}{2}\mathscr{P}_{\pm}(\boldsymbol{\alpha}, \boldsymbol{\beta}),$$

$$\mathcal{W}' = \mathcal{W}'_{\mu,\gamma} := \mathtt{i}\omega(\gamma\mu)^{1/2}\mathbf{I}_8 + \frac{1}{2}\mathscr{P}_{\pm}(\boldsymbol{\beta}, \boldsymbol{\alpha}),$$

$$\begin{aligned}\mathcal{Q} = \mathcal{Q}_{\mu,\gamma} :=& - \omega^2\gamma\mu\mathbf{I}_8 + \frac{1}{4}\operatorname{diag}\left(|\boldsymbol{\alpha}|^2, |\boldsymbol{\beta}|^2\right) + \frac{1}{2}\mathscr{P}_{\mp}(\nabla_\gamma, \nabla_\mu)\mathscr{P}_{\pm}(\boldsymbol{\beta}, \boldsymbol{\alpha}) \\ &+ \mathtt{i}\omega\left(\mathscr{P}_{\pm}(\nabla(\gamma\mu)^{1/2}) + \mathscr{P}_{\mp}(\nabla(\gamma\mu)^{1/2})\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}' = \mathcal{Q}'_{\mu,\gamma} =& - \omega^2\gamma\mu\mathbf{I}_8 + \frac{1}{4}\operatorname{diag}\left(|\boldsymbol{\beta}|^2, |\boldsymbol{\alpha}|^2\right) - \frac{1}{2}\mathscr{P}_{\mp}(\nabla_\mu, \nabla_\gamma)\mathscr{P}_{\pm}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \\ &+ \mathtt{i}\omega\left(\mathscr{P}_{\pm}(\nabla(\gamma\mu)^{1/2}) - \mathscr{P}_{\mp}(\nabla(\gamma\mu)^{1/2})\right),\end{aligned}$$

where with two scalar fields f_1, f_2, we mean $\operatorname{diag}(f_1, f_2)$ by $\operatorname{diag}(f_1\mathbf{I}_4, f_2\mathbf{I}_4)$.

Results

We shall use the following lemmas to convert the Maxwell system to a set of Schrödinger equations.

Lemma 2.2 ([109, 23])**.** *Denote* $X^T = \left(\psi, \mathbf{H}^T, \phi, \mathbf{E}^T\right)$, *with some scalar functions* ψ *and* ϕ, *and three-dimensional vector fields* $\mathbf{H}$ *and* $\mathbf{E}$. *Suppose that* X *satisfies*

$$\left(\mathscr{P}_{\mp}(\nabla) + \mathcal{V}\right) X = 0 \quad \text{in } \Omega. \tag{2.33}$$

Then the following two statements are equivalent.

(1) The scalar functions ψ *and* ϕ *vanish identically in* Ω*;*

(2) The pare of vectorial functions $(\mathbf{E}, \mathbf{H})$ *solves*

$$\operatorname{curl}\mathbf{E} - \mathtt{i}\omega\mu\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{H} + \mathtt{i}\omega\gamma\mathbf{E} = 0, \qquad \text{in } \Omega. \tag{2.34}$$

Lemma 2.3 ([23]). *Let X be the same as in Lemma 2.2, and denote* $Y^T = \left(\mu^{1/2}\psi, \mu^{1/2}\mathbf{H}^T, \gamma^{1/2}\phi, \gamma^{1/2}\mathbf{E}^T\right)$. *Then X satisfies* (2.33) *if and only if Y solves*

$$(\mathscr{P}_{\mp}(\nabla) + \mathcal{W})\, Y = 0. \tag{2.35}$$

Lemma 2.4 ([109, 23, 93]). *There holds*

$$(\mathscr{P}_{\mp}(\nabla) + \mathcal{W})\, (\mathscr{P}_{\mp}(\nabla) - \mathcal{W}') = \Delta \mathbf{I}_8 - \mathcal{Q}, \tag{2.36}$$

and

$$(\mathscr{P}_{\mp}(\nabla) - \mathcal{W}')\, (\mathscr{P}_{\mp}(\nabla) + \mathcal{W}) = \Delta \mathbf{I}_8 - \mathcal{Q}'. \tag{2.37}$$

Lemma 2.5 ([109, 23, 93]). *Let Y and Z be two 8-dimensional fields satisfying*

$$(\mathscr{P}_{\mp}(\nabla) - \mathcal{W}')\, Z = Y.$$

Then Y is a solution to (2.35) *if and only if Z solves*

$$(-\Delta \mathbf{I}_8 + \mathcal{Q})\, Z = 0. \tag{2.38}$$

We end up this subsection by the following concluding result. It is a straightforward consequence of Lemmas 2.2-2.5. Here we omit the proof.

Corollary 2.2. *Let Z be a solution to* (2.38). *Define the* 8*-dimensional field X as*

$$X := \operatorname{diag}\left(\mu^{-1/2}, \gamma^{-1/2}\right) (\mathscr{P}_{\mp}(\nabla) - \mathcal{W}')\, Z,$$

and denote $X^T = \left(\psi, \mathbf{H}^T, \phi, \mathbf{E}^T\right)$. *Then* $(\mathbf{E}, \mathbf{H})$ *solves* (2.34) *if and only if* $\psi \equiv \phi \equiv 0$.

2.4.2 CGO Solutions for the Schrödinger System

In this subsection, CGO solutions to the system (2.38) of eight Schrödinger equations will be constructed.

We first briefly introduce the Sobolev spaces to be used in this section (cf. [8]). For $a \in \mathbb{R}$ and $p \in [1, \infty]$, denote $H_p^a = H_p^a(\mathbb{R}^n)$ as the generalized Sobolev space equipped with the norm

$$\|f\|_{H_p^a(\mathbb{R}^n)} := \left\| \mathscr{F}^{-1}\{\langle\cdot\rangle^a \mathscr{F} f\} \right\|_{L^p(\mathbb{R}^n)},$$

where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$, and $\mathscr{F}$ is the Fourier transform operator.

The aim of this subsection is to prove the following result concerning CGO solutions for the Schrödinger system (2.38). In what follows, given $p \in [1,\infty]$, we shall denote p' as the Hölder conjugate of p.

Theorem 2.7. *Let $\mathcal{Q} = \mathcal{Q}_{\mu,\gamma}$ be the 8-by-8 matrix valued function defined in Section 2.4.1. Given $a \in [0,2]$ and $p \in [4,6)$, there exists a constant C satisfying the following.*

Given any $Z_0 \in \mathbb{C}^8$ and any $\boldsymbol{\rho} \in \mathbb{C}^3$ with $|\boldsymbol{\rho}|$ sufficiently large and $\boldsymbol{\rho}\cdot\boldsymbol{\rho} = 0$, there exists a solution of the following form

$$Z = e^{-\boldsymbol{\rho}\cdot x}\left(Z_0 + \tilde{Z}_{\boldsymbol{\rho},Z_0}\right), \tag{2.39}$$

to the equation

$$(-\Delta \mathbf{I}_8 + \mathcal{Q})\, Z = 0. \tag{2.40}$$

Moreover, there holds

$$\|\tilde{Z}\|_{H^a_p} \le \frac{C|Z_0|}{|\boldsymbol{\rho}|^{6/p-1}}\|\mathcal{Q}\hat{Z}_0\|_{H^a_{p'}}. \tag{2.41}$$

In order to prove Theorem 2.7, we shall make essential use of the following result, which can be obtained from [110].

Lemma 2.6. *Let $a \in \mathbb{R}$ and $p \in [4,6)$. There exists a constant C which satisfies the following.*

For any $\boldsymbol{\rho} \in \mathbb{C}^3$ with sufficiently large imaginary component, there exist a solution operator

$$\mathscr{G} = \mathscr{G}_{\boldsymbol{\rho}} : H^a_{p'} \to H^a_p$$

satisfying

$$(\Delta + 2\boldsymbol{\rho}\cdot\nabla)\,\mathscr{G} f = f,$$

and

$$\|\mathscr{G} f\|_{H^a_p} \le \frac{C}{|\Im\boldsymbol{\rho}|^{6/p-1}}\|f\|_{H^a_{p'}},$$

for any $f \in H^a_{p'}$.

Proof of Theorem 2.7. Let $Z = e^{-\boldsymbol{\rho}\cdot x}(Z_0 + \tilde{Z})$. It is obtained that, Z satisfies (2.40) if and only if $\tilde{Z}$ solves

$$(\Delta + 2\boldsymbol{\rho}\cdot\nabla)\,\tilde{Z} = \mathcal{Q}\left(Z_0 + \tilde{Z}\right).$$

Then by Lemma 2.6 one may find such a $\tilde{Z}$ via the following equation,

$$\tilde{Z} = (\mathbf{I}_8 - \mathscr{G}_{\boldsymbol{\rho}}\mathcal{Q})^{-1}\mathscr{G}_{\boldsymbol{\rho}}\mathcal{Q}Z_0,$$

under the condition that the operator $\mathbf{I}_8 - \mathscr{G}_{\boldsymbol{\rho}}\mathcal{Q}$ is invertible.

Recall from Lemma 2.6 that, for any 8-dimensional function F one has,

$$\|\mathscr{G}_{\boldsymbol{\rho}}\mathcal{Q}F\|_{H_p^a} \le \frac{C_0}{|\boldsymbol{\rho}|^{6/p-1}}\|\mathcal{Q}F\|_{H_{p'}^a}.$$

Notice that $p' < p$ and that $\mathcal{Q}$ is C^2 and compactly support. Then we further obtain the estimate

$$\|\mathscr{G}_{\boldsymbol{\rho}}\mathcal{Q}F\|_{H_p^a} \le \frac{C_1}{|\boldsymbol{\rho}|^{6/p-1}}\|F\|_{H_p^a},$$

with C_1 a constant independent of F. Now we can chose $\boldsymbol{\rho}$ with sufficient large module so that the operator norm of $\mathscr{G}_{\boldsymbol{\rho}}\mathcal{Q}$ on H_p^a is less than one. In this case, the operator $\mathbf{I}_8 - \mathscr{G}_{\boldsymbol{\rho}}\mathcal{Q}$ has a bounded inverse on H_p^a. Moreover, we have

$$\|\tilde{Z}\|_{H_p^a} = \|(\mathbf{I}_8 - \mathscr{G}_{\boldsymbol{\rho}}\mathcal{Q})^{-1}\mathscr{G}_{\boldsymbol{\rho}}\mathcal{Q}Z_0\|_{H_p^a} \le \frac{C}{|\boldsymbol{\rho}|^{6/q-1}}\|\mathcal{Q}\hat{Z}_0\|_{H_{q'}^a}.$$

The proof is complete. □

By taking particular values of Z_0 in Theorem 2.7, we have the following result. The proof will be skipped.

Corollary 2.3. *Let $\boldsymbol{\rho}$, p and a be the same as in Theorem 2.7, and let $\boldsymbol{\zeta} \in \mathbb{C}^3$ satisfy $\boldsymbol{\zeta}\cdot\boldsymbol{\rho} = 0$. Set*

$$Z_0^T = -|\boldsymbol{\rho}|^{-1}\left(c_{11},\, -c_{21}\boldsymbol{\zeta}^T,\, c_{12},\, c_{22}\boldsymbol{\zeta}^T\right), \tag{2.42}$$

with given constants $c_{ij} \in \mathbb{R}$, $i, j = 1, 2$. Then one has

$$-\big(\mathscr{P}_{\mp}(\boldsymbol{\rho})Z_0\big)^T = \big(0,\, c_{12}\hat{\boldsymbol{\rho}}^T + c_{22}(\boldsymbol{\zeta}\times\hat{\boldsymbol{\rho}})^T,\, 0,\, c_{11}\hat{\boldsymbol{\rho}}^T + c_{21}(\boldsymbol{\zeta}\times\hat{\boldsymbol{\rho}})^T\big). \tag{2.43}$$

Moreover, the Schrödinger equations (2.40) *admits a CGO solution of the form* (2.39) *satisfying*

$$\|\tilde{Z}\|_{H_p^a} \le \frac{C}{|\boldsymbol{\rho}|^{6/p}}\|\mathcal{Q}\hat{Z}_0\|_{H_{p'}^a}.$$

Next, we prove our main result, Theorem 2.6, on CGO solutions for Maxwell equations (2.14).

2.4.3 Proof of Theorem 2.6

We present some preliminary results before proving Theorem 2.6.

Lemma 2.7. *Let $\boldsymbol{\rho}$, Z, $\tilde{Z}$ and Z_0 be the same as in Theorem 2.7. Define*

$$Y := (\mathscr{P}_{\mp}(\nabla) - \mathcal{W}')\, Z. \tag{2.44}$$

Then Y has the following representation

$$Y = e^{-\boldsymbol{\rho}\cdot x}\left(Y_0 + \tilde{Y}\right), \tag{2.45}$$

with

$$Y_0 = -\mathscr{P}_{\mp}(\boldsymbol{\rho}) Z_0, \tag{2.46}$$

and

$$\tilde{Y} = \mathscr{P}_{\mp}(\nabla)\tilde{Z} - \mathscr{P}_{\mp}(\boldsymbol{\rho})\tilde{Z} - \mathcal{W}'(\tilde{Z} + Z_0). \tag{2.47}$$

Moreover, if $|\boldsymbol{\rho}|$ is sufficiently large, and

$$(\mathscr{P}_{\mp}(\boldsymbol{\rho})Z_0)^{(1)} = (\mathscr{P}_{\mp}(\boldsymbol{\rho})Z_0)^{(5)} = 0, \tag{2.48}$$

then we have

$$Y^{(1)} = Y^{(5)} = 0. \tag{2.49}$$

Proof. The equations (2.45)-(2.47) can be obtained from the definition (2.44) of Y and the representation (2.39) of Z.

Now we verify the identity (2.49). It is obtained from (2.40), the Schrödinger equation which Z satisfies, that Y is a solution to (2.35), and hence by (2.37) that Y solves

$$(\Delta \mathbf{I}_8 - \mathcal{Q}')\, Y = 0.$$

One can show, by straightforward computation, that $Y^{(1)}$ and $Y^{(5)}$ of $Y = \left(Y^{(j)}\right)_j^8$ satisfy

$$(-\Delta + q_1)Y^{(1)} = (-\Delta + q_5)Y^{(5)} = 0, \tag{2.50}$$

where the compactly supported potentials q_1 and q_2 are given by

$$q_1 = q_\gamma := \frac{1}{4}|\boldsymbol{\alpha}|^2 - \frac{1}{2}\nabla\cdot\boldsymbol{\alpha} - \omega^2\gamma\mu$$

and

$$q_2 = q_\gamma := \frac{1}{4}|\boldsymbol{\beta}|^2 - \frac{1}{2}\nabla\cdot\boldsymbol{\beta} - \omega^2\gamma\mu.$$

Therefore by recalling (2.48) one has $Y_0^{(1)} = Y_0^{(2)} = 0$, and consequently,

$$Y^{(j)} = e^{-\boldsymbol{\rho}\cdot x}\left(Y_0^{(j)} + \tilde{Y}^{(j)}\right) = e^{-\boldsymbol{\rho}\cdot x}\tilde{Y}^{(j)}, \quad j = 1, 5.$$

In addition, the equations in (2.50) suggest that

$$(\Delta + 2\boldsymbol{\rho}\cdot\nabla)\tilde{Y}^{(j)} = q_j\tilde{Y}^{(j)}, \quad j = 1, 5,$$

By applying Lemma 2.6 we have that

$$(\mathbf{I}_8 - \mathscr{G}_{\boldsymbol{\rho}} q_j)\tilde{Y}^{(j)} = \tilde{Y}^{(j)}, \quad j = 1, 5.$$

In a same manner as in the proof of Theorem 2.7, one can deduce that the operators $\mathbf{I}_8 - \mathscr{G}_{\boldsymbol{\rho}} q_j$, $j = 1, 5$, are invertible when $|\boldsymbol{\rho}|$ is sufficiently large. Therefore we deduce that $\tilde{Y}^{(j)} = 0$ and hence $Y^{(j)} = 0$ for each $j = 1, 5$. □

Remark 2.3. It is observed that the condition (2.48) is equivalent to

$$\boldsymbol{\rho}\cdot Z_0^{(2\sim4)} = \boldsymbol{\rho}\cdot Z_0^{(6\sim8)} = 0. \tag{2.51}$$

Corollary 2.4. *Let $\boldsymbol{\rho}$, Z, Z_0 and Y be the same as in Lemma 2.7. Define*

$$X = \left(\psi, \mathbf{H}^T, \phi, \mathbf{E}^T\right) := \operatorname{diag}\left(\mu^{-1/2}, \gamma^{-1/2}\right) Y.$$

Then $\psi = \phi = 0$ and $(\mathbf{E}, \mathbf{H})$ solves the Maxwell system (2.34)*, providing that Z_0 satisfies* (2.51). .

We are now in the position to prove Theorem 2.6.

Proof of Theorem 2.6. Let Z_0 be the vector given in (2.42), which satisfies the condition (2.51). Then the field $(\mathbf{E}, \mathbf{H})$ as specified in Corollary 2.4 solves the Maxwell system (2.34). with the 8-dimensional field Y given by (2.45)-(2.47). Moreover, $\mathbf{E}$ and $\mathbf{H}$ admit the form (2.30) with

$$\tilde{\mathbf{E}}_{\boldsymbol{\rho},\hat{\boldsymbol{\rho}}} = \gamma^{-1/2}\tilde{Y}^{(6\sim8)}, \quad \tilde{\mathbf{H}}_{\boldsymbol{\rho},\mathbf{H}_0} = \mu^{-1/2}\tilde{Y}^{(2\sim4)}, \tag{2.52}$$

and the form (2.31) for $\hat{\boldsymbol{\rho}}$ and $\mathbf{H}_0$ can be derived from (2.46) and (2.43).

We are left to verify the estimates in (2.32). For $a \in (0, 3/q)$ and $q \in [4, 6)$, Corollary 2.3 gives

$$\|\mathscr{P}_{\mp}(\nabla)\tilde{Z}\|_{H^a_q} \leq \frac{C}{|\boldsymbol{\rho}|^{6/q}}\|\mathcal{Q}\hat{Z}_0\|_{H^{a+1}_{q'}},$$

and

$$\|\mathscr{P}_{\mp}(\boldsymbol{\rho})\tilde{Z}\|_{H^a_q} \leq \frac{C}{|\boldsymbol{\rho}|^{6/q-1}}\|\mathcal{Q}\hat{Z}_0\|_{H^a_{q'}}.$$

Recalling the presentation (2.47) one can obtain that

$$\|\tilde{Y}\|_{H^a_q} \leq \frac{C}{|\boldsymbol{\rho}|^{6/q-1}}.$$

Then for $3/p = 3/q - a$, the Sobolev embedding $H^a_q \subset L^p$ in $\mathbb{R}^3$ implies,

$$\|\tilde{Y}\|_{L^p} \leq \frac{C}{|\boldsymbol{\rho}|^{3/p+(3/q+a-1)}}. \tag{2.53}$$

Given any $t \in (0, 1)$, take $a = 3/q - t\,(6/q - 1)$. Then the number p is given by

$$p = \frac{3}{t\,(6/q - 1)}.$$

By proper choices of the two parameters $t \in (0, 1)$ and $q \in [4, 6)$, the value of p takes full range in $(6, \infty)$. Hence by taking $\delta := 3/q + a - 1 = (1 - t)\,(6/q - 1) > 0$ the relation (2.53) can be reformulated as

$$\|\tilde{Y}\|_{L^p} \leq \frac{C}{|\boldsymbol{\rho}|^{3/p+\delta}},$$

The estimate (2.32) is then a consequence of (2.52). □

2.5 Future Work

In Theorems 2.1 and 2.2, we assume admissible conditions for the incident fields to guarantee the corner scattering results for EM wave propagation. However, this is due to the technique in our arguments of the mathematical proof, rather than a fundamental obstruction. In other words, it is not clear yet that whether an "inadmissible" incident wave field characterized in Definition 2.2 will actually be scattered by a medium with corner or not.

One possible research direction is to get rid of some assumptions in Theorems 2.1 and 2.2 for the "admissible" cases. More precisely, I would like to minimize the inadmissible class of incident fields so that, ideally, any inadmissible wave field can be proven to be actually scattered by a cornered medium. It is also possible that the inadmissible class is in fact the empty set, which has been proven to be true in acoustic wave scattering [10, 45, 110, 46]. This is important for further investigation of uniqueness and stability in inverse EM scattering, concerning the shape identification of a cornered medium by a single or a few measurement(s). Analogues for acoustic cases have been done, for instance, in [72, 9].

The assumption of inadmissible classes comes from the requirement of a vanishing property for the Laplacian transform of certain vectorial harmonic functions which are divergence free. The difference compared to the acoustic case owes to the more complicated vectorial form of CGO solutions for Maxwell equations. One possible way to solve this problem is to look for other formulas for CGO solutions, which can be still applied for the non-vanishing property in the EM case. Another possibility is to investigate directly the wave fields in the current inadmissible classes and check whether such incident waves will be scattered or not.

Moreover, it is also worth looking at scattering problems for more general corners or other types of singularities. This is interesting in both forward and inverse scattering. As we discussed in the introduction, the essential gap between non-scattering energies and interior transmission eigenvalues is whether interior transmission eigenfunctions can be extended to the exterior. Scarce results has been shown except for radial or cornered ones. As for inverse problems, applications are twofold. In the case that corners or other kinds of singularities scatters, one might be able to obtain information of those singularities from scattered measurements. Otherwise if certain types of singularities or regularities do not scatter certain kinds of incident fields, then there would be a potential application of partial invisibility cloaking.

Chapter 3

Decoupling of Elastic Fields and its Applications

We consider in this chapter the time-harmonic elastic wave scattering which is governed by the Lamé system. It is known that the elastic or seismic body waves can be decomposed into two kinds, namely, the primary (P-) and the secondary (S-) waves, which are also called, respectively, the compressional and the shear waves. These two kinds of waves are generally coupled due to the presence of interfaces or any other inhomogeneities. However, we shall show in this chapter that P- and the S-waves can be completely decoupled in certain circumstances. In this case, the boundary value or scattering problem for the *Lamé system* can be reduced to the ones for the *Helmholtz equation and the Maxwell system.* The decoupling results hence can be further applied to establish the *uniqueness and stability* for the corresponding *inverse elastic scattering* problems on determining polyhedral scatterers from a minimal number of far-field measurements.

3.1 Introduction

Consider the time-harmonic linearized elastic or seismic wave propagation governed by the following Lamé system,

$$(\Delta^{*} + \omega^2)\mathbf{U} = 0, \tag{3.1}$$

where $\mathbf{U} = (U_j)_{j=1}^3$ is the displacement field, the constant ω stands for the propagation frequency, and the differential operator Δ^* is defined by

$$\begin{aligned}\Delta^* \mathbf{U} :=& \mu\Delta\mathbf{U} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{U})\\ =&(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{U}) - \mu\,\mathrm{curl}\,\mathrm{curl}\,\mathbf{U}\end{aligned}$$

with the Lamé constants $\lambda \geq 0$ and $\mu > 0$. Taking divergence at both sides of the Lamé equation one has

$$(\Delta + k_p^2)\, v = 0 \tag{3.2}$$

with $k_p := \omega/\sqrt{\lambda+2\mu}$ and $v := \nabla\cdot\mathbf{U}$. While taking curl of the Lamé equation one obtains

$$(\Delta + k_s^2)\,\mathbf{E} = 0, \tag{3.3}$$

where $k_s := \omega/\sqrt{\mu}$ and $\mathbf{E} := \mathrm{curl}\,\mathbf{U}$. The equations (3.2) and (3.3) characterize the propagation of, respectively, S- and P-waves; and the constants k_s and k_p are referred to as the S- and P-wave numbers respectively. Notice that the field $\mathbf{E}$ corresponding to the S-wave is divergence free. It is observed that, by denoting $\mathbf{H} := \mathrm{curl}\,\mathbf{E}/(\mathrm{i}k_s)$, the pair $(\mathbf{E}, \mathbf{H})$ satisfies the Maxwell equation

$$\mathrm{curl}\,\mathbf{E} - \mathrm{i}k_s\mathbf{H} = 0 \quad \text{and} \quad \mathrm{curl}\,\mathbf{H} + \mathrm{i}k_s\mathbf{E} = 0. \tag{3.4}$$

The fact that the displacement $\mathbf{U}$ can be decomposed into the P- and the S-part can also be explained by the Helmholtz decomposition. It is known that any rapidly decaying three dimensional vector field $\mathbf{F}$ can be decomposed into a curl-free and a divergence-free component, namely,

$$\mathbf{F} = -\nabla\phi + \mathrm{curl}\,\mathbf{A}, \tag{3.5}$$

which holds in the sense of distribution. Moreover, if $\mathbf{F} \in L^2(\Omega)$ with Ω a bounded domain in $\mathbb{R}^3$, then $\phi \in H^1(\Omega)$ and $\mathbf{A} \in H(\mathrm{curl}, \Omega)$. Writing $\mathbf{U}$ into the form (3.5), one has

$$\nabla\cdot\mathbf{U} = -\Delta\phi \quad \text{and} \quad \mathrm{curl}\,\mathbf{U} = \mathrm{curl}\,\mathrm{curl}\,\mathbf{A}.$$

Then the equation (3.1) yields

$$-\nabla\left(\Delta + k_p^2\right)\phi + \mathrm{curl}\left(\Delta + k_s^2\right)\mathbf{A} = 0,$$

which implies that

$$\left(\Delta + k_p^2\right)\phi = 0 \quad \text{and} \quad \left(\Delta + k_s^2\right)\mathbf{A} = 0,$$

up to a constant. In fact, one has that

$$\mathbf{U}_p = -\nabla\phi \quad \text{and} \quad \mathbf{U}_s = \operatorname{curl}\mathbf{A}.$$

Given a Lipschitz domain Ω in $\mathbb{R}^3$, let $\mathbf{U}$ satisfies the Lamé system (3.1) in Ω. We introduce four kinds of boundary conditions for $\mathbf{U}$ on $\partial\Omega$, which is related to physical properties of the scatterer. The first kind or the Dirichlet boundary condition is given by

$$\mathbf{U} = 0 \quad \text{on} \quad \partial\Omega. \tag{3.6}$$

The second kind ir the Neumann type is characterized as

$$\mathcal{T}\mathbf{U} = 0 \quad \text{on} \quad \partial\Omega, \tag{3.7}$$

with $\mathcal{T}$ is the traction operator defined on $\partial\Omega$ as

$$\mathcal{T}\mathbf{U} := \lambda(\nabla\cdot\mathbf{U})\,\nu + \mu\left(\nabla\mathbf{U} + \nabla^T\mathbf{U}\right)\nu, \tag{3.8}$$

with $\nu \in \mathbb{S}^2$ the outward unit normal vector to ∂D. The boundary conditions of the third and the fourth kinds are characterized by, respectively,

$$\nu\cdot\mathbf{U} = 0 \quad \text{and} \quad \nu\times\mathcal{T}\mathbf{U} = 0 \qquad \text{on} \quad \partial\Omega, \tag{3.9}$$

and

$$\nu\times\mathbf{U} = 0 \quad \text{and} \quad \nu\cdot\mathcal{T}\mathbf{U} = 0 \qquad \text{on} \quad \partial\Omega. \tag{3.10}$$

The third kind boundary condition means that there has only tangent displacement and normal traction on the boundary; and the forth kind says that the displacement is normal and the traction is tangent upon the boundary. For more information on boundary value problems concerning the Lamé system (3.1), we refer to references [81, 85, 28, 99, 49].

It is known that the S- and the P- waves, though can be locally characterized by independent equations (3.2) and (3.3) or (3.4), are in general coupled via boundary conditions or inhomogeneity. However, we shall prove that under certain geometric conditions, the third or fourth kind boundary conditions for the displacement field $\mathbf{U}$, are *equivalent to* certain boundary conditions of the *Helmholtz* equation and *Maxwell* equations, for v and $\mathbf{E}$, respectively, in the whole domain. More precisely, suppose the obstacle Ω is of the fourth kind, then v can be viewed as an acoustic field in the domain with sound-soft boundary, and $\mathbf{E}$ can be referred to as an electric field satisfying the perfect magnetic conducting boundary condition, provided that $\partial\Omega$ is a piecewise minimal surface. If the obstacle is polyhedral and of the third kind, then $\mathbf{E}$ is regarded as an electric field with the perfect electric boundary condition and v solves the sound-hard acoustic equation.

3.2 Decoupling of Elastic Fields

We shall establish in this section the decoupling result for elastic waves. We shall divided our arguments into two categories, regarding whether the boundary data is of the third or the fourth type.

3.2.1 Boundaries Pieces of the Fourth Kind

We first consider decoupling of solutions to the Lamé system which satisfies the fourth kind boundary condition on a boundary piece.

Theorem 3.1. *Given a bounded Lipschitz domain Ω in $\mathbb{R}^3$, let the vector field $\mathbf{U}$ be a solution to*

$$(\Delta^* + \omega^2)\mathbf{U} = 0 \quad \textit{in } \Omega. \tag{3.11}$$

Let Γ be a connected piece of $\partial\Omega$, and let $\mathbf{U}$ satisfy

$$\nu \times \mathbf{U} = 0 \quad \textit{and} \quad \nu \cdot \mathcal{T}\mathbf{U} = 0 \qquad \textit{on} \ \ \Gamma, \tag{3.12}$$

with ν the unit outward normal vector of Ω. Assume that

$$\Theta = \Theta_\Gamma(x) := \sum_{l=1}^{3} \Big[\operatorname{Grad}_\Gamma \nu_l \Big]_l = 0 \quad \textit{for all } x \in \Gamma, \tag{3.13}$$

where $\operatorname{Grad}_\Gamma$ *denotes the surface gradient operator on* Γ *(cf., [30, 106, 76]). Then one has*

$$v = 0 \qquad \textit{on} \ \ \Gamma,$$

where v is defined by $v := \nabla \cdot \mathbf{U}$.

Remark 3.1. We would like first to explain a little more about the geometric condition (3.13). Let

$$x(u) = (x_j(u_1, u_2))_{j=1}^3$$

be a local parametric representation of Γ. Let $g = (g_{jk})_{j,k=1}^2$ be the first fundamental form of Γ which can be represented as

$$g_{jk} := \frac{\partial x}{\partial u_j} \cdot \frac{\partial x}{\partial u_k}, \qquad j,k = 1,2.$$

Then the surface gradient operator Grad_Γ can be formulated as

$$\mathrm{Grad}_\Gamma\, \varphi = \sum_{j,k=1}^{2} g^{jk} \frac{\partial \varphi}{\partial u_j} \frac{\partial x}{\partial u_k},$$

where $g^{-1} = (g^{jk})_{j,k=1}^2$ is the inverse of g. Hence one has that

$$\begin{aligned} \varTheta = \sum_{l=1}^{3} \Big[\mathrm{Grad}_\Gamma\, \nu_l \Big]_l &= \sum_{j,k=1}^{2} \sum_{l=1}^{3} g^{jk} \frac{\partial \nu_l}{\partial u_j} \frac{\partial x_l}{\partial u_k} \\ &= \sum_{j,k=1}^{2} g^{jk} \frac{\partial \nu}{\partial u_j} \cdot \frac{\partial x}{\partial u_k} = - \sum_{j,k=1}^{2} g^{jk} l_{jk}, \end{aligned}$$

where $l = (l_{jk})_{j,k=1}^2$ is the second fundamental form of Γ given by

$$l_{jk} := \frac{\partial^2 x}{\partial u_j \partial u_k} \cdot \nu = - \frac{\partial \nu}{\partial u_j} \cdot \frac{\partial x}{\partial u_k}, \qquad j,k = 1,2.$$

Therefore, the geometric condition (3.13) is equivalent to that

$$0 = \mathrm{tr}(l g^{-1}) = (2 l_{12} g_{12} - l_{11} g_{22} - l_{22} g_{11}) / \det(g). \tag{3.14}$$

Proof. We first compute by direct calculations that

$$\begin{aligned} &\mathrm{Grad}_\Gamma \Big(\nu_l U_j \Big) = \nu_l\, \mathrm{Grad}_\Gamma\, U_j + U_j\, \mathrm{Grad}_\Gamma\, \nu_l \\ =&\nu_l \nabla U_j - \nu_l \sum_{i=1}^{3} \nu_i \partial_i U_j \nu + U_j\, \mathrm{Grad}_\Gamma\, \nu_l, \quad j,l = 1,2,3, \end{aligned}$$

where we have used the relation

$$\nabla\, \varphi = \mathrm{Grad}_\Gamma\, \varphi + (\partial_\nu \varphi)\, \nu,$$

for any φ defined in a neighborhood of Γ. As a consequence one observes on Γ that

$$\sum_{l=1}^{3}\Big[\operatorname{Grad}_\Gamma\left(\nu_l U_j\right)\Big]_l=\sum_{l=1}^{3}\nu_l\partial_l U_j-\sum_{i=1}^{3}\nu_i\partial_i U_j+\varTheta U_j=0 \tag{3.15}$$

and

$$\sum_{l=1}^{3}\Big[\operatorname{Grad}_\Gamma\left(\nu_j U_l\right)\Big]_l=\nu_j\,\nabla\cdot\mathbf{U}-\left(\nu^T\left(\nabla\,\mathbf{U}\right)\nu\right)\nu_j+\mathbf{U}\cdot\operatorname{Grad}_\Gamma\nu_j$$

hold for all $j=1,2,3$.

On the other hand, the first relation in the boundary condition (3.12) implies for each $j=1,2,3$ that

$$\operatorname{Grad}_\Gamma\left(\big[\nu\times\mathbf{U}\big]_j\right)=0\quad\text{on}\;\;\Gamma.$$

Moreover, noticing the the property $\nu\cdot\operatorname{Grad}_\Gamma\nu_j=0$ we have for $j=1,2,3$ that

$$\mathbf{U}\cdot\operatorname{Grad}_\Gamma\nu_j=0\qquad\text{on}\;\;\Gamma. \tag{3.16}$$

Therefore, by combining the equations (3.15)-(3.16) we obtain

$$\nu^T\left(\nabla\,\mathbf{U}\right)\nu=\nabla\cdot\mathbf{U}\qquad\text{on }\Gamma.$$

By further using the second identity in (3.12) one has,

$$0=\nu\cdot\mathcal{T}\mathbf{U}=2\mu\nu^T\left(\nabla\,\mathbf{U}\right)\nu+\lambda\,\nabla\cdot\mathbf{U}=(2\mu+\lambda)v\quad\text{on}\;\;\Gamma.$$

The proof is complete. □

We have derived in Theorem 3.1 a boundary condition related to the P-wave, due to the fourth kind boundary condition for the Lamé system. Next we will prove a consequence which specify the corresponding boundary condition associating to the S-wave.

Corollary 3.1. *Let Ω, Γ, ν and $\mathbf{U}$ be the same as in Theorem 3.1. Then*

$$\nu\times\mathbf{H}=0\quad\textit{on }\Gamma$$

with $\mathbf{H}:=-\mathbf{i}\operatorname{curl}\mathbf{E}/k_s$ and $\mathbf{E}:=\operatorname{curl}\mathbf{U}$.

Proof. We first rewrite the Lamé system (3.11) as

$$-(\lambda + 2\mu)\nabla v - \mu \mathbf{H} + \omega^2 \mathbf{U} = 0,$$

where the scalar function v is defined by $v := \nabla \cdot \mathbf{U}$. Then by using the boundary condition (3.12) and integration by parts, one can obtain, for any $\mathbf{F} \in H(\mathrm{curl}, \Omega)$ such that $\mathrm{curl}\, \mathbf{F} \in H(\mathrm{curl}, \Omega)$ and $\nu \times \mathbf{F} = 0$ on $\partial\Omega \backslash \Gamma$, that

$$\begin{aligned}
\mu \int_\Gamma \mathbf{F} \cdot (\nu \times \mathbf{H}) &= \int_\Gamma \big(\omega^2 \mathbf{U} - \mu \mathbf{H}\big) \cdot (\nu \times \mathbf{F}) \\
&= \int_\Omega \big(\omega^2 \mathbf{U} - \mu \mathbf{H}\big) \cdot (\mathrm{curl}\, \mathbf{F}) - \int_\Omega \mathbf{F} \cdot \mathrm{curl}\, \big(\omega^2 \mathbf{U} - \mu \mathbf{H}\big) \\
&= (\lambda + 2\mu) \int_\Omega \nabla v \cdot (\mathrm{curl}\, \mathbf{F}) \\
&= (\lambda + 2\mu) \left(\int_\Gamma \nu \cdot (\mathrm{curl}\, \mathbf{F})\, v + \int_{\partial\Omega \backslash \Gamma} (\nu \times \mathbf{F}) \cdot \nabla v \right) \\
&= 0,
\end{aligned}$$

where the last identity is implied by Theorem 3.1 that $v = 0$ on Γ. We hence have completed the proof. □

3.2.2 Boundaries Pieces of the Third Kind

Decoupling of solutions to the Lamé system which satisfies the third kind boundary condition on a boundary piece will be given in this subsections. Instead of the geometric assumption (3.13) or (3.14) for the fourth kind case, we shall adapt a more restrictive geometric condition for the third kind boundary case.

First, we have

Theorem 3.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, and let $\mathbf{U}$ be a solution to the Lamé system* (3.11). *Suppose that $\mathbf{U}$ satisfy*

$$\nu \cdot \mathbf{U} = 0 \qquad \text{and} \qquad \nu \times \mathcal{T}\mathbf{U} = 0 \qquad \text{on} \quad \Gamma, \tag{3.17}$$

with ν the unit outward normal vector of Ω, and Γ a connected piece of $\partial\Omega$. If Γ is flat, then one has

$$\nu \times \mathbf{E} = 0 \qquad \text{on} \quad \Gamma, \tag{3.18}$$

where the vector field $\mathbf{E}$ is defined by $\mathbf{E} := \mathrm{curl}\, \mathbf{U}$.

Proof. The first identity in (3.17) implies that

$$0 = \mathrm{Grad}_\Gamma(\nu \cdot \mathbf{U}) = (\nabla \mathbf{U})\, \nu - \left(\nu^T (\nabla \mathbf{U})\, \nu\right) \nu \quad \text{on } \Gamma.$$

The second relation in (3.17) suggests on Γ that

$$0 = (\nu \times \mathcal{T}\mathbf{U}) \times \nu/\mu = \left(\nabla^T \mathbf{U} + \nabla \mathbf{U}\right) \nu - 2\left(\nu^T (\nabla \mathbf{U})\, \nu\right) \nu.$$

Combining the above two equations one has

$$\left(\nabla^T \mathbf{U} - \nabla \mathbf{U}\right) \nu = 0 \quad \text{on } \Gamma,$$

which is observed to be equivalent to (3.18).

The proof is complete. □

Theorem 3.2 shows a boundary condition subjected by the induced field $\mathbf{E}$ corresponding the S-wave which is implied by the third boundary condition of the Lamé system. Next, we shall show a consequence on a boundary relation for the field closely related to the P-wave.

Corollary 3.2. *Let Ω, Γ, ν and $\mathbf{U}$ be the same as in Theorem 3.2. Then*

$$\partial_\nu v = 0 \quad \text{on } \Gamma, \tag{3.19}$$

where the scalar function v is defined as $v := \nabla \cdot \mathbf{U}$.

Proof. Denote $\mathbf{E} := \mathrm{curl}\, \mathbf{U}$. Then Lamé system (3.11) can be written as

$$-(\lambda + 2\mu)\nabla v - \mu\, \mathrm{curl}\, \mathbf{E} + \omega^2 \mathbf{U} = 0.$$

Hence for any $f \in H^1(\Omega)$ such that $f = 0$ on $\partial\Omega \backslash \Gamma$, we have

$$\begin{aligned}
& -(\lambda + 2\mu) \int_\Gamma f\, \partial_\nu v \\
=& \int_\Gamma f \left(\omega^2 \nu \cdot \mathbf{U} - (\lambda + 2\mu)\partial_\nu v\right) \\
=& \int_\Omega \nabla f \cdot \left(\omega^2 \mathbf{U} - (\lambda + 2\mu)\, \nabla v\right) \int_\Omega f\, \nabla \cdot \left(\omega^2 \mathbf{U} - (\lambda + 2\mu)\, \nabla v\right) \\
=& \mu \int_\Omega \nabla f \cdot (\mathrm{curl}\, \mathbf{E}) \\
=& \mu \int_\Gamma (\nu \times \mathbf{E}) \cdot \nabla f - \mu \int_{\partial\Omega \backslash \Gamma} v \cdot (\mathrm{curl}\, \mathbf{E})\, f \\
=& 0,
\end{aligned}$$

where in the last identity we have used the result from Theorem 3.2 that $\nu \times \mathbf{E} = 0$ on Γ.

The proof is complete. □

3.2.3 Elastic Decoupling in Boundary Value Problems

We summarize in the following the decoupling results derived in Sections 3.2.1 and 3.2.2 for boundary value problems.

Theorem 3.3. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. We have the following conclusions:*

(1) Suppose that $\partial\Omega = \sum_{j=1}^{m} \Gamma_j$ where m is a constant and each boundary piece Γ_j satisfies the condition (3.13)*. Let $\mathbf{U}$ be the solution to the constant coefficients Lamé system with the fourth kind boundary condition:*

$$\begin{cases} (\Delta^* + \omega^2)\mathbf{U} = 0 & \text{in } \Omega, \\ \nu \times \mathbf{U} = 0 \quad \text{and} \quad \nu \cdot \mathcal{T}\mathbf{U} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

Then one has

$$\begin{cases} (\Delta^* + k_p^2)v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.21}$$

and

$$\begin{cases} \operatorname{curl}\mathbf{E} - \mathtt{i}k_s\mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{H} + \mathtt{i}k_s\mathbf{E} = 0 & \text{in } \Omega, \\ \nu \times \mathbf{H} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.22}$$

where $v := \nabla \cdot \mathbf{U}$, $\mathbf{E} := \operatorname{curl}\mathbf{U}$ and $\mathbf{H} := \operatorname{curl}\mathbf{E}/(\mathtt{i}k_s)$.

(2) If Ω is a polyhedron and $\mathbf{U}$ satisfies

$$\begin{cases} (\Delta^* + \omega^2)\mathbf{U} = 0 & \text{in } \Omega, \\ \nu \cdot \mathbf{U} = 0 \quad \text{and} \quad \nu \times \mathcal{T}\mathbf{U} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.23}$$

Then one has

$$\begin{cases} (\Delta^* + k_p^2)v = 0 & \text{in } \Omega, \\ \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.24}$$

and

$$\begin{cases} \operatorname{curl}\mathbf{E} - \mathtt{i}k_s\mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl}\mathbf{H} + \mathtt{i}k_s\mathbf{E} = 0 & \text{in } \Omega, \\ \nu \times \mathbf{E} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.25}$$

We have shown in Theorem 3.3 that, in the case when P- and S- waves are decoupled, the corresponding boundary value problem (BVP) for Lamé systems can be simplified to two BVPs, one for Helmholtz equations and the other for Maxwell systems. Therefore, related results for BVPs associated to Helmholtz equations or Maxwell systems may be applied to obtain parallel results for corresponding problems of Lamé systems.

3.3 Scattering for Linearized Elasticity

We will discuss in this section the application of the decoupling results we have derived in Section 3.2 to elastic or seismic scattering problems.

3.3.1 Backgrounds

We introduce in this section some preliminaries of the scattering theory for linearized elastic, or seismic, waves. We shall refer an obstacle as a bounded domain in $\mathbb{R}^3$ whose complement is connected from now on.

Let D be a third or fourth kind elastic obstacle with the Lamé coefficients λ and μ. Given a time-harmonic incident field $\mathbf{U}^{\mathrm{inc}}$ which is an solution to the Navier equation (3.1) in the whole space $\mathbb{R}^3$, the corresponding elastic or seismic scattering problem for D can be characterized as the following system,

$$\begin{cases} (\Delta^* + \omega^2)\mathbf{U} = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{D}, \\ \mathbf{U} = \mathbf{U}^{\mathrm{inc}} + \mathbf{U}^{\mathrm{sca}} \quad \text{in } \mathbb{R}^3 \backslash \overline{D}, \\ \mathbf{U} \text{ satisfies certain boundary condition (B.C.) on } \partial D, \\ \mathbf{U}^{\mathrm{sca}} \text{ satisfies the Kupradze radiation condition at infinity,} \end{cases} \tag{3.26}$$

where $\mathbf{U}$ is the total displacement field, and $\mathbf{U}^{\mathrm{sca}}$ is the scattered field. The Kupradze radiation condition in (3.26) guaranteeing $\mathbf{U}^{\mathrm{sca}}$ to propagate outwards is given by

$$\begin{cases} \lim\limits_{|x|\to\infty} \left(x \times \operatorname{curl}\operatorname{curl}\mathbf{U}^{\mathrm{sca}} + \mathtt{i}k_s|x| \operatorname{curl}\mathbf{U}^{\mathrm{sca}} \right) = 0, \\ \lim\limits_{|x|\to\infty} \left(x \cdot \nabla(\nabla \cdot \mathbf{U}^{\mathrm{sca}}) - \mathtt{i}k_p|x| \, \nabla \cdot \mathbf{U}^{\mathrm{sca}} \right) = 0, \end{cases} \tag{3.27}$$

which is satisfied uniformly for all directions $\hat{x} \in \mathbb{S}^2$. Solutions to the exterior Lamé system which satisfies the Kupradze radiation condition are called radiating solutions.

In terms of P- and S-waves, the Kupradze radiation condition (3.27) is equivalent to the Sommerfeld radiation conditions,

$$\begin{cases} \lim\limits_{|x|\to\infty} \left(x \cdot \nabla \mathbf{U}_p^{\mathrm{sca}} - \mathtt{i}k_p|x|\mathbf{U}_p^{\mathrm{sca}}\right) = 0, \\ \lim\limits_{|x|\to\infty} \left(x \cdot \nabla \mathbf{U}_s^{\mathrm{sca}} - \mathtt{i}k_s|x|\mathbf{U}_s^{\mathrm{sca}}\right) = 0, \end{cases} \quad \text{uniformly for all } \hat{x} \in \mathbb{S}^2. \tag{3.28}$$

As a consequence, one has the following asymptotic behaviors as $|x| \to +\infty$,

$$\begin{aligned} \mathbf{U}_p^{\mathrm{sca}}(x) &= |x|^{-1}\left(e^{\mathtt{i}k_p|x|}\mathbf{U}_p^{\infty}(\hat{x}) + \mathcal{O}\left(|x|^{-1}\right)\right), \\ \mathbf{U}_s^{\mathrm{sca}}(x) &= |x|^{-1}\left(e^{\mathtt{i}k_s|x|}\mathbf{U}_s^{\infty}(\hat{x}) + \mathcal{O}\left(|x|^{-1}\right)\right), \end{aligned} \tag{3.29}$$

where $\mathbf{U}_p^{\infty}$ and $\mathbf{U}_s^{\infty}$ are known as, respectively, the longitudinal (or P-part of) and the transversal (or S-part of) the far-field pattern of $\mathbf{U}^{\mathrm{sca}}$. The total scattering amplitude $\mathbf{U}^{\infty}$ is defined as

$$\mathbf{U}^{\infty} := \mathbf{U}_p^{\infty} + \mathbf{U}_s^{\infty}. \tag{3.30}$$

Moreover, one has the following property (cf. [65])

$$\hat{x} \times \mathbf{U}_p^{\infty} = 0 \quad \text{and} \quad \hat{x} \cdot \mathbf{U}_s^{\infty} = 0. \tag{3.31}$$

For more results concerning the scattering problem (3.26), we refer to [81, 85, 4].

We recall that the S- and P- waves as characterized in Theorem 3.3 can be decoupled and formulated into corresponding problems for Helmholtz equations and Maxwell systems. We hence introduce some backgrounds on scattering problems for, respectively, Helmholtz equations and Maxwell systems.

Given an impenetrable acoustic obstacle D and an incident wave v^{inc}, the corresponding acoustic scattering problem can be formulated as

$$\begin{cases} (\Delta + k^2)v = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{D}, \\ v = v^{\mathrm{inc}} + v^{\mathrm{sca}} \quad \text{in } \mathbb{R}^3\backslash\overline{D}, \\ \text{B.C. for } v \text{ on } \partial D, \\ v^{\mathrm{sca}} \text{ satisfies the Sommerfeld radiation condition at infinity,} \end{cases} \tag{3.32}$$

where k is the acoustic wave number. The Sommerfeld radiation condition in (3.32) is given by

$$\lim_{|x|\to\infty} (x \cdot \nabla v^{sc} - \mathtt{i}k|x|v^{sc}) = 0, \quad \text{uniformly for all } \hat{x} \in \mathbb{S}^2, \tag{3.33}$$

which results into the far-field pattern v^∞ at infinity such that

$$v^{\mathrm{sca}}(x) = |x|^{-1} \left(e^{\mathtt{i}k|x|} v^\infty(\hat{x}) + \mathcal{O}(|x|^{-1})\right). \tag{3.34}$$

Depending on the physical property of the obstacle, the boundary condition in (3.32) is usually of two forms. One is the Dirichlet boundary condition,

$$v = 0 \qquad \text{on } \partial D, \tag{3.35}$$

which is imposed when the obstacle is sound-soft. The other is the Neumann boundary condition corresponding to sound-hard obstacles,

$$\partial_\nu v = 0 \qquad \text{on } \partial D. \tag{3.36}$$

Analogously, the time harmonic electromagnetic obstacle scattering problem with wave number k can be characterized as

$$\begin{cases} \operatorname{curl} \mathbf{E} - \mathtt{i}k\mathbf{H} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{H} + \mathtt{i}k\mathbf{E} = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{D}, \\ (\mathbf{E}, \mathbf{H}) = (\mathbf{E}^{\mathrm{inc}}, \mathbf{H}^{\mathrm{inc}}) + (\mathbf{E}^{\mathrm{sca}}, \mathbf{H}^{\mathrm{sca}}) \quad \text{in } \mathbb{R}^3 \backslash \overline{D}, \\ \text{B.C. for } \mathbf{E} \text{ or } \mathbf{H} \text{ on } \partial D, \\ (\mathbf{E}^{\mathrm{sca}}, \mathbf{H}^{\mathrm{sca}}) \text{ satisfies certain radiation condition at infinity.} \end{cases} \tag{3.37}$$

The radiation condition in (3.37) is the Silver-Müller radiation condition given by

$$\lim_{|x|\to\infty} (x \times \mathbf{H}^{\mathrm{sca}} + |x|\mathbf{E}^{\mathrm{sca}}) = 0, \quad \text{uniformly for all } \hat{x} \in \mathbb{S}^2, \tag{3.38}$$

or equivalently,

$$\lim_{|x|\to\infty} (x \times \mathbf{E}^{\mathrm{sca}} - |x|\mathbf{H}^{\mathrm{sca}}) = 0, \quad \text{uniformly for all } \hat{x} \in \mathbb{S}^2. \tag{3.39}$$

As a consequence, the scattered field subjects to the following asymptotic behavior

$$\big(\mathbf{E}^{\mathrm{sca}}, \mathbf{H}^{\mathrm{sca}}\big)(x) = |x|^{-1}\Big(e^{\mathtt{i}k|x|}\big(\mathbf{E}^\infty, \mathbf{H}^\infty\big)(\hat{x}) + \mathcal{O}(|x|^{-1})\Big), \tag{3.40}$$

where $\mathbf{E}^\infty$ and $\mathbf{H}^\infty$ are called, respectively, the electric and the magnetic far-field pattern. Moreover, one has that (cf. [76, 30])

$$\mathbf{H}^\infty = \hat{x} \times \mathbf{E}^\infty \quad \text{and} \quad \hat{x} \cdot \mathbf{E}^\infty = \hat{x} \cdot \mathbf{H}^\infty = 0. \tag{3.41}$$

The types of electromagnetic obstacle we consider are perfect electric conductors (PECs) described by the boundary condition

$$\nu \times \mathbf{E} = 0 \quad \text{on } \partial D, \tag{3.42}$$

and perfect magnetic conductors (PMCs) satisfying

$$\nu \times \mathbf{H} = 0 \quad \text{on } \partial D. \tag{3.43}$$

3.3.2 Elastic Decoupling in Forward Scattering

In this section, decoupling properties for S- and P-waves of the time-harmonic elastic/seismic scattering system (3.1) with the third or fourth kind boundary condition. The following two theorems can be deduced from the previous arguments and results, in particular, Theorem 3.3. We omit the proofs.

Theorem 3.4. *Let D be a fourth kind Lipschitz obstacle with $\partial D = \sum_{j=1}^m \Gamma_j$, where m is a constant and each boundary piece Γ_j satisfies the condition* (3.13). *Given a incident field $\mathbf{U}^{\rm inc}$, let $\mathbf{U}$ be the solution (total wave) to the scattering problem* (3.26) *with the fourth kind boundary condition*

$$\nu \times \mathbf{U} = 0 \quad \textit{and} \quad \nu \cdot \mathcal{T}\mathbf{U} = 0 \qquad \textit{on } \ \partial D. \tag{3.44}$$

Then one has that $v := \nabla \cdot \mathbf{U}$ satisfies the Helmholtz scattering problem (3.32) *with the incident field $v^{\rm inc} = \nabla \cdot \mathbf{U}^{\rm inc}$, the wave number k_p and the sound-soft boundary condition* (3.35)*; and $(\mathbf{E}, \mathbf{H})$ defined by $\mathbf{E} := \operatorname{curl} \mathbf{U}$ and $\mathbf{H} := -\mathtt{i} \operatorname{curl} \mathbf{E}/k_s$ solves the Maxwell scattering problem* (3.37) *with the wave number k_s, the PMC boundary condition* (3.43)*, and the incident waves $\mathbf{E}^{\rm inc} = \operatorname{curl} \mathbf{U}^{\rm inc}$ and $\mathbf{H}^{\rm inc} = -\mathtt{i} \operatorname{curl} \mathbf{E}^{\rm inc}/k_s$.*

Theorem 3.5. *Let D be a third kind Lipschitz polyhedral obstacle. Let $\mathbf{U}$ be the total wave of* (3.26) *with the third kind boundary condition*

$$\nu \cdot \mathbf{U} = 0 \quad \textit{and} \quad \nu \times \mathcal{T}\mathbf{U} = 0 \qquad \textit{on } \ \partial D, \tag{3.45}$$

and the incident field of the form (3.46)*. Let v, $\mathbf{E}$ and $\mathbf{H}$ be defined as the same in Theorem 3.4. Then one has that v satisfies the Helmholtz scattering problem* (3.32) *with the sound-hard boundary condition* (3.36)*, and that* $(\mathbf{E}, \mathbf{H})$ *solves the Maxwell scattering problem* (3.37) *with the PEC boundary condition* (3.42)*, where the corresponding wave numbers and incident fields are the same as in Theorem 3.4.*

We introduce here elastic incident plane waves of the following general form

$$\begin{aligned} \mathbf{U}^{\mathrm{inc}}(x) &= \mathbf{U}^{\mathrm{inc}}(x; d, d^{\perp}, k_p, k_s \alpha_p, \alpha_s) \\ &= \alpha_p d e^{\mathtt{i} k_p d \cdot x} + \alpha_s d^{\perp} e^{\mathtt{i} k_s d \cdot x} =: \mathbf{U}_p^{\mathrm{inc}} + \mathbf{U}_s^{\mathrm{inc}}, \end{aligned} \tag{3.46}$$

where α_p and α_s are arbitrary complex constants, $d \in \mathbb{S}^2$ is the incident direction, and $d^{\perp} \in \mathbb{S}^2$ satisfying $d \cdot d^{\perp} = 0$ is the polarization direction. Hereinafter, given a vector field $d \in \mathbb{S}^2$, the notation $d^{\perp}$ shall refer to as a unit perpendicular vector to d.

The following corollary concerns the particular case for Theorems 3.4 and 3.5 when the elastic/seismic incident field is given by (3.46).

Corollary 3.3. *Consider the time-harmonic incident wave $\mathbf{U}^{\mathrm{inc}}$ with the form* (3.46) *in Theorems 3.4 and 3.5. Then the results of Theorems 3.4 and 3.5 hold for the acoustic incident wave*

$$v^{\mathrm{inc}} = \nabla \cdot \mathbf{U}^{\mathrm{inc}} = \mathtt{i} \alpha_p k_p e^{\mathtt{i} k_p d \cdot x}, \tag{3.47}$$

and the electromagnet incident fields

$$\mathbf{E}^{\mathrm{inc}} = \mathtt{i} \alpha_s k_s d \times d^{\perp} e^{\mathtt{i} k_s d \cdot x} \quad \text{and} \quad \mathbf{H}^{\mathrm{inc}} = -\mathtt{i} \alpha_s k_s d^{\perp} e^{\mathtt{i} k_s d \cdot x}. \tag{3.48}$$

By reviewing the decoupling results in Theorems 3.4 and 3.5, one observes that a fourth kind seismic obstacle which satisfies the geometry condition in Theorem 3.4 can be theoretically viewed as a sound-soft acoustic scatterer, or a perfect magnetic conductor; and a third kind polyhedral elastic obstacle can be viewed as a sound-soft acoustic obstacle, or a perfect electric conductor. Therefore, the corresponding known results on inverse scattering for acoustic and electromagnetic waves can be extended to their counterparts in inverse scattering problems for seismic/elastic waves. Moreover, one can see from previous arguments that the field satisfies the acoustic scattering problem involves only the P-part of the elastic field, and that the one subjected to the electromagnetic scattering problem corresponds to pure S-waves. This is also of significant importance from the practical view.

Before proceeding to inverse scattering problems for elastic/seismic waves, we present a lemma which reveals the one-to-one correspondences between the far-field patterns of elastic waves and those of acoustic and electromagnetic fields. It can be found in references such as [65].

Lemma 3.1. *Let* $\mathbf{U}^{\text{sca}}$ *be a radiating solution to the Lamé system. Define* $v^{\text{sca}} := \nabla \cdot \mathbf{U}^{\text{sca}}$, $\mathbf{E}^{\text{sca}} := \operatorname{curl} \mathbf{U}^{\text{sca}}$ *and* $\mathbf{H}^{\text{sca}} := \operatorname{curl} \mathbf{E}^{\text{sca}}/(\mathtt{i}k_s)$. *Then* v^{sca} *is a radiating solution to the Helmholtz equation, and* $(\mathbf{E}^{\text{sca}}, \mathbf{H}^{\text{sca}})$ *is a radiating solution to the Maxwell system. Moreover, one has that*

$$\mathbf{U}_p^{\infty} = v^{\infty}\hat{x}/(\mathtt{i}k_p) \quad \textit{and} \quad \mathbf{U}_s^{\infty} = \mathbf{H}^{\infty}/(\mathtt{i}k_s).$$

3.4 Inverse Obstacle Scattering in Elasticity

In this section, we consider the determination of a polyhedral elastic obstacle D by measuring a minimum number of P- or S-far-field measurement. This is a typical problem in inverse scattering. For study concerning inverse problems on elastic/seismic fields, we refer to [65, 105, 49, 5, 68, 69, etc.].

Our results are valid for a very general admissible class of polyhedral obstacles. In principle, obstacles in the admissible class may contain several disjointed components. Moreover, two-dimensional flat surfaces, which are called referred to as "screen-like" pieces, are also allowed to appear in the admissible obstacle. But the minimal number of measurements needed in uniquely recovering the obstacle varies for obstacles containing screen-like pieces or for those which do not contain two-dimensional pieces.

Based on the decoupling results in previous sections, in particular those in Section 3.3.2, uniqueness and stability results for elastic/seismic inverse scattering in this section will be proved by the counterpart results for acoustic and electromagnetic inverse scattering problems. It has been proven that acoustic sound-soft/hard or EM perfect electric/magnetic conducing (PEC/PMC) polyhedral obstacles can be uniquely determined by a single, or $N-1$, or N scattering measurement(s); see, [26, 3, 27, 47, 96, 97, 94, 48, 89, 95]. The numbers of measurements depend on different physical and geometrical assumptions on the obstacles and have been shown to be *optimal* to guarantee the uniqueness therein. Corresponding stability estimates for these inverse acoustic or electromagnetic scattering problems can be found in [114, 123, 91, 90].

3.4.1 The Uniqueness

Uniqueness results on determining a polyhedral obstacle by a few scattering measurements will be given in this section. Depending on whether the targeted obstacle contains flat pieces, uniqueness results for inverse elastic/seismic obstacle scattering problems shall be divided into two parts. We would like to remark here that, though the unique recovery results presented in the following theorems concern only far-field measurements, we know from unique continuation that all these results are also valid when near-field measurements of the scattered waves are used instead.

We first give the uniqueness in the case when the obstacle has no flat pieces, which are also referred to as screen-like/type pieces. We shall prove that a single measurement uniquely identifies a third or fourth kind obstacle.

Theorem 3.6. *Let D be a polyhedral obstacle of either the third or the fourth kind which does not contain two-dimensional flat pieces. Then the obstacle can be uniquely determined by* a single far-field measurement $\mathbf{U}^\infty$ *corresponding to the incident wave* $\mathbf{U}^{\text{inc}}(\cdot; d, d^\perp, k_p, k_s, \alpha_p, \alpha_s)$ *as introduced in* (3.46) *with fixed constants α_p, α_s, wavenumber k_p, k_s, and unit vectors $d, d^\perp$.*

Moreover, if $\alpha_p \neq 0$ (or $\alpha_s \neq 0$) then the obstacle along with its boundary type is uniquely identified by a single longitudinal far-field pattern $\mathbf{U}_p^\infty$ (or respectively, transversal far-field data $\mathbf{U}_s^\infty$).

When the obstacle might contain screen-like pieces, the number of measurement needed for the uniqueness result might be two or three, depending on the kind of boundary condition as well as the type of far-field measurements (longitudinal, transversal, or total).

Theorem 3.7. *Let D be a polyhedral obstacle which might* contain flat pieces. *We have the following uniqueness results.*

(i) If D is of the fourth kind, then it is uniquely identifiable by a single far-field measurement $\mathbf{U}^\infty$ or $\mathbf{U}_p^\infty$ with an incident wave $\mathbf{U}^{\mathrm{inc}}$ given in (3.46) *satisfying $\alpha_p \neq 0$.*

(ii) Suppose that D is of either the third or the fourth type, which is known a priori. Then D can be uniquely determined by two far-field measurements $\mathbf{U}_j^\infty$ (or the transversal $\mathbf{U}_{j,s}^\infty$), $j = 1, 2$, for incident fields $\mathbf{U}_j^{\mathrm{inc}}$ given in (3.46) *with $\alpha_s \neq 0$ and the same wavenumber k_s and incident direction d but different $d_1^\perp$ and $d_2^\perp$, satisfy that $d_1^\perp$, $d_2^\perp$ and d are linearly independent.*

(iii) If D is of the third kind, then it can also be uniquely identified by three longitudinal far-field measurements $\mathbf{U}_{j,p}^\infty$, $j = 1, 2, 3$, for incident fields with $\alpha_p \neq 0$ and three linearly independent incident directions d_1, d_2 and d_3.

Proof of Theorems 3.6 and 3.7. Let D_j, $j = 1, 2$, be two polyhedral obstacles which has the same (longitudinal or transversal) far-field pattern for the incident wave $\mathbf{U}^{\mathrm{inc}} = \mathbf{U}^{\mathrm{inc}}(\cdot; d, d^\perp, k_p, k_s, \alpha_p, \alpha_s)$ as in (3.46).

It is observed from (3.30) and (3.31) that, if two total far-field patterns $\mathbf{U}_j^\infty$, $j = 1, 2$, are the same, then the corresponding longitudinal and the transversal far-field data will coincide as well, namely, $\mathbf{U}_{1,p}^\infty = \mathbf{U}_{2,p}^\infty$ and $\mathbf{U}_{1,s}^\infty = \mathbf{U}_{2,s}^\infty$. Therefore, we need only to prove the cases when the pure longitudinal or transversal far-field measurements are the same for D_1 and D_2.

If $\alpha_p \neq 0$ and $\mathbf{U}_{1,p}^\infty = \mathbf{U}_{2,p}^\infty$, then we obtain from Lemma 3.1 that $v_1^\infty = v_2^\infty$, where v_j^∞ is the corresponding far-field of $\nabla \cdot \mathbf{U}_j$ given in Theorems 3.4 and 3.5, for each $j = 1, 2$. Recall that v_j satisfies the acoustic scattering system (3.32) with the sound-soft boundary condition if D is of the fourth kind, and with sound-hard one if D is the third kind.

In the case that the obstacles are solid, that is, they do not contain two-dimensional flat pieces, the results in [3] and [48] ensure the unique determination of a sound-soft, or respectively a sound-hard obstacle by a single measurement, and hence imply our uniqueness in determining a fourth kind, or respectively a third kind elastic obstacle from one measurement.

Otherwise if the scatterers D_1 and D_2 might contain screen-like components, then the uniqueness in recovering a polyhedral sound-soft scatterer [3] suggests $D_1 = D_2$ with only one consistent measurements, when D_1 and D_2 are of the fourth kind. If they are of the third kind, then the uniqueness is ensured by using three measurements with linearly independent directions d_j, $j = 1, 2, 3$, as suggested in [96] for sound-soft scatterers.

If $\alpha_s \neq 0$ and $\mathbf{U}_{1,s}^\infty = \mathbf{U}_{2,s}^\infty$, then again by Lemma 3.1 we have $(\mathbf{E}_1^\infty, \mathbf{H}_1^\infty) = (\mathbf{E}_2^\infty, \mathbf{H}_2^\infty)$ with the new far-field patterns related to $\operatorname{curl} \mathbf{U}_j$, $j = 1, 2$ given in Theorems 3.4 and 3.5. Then uniqueness results in inverse electromagnetic scattering for impenetrable polyhedral scatterers implies our corresponding statement when using transversal far-field data $\mathbf{U}_s^\infty$. In particular, [89] gives the unique determination result for solid polyhedral PEC/PMC obstacle by one measurement, which implies the statement in Theorem 3.6 when $\alpha_s \neq 0$; and [94] provides the counterpart for polyhedral PEC/PMC scatterers which might contain two-dimensional flat pieces, which yields Statement (ii) in Theorem 3.7.

The proof is complete. □

3.4.2 The Stability

In this section, we give the stability results corresponding to the uniqueness established in Section 3.4.1 for inverse elastic scattering.

Stability for inverse problems is important from both the analytic and the practical point of view. Since the measurements one can collect usually have errors, it is necessary to require a reasonable stability estimate, to guarantee that the scatterer reconstructed through the perturbed data is sufficiently close to the actual scatterer. Stability estimates for different kinds of inverse (scattering) problems have been extensively studied, see for instance, [125, 74, 75, 113, 2, 114, 115].

Definition 3.1. Given positive constants r, L, R and h, we say that an open set D in $\mathbb{R}^3$ belongs to the admissible class $\mathscr{A}_h = \mathscr{A}_h(r, L, R)$ if and only if D is composed of a finite number of solid polyhedrons with each polyhedron a Lipschitz domain with constants r and L whose boundary is also composed of Lipschitz domains in $\mathbb{R}^2$ with constants h and L.

For a more detailed definition and discussion of the admissible class of scatterers, we refer to [92]; see also [90] and [91].

The Hausdorff distance d_H given by

$$d_H(D_1, D_2) := \max\left\{\sup_{x\in D_1} \operatorname{dist}(x, D_2),\ \sup_{x\in D_1} \operatorname{dist}(x, D_2)\right\},$$

shall be applied to measure the difference between two shapes D_1 and D_2.

Theorem 3.8. *Let $D_j \in \mathscr{A}_h(r, L, R)$, $j = 1, 2$, be two polyhedral obstacles of either the third or the fourth kind. Then there exist positive constants a, α, C and $\epsilon_0 = \epsilon_0(h; \mathrm{K}) > 0$ satisfying the following.*

Given an incident field $\mathbf{U}^{\mathrm{inc}}$ as in (3.46), suppose that

$$\left\|\mathbf{U}^\infty(\cdot; \mathbf{U}^{\mathrm{inc}}, D_1) - \mathbf{U}^\infty(\cdot; \mathbf{U}^{\mathrm{inc}}, D_2)\right\|_{L^2(\mathbb{S}^2)} \le \epsilon < \epsilon_0. \tag{3.49}$$

Then one has

$$d_H(D_1, D_2) \le C f^\alpha(a\epsilon), \tag{3.50}$$

where the function $f : (0, 1/e) \to (0, 1)$ is defined by

$$f(t) := e^{\left(-(\log(-\log t))^{1/2}\right)}.$$

Remark 3.2. For notational simplicity, we present here only the stability result concerning the error in the total far-field data $\mathbf{U}^\infty$. In fact, it can be replaced as the difference in the transversal $\mathbf{U}_s^\infty$ or the longitudinal $\mathbf{U}_p^\infty$, in the same manner as in the uniqueness result Theorem 3.6.

Moreover, stability estimates in recovering scatterers containing screen-like pieces, in accordance with the uniqueness in Theorem 3.7, can also be established, by using the counterpart estimates for inverse acoustic or electromagnetic scattering in references [114, 90, 91]. The main difference compared to the current stability in Theorem 3.8 is the class for admissible scatterers. Owing to the appearance of two-dimensional flat pieces in scatterers, the definition of the corresponding admissible class is more technical and needs more careful characterizations.

Proof. By recalling the relation (3.31) one has

$$\|\mathbf{U}^\infty\|_{L^2(\mathbb{S}^2)} = \|\mathbf{U}_p^\infty\|_{L^2(\mathbb{S}^2)} + \|\mathbf{U}_s^\infty\|_{L^2(\mathbb{S}^2)}.$$

Hence the inequality (3.49) along with Lemma 3.1 implies that

$$\|\mathbf{E}^\infty(\cdot;\mathbf{E}^{\mathrm{inc}}, D_1) - \mathbf{E}^\infty(\cdot;\mathbf{E}^{\mathrm{inc}}, D_2)\|_{L^2(\mathbb{S}^2)} \leq \epsilon,$$

and

$$\|v^\infty(\cdot;v^{\mathrm{inc}}, D_1) - v^\infty(\cdot;v^{\mathrm{inc}}, D_2)\|_{L^2(\mathbb{S}^2)} \leq \epsilon,$$

where v^{inc} and $\mathbf{E}^{\mathrm{inc}}$ are the incident fields given in, respectively, (3.47) and (3.48), and v^∞ and $\mathbf{E}^\infty$ are the corresponding far-field patterns of the acoustic or electromagnetic scattering problems specified in Theorems 3.4 and 3.5.

Now, the stability estimate (3.50) is seen from the stability results for corresponding inverse acoustic scattering on sound-soft scatterers in [114], and for sound-hard ones in [90]. It is also obtained, in particular when the transversal far-field data is of use, from the stability estimates for corresponding inverse electromagnet scattering in [91].

The proof is complete.

□

Chapter 4

A Fractional Operator and its Calderón Problem

The fractional (nonlocal) Laplacian has found important applications in fields such as anomalous diffusion, quantum mechanics and image processing among others Inverse problems involving fractional Laplacian have attracted attentions in recent couple of years. Existing results on such inverse problems mainly concern with the non-variant Laplacian, whose fractional operator can be defined naturally by Fourier transforms. In this chapter, variable fractional Laplacian and an associated inverse problem will be discussed. We will give a proper definition of the variable fractional Laplacian, prove the well-posedness of corresponding forward problems, define the Dirichlet to Neumann map, and show the uniqueness of the associated inverse problem.

4.1 Introduction

In this chapter, a fractional (and nonlocal) anisotropic operator $\mathcal{L}^s$ with $s \in (0,1)$, as well as an associated inverse problem, will be discussed. Hereinafter, $\mathcal{L}$ denotes a uniform elliptic and bounded operator given by $\mathcal{L}u = \mathcal{L}_\gamma u := \nabla \cdot (\gamma \nabla u)$. Originating from the spectral theorem for densely defined self-adjoint operators, the fractional operator $\mathcal{L}^s$ we shall define has the form

$$\langle \mathcal{L}^s v, w \rangle = \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (v(x) - v(z))(w(x) - w(z)) \mathcal{K}_s(x,z) dx dz, \tag{4.1}$$

which also admits the pointwise presentation

$$\left(\mathcal{L}^s v\right)(x) = \lim_{\epsilon \to 0^+} \int_{|x-z|>\epsilon} (v(x) - v(z))\mathcal{K}_s(x,z)dz. \tag{4.2}$$

It gives a bounded operator

$$\mathcal{L}^s : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n).$$

Nonlocal operators of the form (4.1) have got considerable attention in recent decades due to practical applications. Fields of applications include, phase transition [7], anomalous diffusion [124, 42], random walk and Lévy flights [100, 101], image processing [79, 57, 12], machine learning [118], shape optimization [34] and population dynamics [98] among others. Mathematical study for the fractional Laplacian or more general fractional/nonlocal operators as in the form (1.1) can be found in [42, 36, 13, 51, 52, 116, 117, 14, 44, 40, etc].

We shall consider in this chapter the Calderón problem of recovering the potential q from the Dirichlet to Neumann (DtN) map of the following "boundary" value problem

$$(\mathcal{L}^s + q)u = 0 \text{ in } \Omega \quad \text{with } u = g \text{ in } \Omega_e. \tag{4.3}$$

Owning to the nonlocal property of the operator $\mathcal{L}^s$, the "boundary" data g is given in the exterior, instead of data on the boundary as a standard setting for local differential operators. We shall verify well-posedness of the forward problem (4.3) and define the DtN map with appropriate mapping properties. The uniqueness in the inverse problem of identifying q from the DtN map shall be established. A crucial tool for proving the uniqueness result is an important and interesting property of $\mathcal{L}^s$. We shall show that if both u and $\mathcal{L}^s u$ vanish in an open set, then u must be zero everywhere. This is not valid for local differential operators.

The Calderón problem for fractional operators was first introduced in [55], where the authors considered to recover the potential q in a bounded domain Ω, from the corresponding DtN map of the nonlocal operator $(-\Delta)^s + q$ in Ω. It was proved in [55] that the potential q can be uniquely recovered from partial DtN data. Several related work on inverse problems concerning the fractional Laplacian has been done afterwards, see the review paper [120].

The standard Calderón problem was first introduced in [22], trying to recover the interior electrical conductivity by measuring voltage and current data at the boundary. Mathematically speaking, one tries to determine the conductivity or admittivity (matrix) function γ in a bounded domain Ω, from the Dirichlet to Neumann (DtN) map

$$\Lambda_\gamma : u|_{\partial\Omega} \mapsto \nu \cdot (\gamma \nabla u)\,,$$

where ν is the unit normal of $\partial\Omega$, and u is the electric potential or voltage which satisfies the partial differential equation (PDE)

$$\mathcal{L}_\gamma u := -\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in } \Omega. \tag{4.4}$$

Another inverse problem which is closely related to the original Calderón problem is to recover the potential q from the DtN map associated with the following PDE:

$$(\mathcal{L}_\gamma + q)\, u = 0 \quad \text{in } \Omega. \tag{4.5}$$

In the case that the conductivity is a scalar function which is smooth enough, the Calderón problem concerning to recover γ in (4.4) can be reformulated into the recovery of q in (4.5), by taking $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$ and $\mathcal{L} = -\Delta$.

According to whether the material parameters depend on directions, there are two types of materials in general. For example, when the conductivity γ is a scalar function, the material is called *isotropic*; otherwise if $\gamma = (\gamma_{ij})$ is a matrix valued function which is not reducible to a scalar function, then it is referred to as *anisotropic*. Many medium or materials in practice are anisotropic, for instance, muscle tissue and the inner core of the Earth.

There is a considerable literature concerning the Calderón problem in the recent three decades. In the case that the material is isotropic, results are almost complete for any dimensions $n \geq 2$ since the first work [127], which showed the global uniqueness of determining q from the DtN map associated with $-\Delta + q$ for dimensions $n \geq 3$. For two-dimensional case, [104] was the first uniqueness result of the Calderón problem. Later on, there are many articles suggesting improvement for the uniqueness results in [127] and [104] by, for instance, recovering parameters with less regularity. The main tool for uniqueness of the Calderón problem is complex geometrical optics (CGO) solutions, which has also found crucial importance in other types of inverse problems. For more details, please see the survey [131] and papers [64, 24, etc.] afterwards.

When the conductivity is anisotropic, the known uniqueness results (and techniques involved) for the Calderón problem in dimensions $n \geq 3$ or $n = 2$ are very different. For dimension two, the best known result is due to [6] which showed the unique determination of a general L^∞ anisotropic conductivity up to a diffeomorphism. However for three or higher dimensions, there are only results for special kinds of conductivities, for example, those with real-analytic coefficients and some kind of transversally anisotropic conductivities; see [87, 41, 78, etc.].

Unlike the results for standard Calderón problem concerning local differential operators, we are able to establish the uniqueness in the Calderón problem of recovering q for the fractional anisotropic operator $\mathcal{L}_\gamma^s$. Moreover, the result is valid under very general partial data setting.

4.2 Definition of the Fractional Operator

In this section, we will give a proper definition to the fractional operator $\mathcal{L}^s = \mathcal{L}_\gamma^s$ with $s \in (0,1)$ and $\mathcal{L}_\gamma = -\nabla \cdot (\gamma \nabla)$. We shall also denote $\mathcal{L} = \mathcal{L}_\gamma$ for notational simplicity. In the rest of the chapter, unless otherwise specified, all the arguments will be made under the following condition.

Assumption A. Assume that s is a constant in $(0,1)$, and $\gamma = (\gamma_{ij}(x))_{i,j=1}^n$ is an n-by-n symmetric matrix which is C^∞ smooth and satisfies

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^n \gamma_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^n,$$

with some positive constant λ.

4.2.1 Sobolev Spaces

In this subsection, the definition and some properties of certain (fractional) Sobolev spaces will be given. Most of these definitions and facts can be found in standard monographs on functional analysis or Sobolev spaces, for instance, [8, 99, 1]. For notational simplicity, we shall consider, without lose of generality, only real-valued function spaces throughout this chapter.

For any $a \in \mathbb{R}$, let $H^a(\mathbb{R}^n)$ be the L^2-based (fractional) Sobolev space endowed with the norm

$$\|u\|_{H^a(\mathbb{R}^n)} := \left\|\mathscr{F}^{-1}\{\langle\cdot\rangle^a \mathscr{F}u\}\right\|_{L^2(\mathbb{R}^n)},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$, and $\mathscr{F}$ is the Fourier transform operator. Notice that $H^a(\mathbb{R}^n)$ is a Hilbert space equipped with the inner product

$$(u,v)_{H^a(\mathbb{R}^n)} = \left(\mathscr{F}^{-1}\{\langle\xi\rangle^a \mathscr{F}u\}, \mathscr{F}^{-1}\{\langle\xi\rangle^a \mathscr{F}v\}\right)_{L^2(\mathbb{R}^n)}. \tag{4.6}$$

It is known that for $s \in (0,1)$, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ has the following equivalent form

$$\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + [u]_{H^s(\mathbb{R}^n)}$$

with

$$[u]^2_{H^s(\mathcal{O})} := \int_{\mathcal{O}\times\mathcal{O}} \frac{|u(x)-u(z)|^2}{|x-z|^{n+2s}} dxdz$$

for any open set $\mathcal{O}$ of $\mathbb{R}^n$. It is noted that $[\cdot]_{H^s(\mathbb{R}^n)}$ is a semi-norm of $H^s(\mathbb{R}^n)$.

Given any open set $\mathcal{O}$ of $\mathbb{R}^n$ and $a \in \mathbb{R}$, the following Sobolev spaces shall also be used in this chapter:

$$\begin{aligned} H^a(\mathcal{O}) &:= \{u|_{\mathcal{O}};\, u \in H^a(\mathbb{R}^n)\}, \\ \widetilde{H}^a(\mathcal{O}) &:= \text{the closure of } C_c^\infty(\mathcal{O}) \text{ in } H^a(\mathbb{R}^n), \\ H^a_{\overline{\mathcal{O}}} = H^a_{\overline{\mathcal{O}}}(\mathbb{R}^n) &:= \{u \in H^a(\mathbb{R}^n);\, \mathrm{supp}(u) \subset \overline{\Omega}\}. \end{aligned}$$

One can see from the definition that both $\widetilde{H}^a(\mathcal{O})$ and $H^a_{\overline{\mathcal{O}}}$ are closed subspaces of $H^a(\mathbb{R}^n)$. Moreover, $H^a_{\overline{\mathcal{O}}}$ is also a Hilbert space with the same inner product introduced in (4.6). Then one has

$$H^a(\mathbb{R}^n) = H^a_{\overline{\mathcal{O}}} \oplus \left(H^a_{\overline{\mathcal{O}}}\right)^\perp.$$

It is known that the Sobolev space $H^a(\mathcal{O})$ is complete under the norm:

$$\|u\|_{H^a(\mathcal{O})} := \inf \left\{ \|v\|_{H^a(\mathbb{R}^n)}; v \in H^a(\mathbb{R}^n) \text{ and } v|_{\mathcal{O}} = u \right\}. \tag{4.7}$$

As a consequence, one can also define the following closed subspace of $H^a(\mathcal{O})$,

$$H_0^a(\mathcal{O}) := \text{the closure of } C_c^\infty(\mathcal{O}) \text{ in } H^a(\mathcal{O}).$$

In addition, one has

$$H^a(\mathcal{O}) \cong \left(H^a_{\mathbb{R}^n \setminus \mathcal{O}} \right)^\perp.$$

When the domain Ω is smooth enough, Lipschitz for instance, the corresponding Sobolev spaces have some further relations.

Lemma 4.1. *Let $a \in \mathbb{R}$ and Ω be a Lipschitz domain in $\mathbb{R}^n$. Then one has*

$$\begin{aligned} &\widetilde{H}^a(\Omega) = H^a_{\overline{\Omega}} \subseteq H_0^a(\Omega), \\ &(H^a(\Omega))^* = \widetilde{H}^{-a}(\Omega) \text{ and } \left(\widetilde{H}^a(\Omega)\right)^* = H^{-a}(\Omega). \end{aligned}$$

4.2.2 The Heat Kernel

Consider the heat equation

$$\begin{aligned} \partial_t U + \mathcal{L} U = 0 \quad &\text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ U|_{t=0} = f \quad &\text{in } \mathbb{R}^n. \end{aligned} \tag{4.8}$$

It is observed that a solution to (4.8) can formally be written as

$$U = e^{-t\mathcal{L}} f.$$

In fact, $\{e^{-t\mathcal{L}}\}_t$ is called heat semi-group. If there exists a function $p_t = p_{\gamma,t}$ such that

$$e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} p_t(x,z) f(z) dz, \tag{4.9}$$

then p_t is called the heat kernel of the operator $\mathcal{L}$ or the semi-group $\{e^{-t\mathcal{L}}\}_t$. It is known that for $\mathcal{L} = \mathcal{L}_\gamma$ with γ satisfying Assumption A, the heat kernel p_t admits a upper and a lower bounds (see, [35, Chapter 3]):

$$c_1 e^{-b_1 \frac{|x-z|^2}{t}} t^{-\frac{n}{2}} \le p_t(x,z) \le c_2 e^{-b_2 \frac{|x-z|^2}{t}} t^{-\frac{n}{2}}, \quad x, z \in \mathbb{R}^n, \tag{4.10}$$

with b_j and c_j, $j = 1, 2$, positive constants. Moreover, p_t can be chosen to be symmetric [63, Chapter 9], namely,

$$p_t(x, z) = p_t(z, x), \quad \text{for all } x, z \in \mathbb{R}^n \text{ and all } t > 0; \tag{4.11}$$

Now we define $\mathcal{K}_s = \mathcal{K}_{\gamma,s}$ as

$$\mathcal{K}_s(x, z) := \frac{1}{\Gamma(-s)} \int_0^\infty p_t(x, z) \frac{dt}{t^{1+s}}. \tag{4.12}$$

Lemma 4.2. *There exist positive constants C_j, $j = 1, 2$, such that*

$$\frac{C_1}{|x - z|^{n+2s}} \leq \mathcal{K}_s(x, z) = \mathcal{K}_s(z, x) \leq \frac{C_2}{|x - z|^{n+2s}}. \tag{4.13}$$

Proof. Using the Gamma function we have for given $\lambda > 0$ that

$$\begin{aligned} \Gamma(\lambda) &:= \int_0^\infty e^{-t} t^{\lambda-1} dt = \int_0^\infty e^{-1/t} \frac{dt}{t^{1+\lambda}} \\ &= c^\lambda \int_0^\infty e^{-c/t} \frac{dt}{t^{1+\lambda}}, \quad c > 0. \end{aligned}$$

As a consequence we have

$$\int_0^\infty e^{-b\frac{|x-z|^2}{t}} \frac{dt}{t^{1+s+n/2}} = \frac{\Gamma(s + N/2)}{b^{s+n/2}} \frac{1}{|x - z|^{n+2s}}.$$

The proof is completed by recalling (4.10) and (4.11). □

4.2.3 Definition of the Operator $\mathcal{L}^s$

We define the following operator

$$\langle \mathcal{L}^s v, w \rangle := \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (v(x) - v(z))(w(x) - w(z)) \mathcal{K}_s(x, z) dx dz. \tag{4.14}$$

It is seen from (4.13) that

$$\begin{aligned} |\langle \mathcal{L}^s v, w \rangle| &\leq C \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(z)||w(x) - w(z)|}{|x - z|^{n+2s}} dx dz \\ &\leq C[v]_s [w]_s \leq C \|v\|_{H^s(\mathbb{R}^n)} \|w\|_{H^s(\mathbb{R}^n)}. \end{aligned} \tag{4.15}$$

Therefore, the equation (4.14) in fact defines a bounded linear operator:

$$\mathcal{L}^s : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n). \tag{4.16}$$

The symbol $\langle \cdot, \cdot \rangle$ can hence be regarded as the naturally dual pairing between $H^{-s}(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$. In the rest of this chapter, we will always adapt this notation to represent the dual operation of other pairs, for example, the duality between $\widetilde{H}^s(\Omega)$ and $\left(\widetilde{H}^s(\Omega)\right)^*$, and the one between $L^2(\Omega)$ and itself. It is observed from the definition (4.14) that $\mathcal{L}^s$ is self-adjoint, namely,

$$\langle \mathcal{L}^s v, w\rangle = \langle \mathcal{L}^s w, v\rangle, \quad \text{for any } v, w \in H^s(\mathbb{R}^n). \tag{4.17}$$

It is observed that

$$\begin{aligned}\langle \mathcal{L}^s v, w\rangle =&\frac{1}{2}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^n}\int_{|x-z|>\epsilon}(v(x)-v(z))(w(x)-w(z))\mathcal{K}_s(x,z)dxdz\\ =&\int_{\mathbb{R}^n} g(x)\lim_{\epsilon\to 0^+}\int_{|x-z|>\epsilon}(v(x)-v(z))\mathcal{K}_s(x,z)dzdx.\end{aligned}$$

Therefore, the fractional operator $\mathcal{L}^s$ admits the following pointwise form

$$\left(\mathcal{L}^s v\right)(x) = \lim_{\epsilon\to 0^+}\int_{|x-z|>\epsilon}(v(x)-v(z))\mathcal{K}_s(x,z)dz. \tag{4.18}$$

We would like to remark here that the definition (4.14) and the pointwise form (4.18) of the fractional operator $\mathcal{L}^s = \mathcal{L}^s_\gamma$ coincide with the case for $(-\Delta)^s$, when γ is taken as the identical matrix. In fact, the symmetric "kernel" $\mathcal{K}_s$ for $(-\Delta)^s$ is simply

$$\mathcal{K}_s(x,z) = \frac{c_{n,s}}{|x-z|^{n+2s}},$$

which also obeys the estimate (4.13).

Another remark for the definition (4.14) of the fractional operator $\mathcal{L}^s$ is that it comes from the spectral theory for non-negative definite, self-adjoint and densely defined self-adjoint operators. More precisely, given ϕ be a real-valued measurable function defined on the spectrum of $\mathcal{L}$, the operator function $\phi(\mathcal{L})$ is defined by

$$\phi(\mathcal{L}) := \int_0^\infty \phi(\lambda)\, dE_\lambda,$$

where $\{E_\lambda\}$ is called the spectral resolution of $\mathcal{L}$ (see for instance, [63]). In our case, the function ϕ is given by $\phi(\lambda) = \lambda^s$. By noticing

$$\lambda^s = \frac{1}{\Gamma(-s)}\int_0^\infty (e^{-t\lambda}-1)t^{-1-s}dt$$

for $s \in (0,1)$ with Γ the Gamma function, we can write

$$\mathcal{L}^s = \int_0^\infty \lambda^s\, dE_\lambda = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{-t\mathcal{L}} - \mathrm{Id}\right)\frac{dt}{t^{1+s}}, \tag{4.19}$$

where $e^{-t\mathcal{L}}$ is the same as introduced in (4.9). Then by noticing

$$\langle \mathcal{L}^s f, g\rangle_{L^2} = \frac{1}{\Gamma(-s)} \int_0^\infty \left\langle \left(e^{-t\mathcal{L}} f - f\right), g\right\rangle_{L^2} \frac{dt}{t^{1+s}},$$

one can obtain our definition (4.14) of the fractional operator $\mathcal{L}^s$.

The main reason that we prefer the weak form (4.14) to the spectral definition (4.19) is the mapping property. Recall that with the weak form (4.14) definition, we have the mapping property (4.16), which is generally not true for the spectral definition (4.19). In particular,the domain of the operator defined in (4.19) could be expected as $H^{2s}(\mathbb{R}^n)$, which has been proved for $s = 1/2$; see, [35, 63].

4.3 The Dirichlet Problem

In this section, we will formulate the "boundary" value problem associated with the fractional operator $\mathcal{L}^s$. The quotation mark to the word "boundary" is because our "boundary" value is in fact given in an exterior open set rather than on the boundary of the domain, where the latter is the typical setting in standard boundary value problems for classical differential operators. We will also derive the corresponding well-posedness for the Dirichlet problem. As an important consequence, we are able to define the corresponding Dirichlet to Neumann (DtN) map.

4.3.1 Well-Posedness of the Dirichlet Problem

We will consider in this subsection the following nonlocal Dirichlet problem

$$\begin{cases} (\mathcal{L}^s + q)u = f & \text{in } \Omega, \\ u = g & \text{in } \Omega_e, \end{cases} \tag{4.20}$$

and establish its well-posedness. In order to guarantee the well-posedness of the forward problem (4.20), the "boundary" value g is given in the exterior domain Ω_e. See Theorem 4.1 for more details. In what follows, we will call f in (4.20) as the right hand side (RHS) function, and g as the Dirichlet data of (4.20). Here and thereafter, we shall denote $\Omega \subseteq \mathbb{R}^n$ as a bounded *Lipschitz* domain, q as a potential in $L^\infty(\Omega)$, and $s \in (0, 1)$ as a given constant.

We first give the weak formulation of the Dirichlet problem (4.20). Define $\mathcal{B}_q = \mathcal{B}_{q,\gamma}$ as

$$\mathcal{B}_q(v, w) := \langle \mathcal{L}^s v, w\rangle + \int_\Omega q(x)v(x)w(x)\, dx, \quad v, w \in H^s(\mathbb{R}^n) \tag{4.21}$$

with $\mathcal{L}^s$ given in (4.14). Since $\mathcal{L}^s$ is linear and symmetric, one has that B_q is a symmetric bilinear form. Moreover, one can obtain from (4.15) that

$$|\mathcal{B}_q(v, w)| \le C\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}, \quad v, w \in H^s(\mathbb{R}^n). \tag{4.22}$$

Recall that the Sobolev space $\widetilde{H}^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{H^s(\mathbb{R}^n)}$. Therefore, $\mathcal{B}_q$ is also a symmetric bounded bilinear form in the space $\widetilde{H}^s(\Omega)$.

We are now in the position of defining the notion of (weak) solutions to the nonlocal Dirichlet problem (4.20). Hereinafter, the notation $\mathcal{L}^s g|_\Omega$ means $(\mathcal{L}^s g)|_\Omega$.

Definition 4.1. Given $f \in \left(\widetilde{H}^s(\Omega)\right) = H^{-s}(\Omega)$, we say that $u \in \widetilde{H}^s(\Omega)$ is a (weak) solution to

$$\begin{cases} (\mathcal{L}^s + q)u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega_e, \end{cases} \tag{4.23}$$

if there holds

$$\mathcal{B}_q(u, v) = (f, v)_{L^2(\Omega)}, \quad \text{for any } v \in \widetilde{H}^s(\Omega). \tag{4.24}$$

Given further $g \in H^s(\mathbb{R}^n)$, we say that $u \in H^s(\mathbb{R}^n)$ is a (weak) solution to (4.20) if $\widetilde{u}_g := u - g$ belongs to the Sobolev space $\widetilde{H}^s(\Omega)$ and is a (weak) solution to the following homogeneous Dirichlet problem

$$\begin{cases} (\mathcal{L}^s + q)\widetilde{u}_g = f - qg - \mathcal{L}^s g|_\Omega & \text{in } \Omega, \\ u = 0 & \text{in } \Omega_e. \end{cases} \tag{4.25}$$

By rewriting (4.25) into the weak form as (4.24), one has

$$\mathcal{B}_q(\widetilde{u}_g, v) = \big(f - (\mathcal{L}^s + q)g, v\big)_{L^2(\Omega)}, \quad \text{for any } v \in \widetilde{H}^s(\Omega).$$

Therefore, we have the following equivalent definition of solutions to the non-homogeneous Dirichlet problem (4.20).

Definition 4.2'. Given $f \in H^{-s}(\Omega)$ and $g \in H^s(\mathbb{R}^n)$, we say that $u \in H^s(\mathbb{R}^n)$ is a (weak) solution to (4.20) if $u - g \in \widetilde{H}^s(\Omega)$ and

$$\mathcal{B}_q(u, v) = (f, v)_{L^2(\Omega)}, \quad \text{for any } v \in \widetilde{H}^s(\Omega). \tag{4.26}$$

The following result reveals the well-posedness of Dirichlet problems associated with the nonlocal operator $\mathcal{L}^s + q$.

Theorem 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ and let $q \in L^\infty(\Omega)$. Denote $q_-(x) := \min\{0, q(x)\}$ and $\lambda_0 := -\|q_-\|_{L^\infty(\Omega)} \geq 0$. Then there is a countable set $\Sigma = \{\lambda_j\}_{j=1}^\infty \subset (\lambda_0, +\infty)$ such that, for any given $\lambda \in \mathbb{R} \setminus \Sigma$, $f \in H^{-s}(\Omega)$ and $g \in H^s(\mathbb{R}^n)$, there is a unique solution $u \in H^s(\mathbb{R}^n)$ to the nonlocal Dirichlet problem*

$$\begin{cases} (\mathcal{L}^s + q - \lambda)u = f & \text{in } \Omega, \\ u = g & \text{in } \Omega_e. \end{cases} \tag{4.27}$$

Moreover, the solution u has the following estimate

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C \left(\|f\|_{H^{-s}(\Omega)} + \|g\|_{H^s(\mathbb{R}^n)} \right), \tag{4.28}$$

with some positive constant C independent of f and g.

Remark 4.1. One can seen from the proof of Theorem 4.1 that the set Σ is in fact the set of eigenvalues of the homogeneous Dirichlet problem. For study of eigenvalues for the fractional Laplacian $(-\Delta)^s$ in the unit ball, please see [43].

Proof. We first show the unique solvability of the following problem

$$\begin{cases} (\mathcal{L}^s + q - \lambda_0)u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega_e. \end{cases} \tag{4.29}$$

The associated bilinear form $\mathcal{B}_q(\cdot, \cdot) - \lambda_0(\cdot, \cdot)_{L^2(\Omega)}$ is bounded over $H^s(\mathbb{R}^n)$ as well as $\widetilde{H}^s(\Omega)$, since $\mathcal{B}_q$ is bounded over both of these spaces. One can obtain from (4.13) that, for any $v \in H^s(\mathbb{R}^n)$,

$$\begin{aligned} \langle \mathcal{L}^s v, v \rangle =& \frac{1}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} (v(x) - v(z))^2 \mathcal{K}_s(x, z) dx dz \\ \geq& C \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(v(x) - v(z))^2}{|x - z|^{n+2s}} dx dz \\ =& C \|(-\Delta)^{s/2} v\|_{L^2(\mathbb{R}^n)}^2 = C[v]_{H^s(\mathbb{R}^n)}^2 \end{aligned}$$

holds with some constant $C > 0$ independent of v. From the Hardy-Littlewood-Sobolev inequality we observes for any $v \in \widetilde{H}^s(\Omega)$ that

$$\|(-\Delta)^{s/2}v\|_{L^2(\mathbb{R}^n)} \geq c\|v\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)} = c\|v\|_{L^{\frac{2n}{n-2s}}(\Omega)}.$$

Furthermore, we have from the Hölder inequality that

$$\begin{aligned}\|v\|^2_{L^2(\Omega)} = \|v^2\|_{L^1(\Omega)} &\leq \|1\|_{L^{\frac{n}{2s}}(\Omega)}\|v^2\|_{L^{\frac{n}{n-2s}}(\Omega)} \\ &= |\Omega|^{2s/n}\|v\|^2_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}.\end{aligned}$$

By combining the above three estimates we obtain that

$$\langle \mathcal{L}^s v, v\rangle \geq \frac{Cc}{|\Omega|^{2s/n}+c}\|v\|^2_{H^s(\mathbb{R}^n)}, \quad \text{for any } v \in \widetilde{H}^s(\Omega).$$

As a consequence, one has for any $v \in \widetilde{H}^s(\Omega)$ that

$$\mathcal{B}_q(v,v) - \lambda_0(v,v)_{L^2} \geq \langle \mathcal{L}^s v, v\rangle \geq c_0\|v\|^2_{\widetilde{H}^s(\Omega)}.$$

Hence, we have deduce by far that $\mathcal{B}_q(\cdot,\cdot) - \lambda_0(\cdot,\cdot)_{L^2}$ is a bounded coercive bilinear form over the Hilbert space $\widetilde{H}^s(\Omega)$. Therefore, by the Lax-Milgram theorem we have that, given any $f \in H^{-s}(\Omega) = \left(\widetilde{H}^s(\Omega)\right)^*$, there is a unique solution $u \in \widetilde{H}^s(\Omega)$ to (4.29). Moreover, there holds the following estimate

$$\|u\|_{\widetilde{H}^s(\Omega)} \leq C\|f\|_{H^{-s}(\Omega)}$$

with some constant C independent of f.

We denote $\mathcal{G}_0$ as the solution operator of the Dirichlet problem (4.29), which maps f to the solution u. Recall from previous arguments that the map

$$\mathcal{G}_0 : H^{-s}(\Omega) \to \widetilde{H}^s(\Omega)$$

is bounded and has a bounded inverse. Moreover, one can obtain that

$$\langle f_1, \mathcal{G}_0 f_2\rangle = \langle f_2, \mathcal{G}_0 f_1\rangle, \quad \text{for any } f_1, f_2 \in H^{-s}(\Omega).$$

It is noticed from Sobolev compact embedding theorems that the following natural embedding operators Id_1 and Id_2 are compact

$$\mathrm{Id}_1 : L^2(\Omega) \to H^{-s}(\Omega), \quad \mathrm{Id}_2 : \widetilde{H}^s(\Omega) \to L^2(\Omega).$$

Therefore, we have concluded that the operator

$$\widetilde{\mathcal{G}}_0 := \mathrm{Id}_2 \circ \mathcal{G}_0 \circ \mathrm{Id}_1 : L^2(\Omega) \to L^2(\Omega)$$

is linear, compact, positive-definite and self-adjoint.

Now we consider the well-posedness of the problem

$$\begin{cases} (\mathcal{L}^s + q - \lambda)u = f & \text{in } \Omega, \\ u = 0 & \text{in } \Omega_e. \end{cases} \tag{4.30}$$

Notice that if $\lambda \leq \lambda_0$, one can deduce from the previous arguments that the Dirichlet problem (4.30) is well-posed. We are left to investigate the case when $\lambda > \lambda_0$. It is observed that, a solution u to (4.30) satisfies the following identity

$$u = \mathcal{G}_0 \left(f + (\lambda - \lambda_0)u \right),$$

which implies

$$\left(\mathrm{Id}_0 - (\lambda - \lambda_0)\widetilde{\mathcal{G}}_0 \right) \mathrm{Id}_2 u = \mathrm{Id}_2 \circ \mathcal{G}_0 f,$$

where Id_0 denotes the identity map on $L^2(\Omega)$. Recall that $\widetilde{\mathcal{G}}_0$ is self-adjoint and compact on $L^2(\Omega)$. Thus by spectral properties of self-adjoint compact operators, the spectrum of $\widetilde{\mathcal{G}}_0$ contains only countable number of eigenvalues $\{\frac{1}{\lambda_j - \lambda_0}\}_{j=1}^{\infty}$ with $\lambda_j > \lambda_0$ for all $j = 1, 2, \ldots$. Denote $\Sigma := \{\lambda_j\}_{j=1}^{\infty}$. Then by the Fredholm alternative, one has for any $\lambda \notin \Sigma$, the operator

$$\left(\mathrm{Id}_0 - (\lambda - \lambda_0)\widetilde{\mathcal{G}}_0 \right) : L^2(\Omega) \to L^2(\Omega),$$

is injective and has a bounded inverse. Hence the operator

$$(\mathrm{Id} - (\lambda - \lambda_0)\mathcal{G}_0 \circ \mathrm{Id}_1 \circ \mathrm{Id}_2) : \widetilde{H}^s(\Omega) \to \widetilde{H}^s(\Omega)$$

is also injective and has a bounded inverse. Therefore, for any $\lambda \notin \Sigma$ and any $f \in H^{-s}(\Omega)$, the problem (4.30) is uniquely solvable, and the unique solution u satisfies

$$\|u\|_{\widetilde{H}^s(\Omega)} \leq C\|f\|_{H^{-s}(\Omega)},$$

with some positive constant C independent of u and f.

Finally, the non-homogeneous Dirichlet problem (4.27) can be reformulated into the problem (4.30), with u and f replaced by, respectively, $u - g$ and $f - \mathcal{L}^s g|_\Omega + (q - \lambda)g$. We shall omit the details and complete the proof. $\square$

As a direct consequence of Theorem 4.1, we have the following result regarding the Dirichlet problem (4.20) for $\mathcal{L}^s + q$.

Corollary 4.1. *Suppose that* 0 *is not a Dirichlet eigenvalue of* $\mathcal{L}^s + q$ *in* Ω. *Then for any given* $f \in H^{-s}(\Omega)$ *and* $g \in \widetilde{H}^s(\Omega)$, *the nonlocal Dirichlet problem* (4.20) *admits a unique solution* $u \in H^s(\mathbb{R}^n)$, *which satisfies the estimate* (4.28).

The following result can be obtained from the proof of Theorem 4.1, by taking $\lambda_0 = 0$.

Corollary 4.2. *If* $q \geq 0$ *a.e. in* Ω, *then* 0 *is not a Dirichlet eigenvalue of* $\mathcal{L}^s + q$ *in* Ω.

Notice that in the Dirichlet problem (4.27), the full data of g is given in the space $H^s(\mathbb{R}^n)$. However, only the data of g in the exterior Ω_e is used. In fact, we will end up this subsection with the following results revealing that, the solution u of (4.33) does not depend on the data of g in Ω.

Lemma 4.3. *Suppose that the problem* (4.27) *is well-posed. For* $j = 1, 2$, *given the RHS function* $f \in H^{-s}(\Omega)$ *and the Dirichlet data* $g_j \in H^s(\mathbb{R}^n)$, *let* u_j *be the corresponding solution to* (4.20). *If there holds* $g_1 - g_2 \in H^s_{\overline{\Omega}} = \widetilde{H}^s(\Omega)$, *then one must have* $u_1 = u_2$.

Proof. Define $\widetilde{u} := u_1 - u_2$. It is observed that $\widetilde{u} \in \widetilde{H}^s(\Omega)$. Moreover, $\widetilde{u}$ solves (4.27) with zero RHS and exterior data. By the well-posedness of (4.27), we conclude that $\widetilde{u} = 0$ and hence complete the proof. □

Remark 4.2. As a consequence of Lemma 4.3, we can consider the nonlocal problem (4.27) with the Dirichlet data g given in the quotient space

$$X := H^s(\mathbb{R}^n)/H^s_{\overline{\Omega}} \cong \left(H^s_{\overline{\Omega}}\right)^{\perp} \cong H^s(\Omega_e). \tag{4.31}$$

Corollary 4.3. *Let* Ω, q *and* Σ *be the same as in Theorem 4.1, and let* $\lambda \in \mathbb{R}^n \backslash \Sigma$. *Then given any* $f \in H^{-s}(\Omega)$ *and* $g \in H^s(\Omega_e)$, *there is a unique solution* $u \in H^s(\mathbb{R}^n)$ *to the nonlocal Dirichlet problem* (4.27). *Moreover, there exists a constant* C, *independent of* g *and* f, *such that*

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C \left(\|f\|_{H^{-s}(\Omega)} + \|g\|_{H^s(\Omega_e)}\right). \tag{4.32}$$

Proof. Let $\tilde{g} \in H^s(\mathbb{R}^n)$ such that $\tilde{g}|_{\Omega_e} = g$. Then by Theorem 4.1, there is a unique $u_{\tilde{g}} \in H^s(\mathbb{R}^n)$ solving (4.27) with data f and $\tilde{g}$. We are claim that $u_{\tilde{g}_1} = u_{\tilde{g}_2}$ for any $\tilde{g}_1, \tilde{g}_2 \in H^s(\mathbb{R}^n)$ such that $\tilde{g}_1|_{\Omega_e} = \tilde{g}_2|_{\Omega_e} = g$. In fact, it is noticed that the new function $\tilde{g}_0 := \tilde{g}_1 - \tilde{g}_2$ belongs to $H^s(\mathbb{R}^n)$ and is supported in $\overline{\Omega}$, and hence is an element in $H^s_{\overline{\Omega}}$. Then following Lemma 4.3 we have $u_{\tilde{g}_1} = u_{\tilde{g}_2}$.

We are left to verify the stability estimate (4.32). First by (4.28) we have

$$\|u\|_{H^s(\mathbb{R}^n)} \leq C_0 \left(\|f\|_{H^{-s}(\Omega)} + \|\tilde{g}\|_{H^s(\mathbb{R}^n)} \right),$$

for any $\tilde{g} \in H^s(\mathbb{R}^n)$ such that $\tilde{g}|_{\Omega_e} = g$. Then by recalling the norm (4.7) of the Sobolev space $H^s(\Omega_e)$ we complete the proof. □

4.3.2 The DtN Map

Next we consider the Dirichlet problem (4.20) with a zero right hand side, namely,

$$\begin{cases} (\mathcal{L}^s + q)u = 0 & \text{in } \Omega, \\ u = g & \text{in } \Omega_e. \end{cases} \tag{4.33}$$

By looking into the bilinear form $\mathcal{B}_q$, we will give in this subsection a proper definition of the Neumann data of the solution to (4.33), as well as its Dirichlet to Neumann (DtN) map. It has been obtained from Theorem 4.1 or Corollary 4.1 that, as long as 0 is not a Dirichlet eigenvalue of $\mathcal{L}^s + q$ in Ω, the problem (4.33) is well-posed with any given Dirichlet data $g \in H^s(\mathbb{R}^n)$, or $g \in H^s(\Omega_e)$ in view of Corollary 4.3.

In order to guarantee the well-posedness of the Dirichlet problem (4.33), we shall assume henceforth that

Assumption B. Assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, q is an L^∞ function, and 0 is not a Dirichlet eigenvalue of $\mathcal{L}^s + q$ in Ω.

Definition 4.2. We define $\Lambda_q = \Lambda^s_{\gamma,q}$ as the following:

$$\langle \Lambda_q g, h \rangle := \mathcal{B}_q(u_g, \widetilde{h}), \quad \text{for any } g, h \in H^s(\Omega_e), \tag{4.34}$$

where $u_g \in H^s(\mathbb{R}^n)$ is the solution to (4.33) with the Dirichlet data g, and $\widetilde{h} \in H^s(\mathbb{R}^n)$ is a function satisfying $\widetilde{h}|_{\Omega_e} = h$.

We shall show in the next result that the RHS of (4.34) is independent of the specific $\widetilde{h}$ one chooses, and hence the left hand side (LHS) is well-defined. Moreover, it gives definition of a bounded linear operator Λ_q.

Lemma 4.4. *The equation* (4.34) *defines a bounded linear operator*

$$\Lambda_q : H^s(\Omega_e) \to (H^s(\Omega_e))^* = \widetilde{H}^{-s}(\Omega_e), \tag{4.35}$$

which is symmetric in the sense that

$$\langle \Lambda_q g, h \rangle = \langle \Lambda_q h, g \rangle , \quad \textit{for any } g, h \in H^s(\Omega_e). \tag{4.36}$$

Proof. We first show that the RHS of (4.34) is independent of the specific $\widetilde{h}$. In other words, we prove

$$\mathcal{B}_q(u_g, \widetilde{h}_1) = \mathcal{B}_q(u_g, \widetilde{h}_2),$$

for any $\widetilde{h}_1, \widetilde{h}_2 \in H^s(\mathbb{R}^n)$ such that $h_1|_{\Omega_e} = h_2|_{\Omega_e} = h$. Notice that the new function $\widetilde{h}_0 := \widetilde{h}_1 - \widetilde{h}_2 \in H^s(\mathbb{R}^n)$ is supported in $\overline{\Omega}$, and hence is an element in $H^s_{\overline{\Omega}} = \widetilde{H}^s(\Omega)$. Since u_g is the solution to (4.33), we have from (4.26) that

$$\mathcal{B}_q(u_g, \widetilde{h}_0) = 0.$$

The linearity of Λ_q follows directly from that of $\mathcal{B}_q$. We are left to show the mapping property (4.35). From the boundedness (4.22) of $\mathcal{B}_q$ we have

$$|\langle \Lambda_q g, h \rangle| \le C_0 \|u_g\|_{H^s(\mathbb{R}^n)} \|\widetilde{h}\|_{H^s(\mathbb{R}^n)},$$

for any $\widetilde{h} \in H^s(\mathbb{R}^n)$ such that $\widetilde{h}|_{\Omega_e} = h$. By recalling the norm (4.7) of the Sobolev space $H^s(\Omega_e)$ as well as the stability estimate (4.32) for the solution u_g, we arrive at

$$|\langle \Lambda_q g, h \rangle| \le C \|g\|_{H^s(\Omega_e)} \|h\|_{H^s(\Omega_e)},$$

with a constant C independent of g and h. Hence we conclude that $\Lambda_q : g \in H^s(\Omega_e) \mapsto \Lambda_q g \in (H^s(\Omega_e))^*$ is a bounded operator. □

Remark 4.3. We obtain from Lemma 4.4 and its proof that the notation $\langle \cdot, \cdot \rangle$ in (4.34) is in fact a pairing between $H^s(\Omega_e)$ and its dual space. Moreover, one can derive that

$$\Lambda_q g = \mathcal{L}^s u_g \big|_{\Omega_e}. \tag{4.37}$$

We end this section by deriving couple of results regarding the integral identity in our case.

Lemma 4.5. *Let Ω and q_1, q_2 satisfy Assumption B. Then the following identity holds true for any $g_1, g_2 \in H^s(\Omega_e)$:*

$$\big\langle (\Lambda_{q_1} - \Lambda_{q_2}) g_1, g_2 \big\rangle = \int_\Omega (q_1 - q_2) u_{g_1,q_1} u_{g_2,q_2} dx, \tag{4.38}$$

where for each $j = 1, 2$, $u_{g_j,q_j} \in H^s(\mathbb{R}^n)$ is the solution to (4.33) with the potential q_j and the Dirichlet data g_j.

Proof. By recalling the symmetry (4.36) and the definition (4.34) of DtN maps we have

$$\begin{aligned}\big\langle (\Lambda_{q_1} - \Lambda_{q_2}) g_1, g_2 \big\rangle =& \langle \Lambda_{q_1} g_1, g_2 \rangle - \langle \Lambda_{q_2} g_2, g_1 \rangle \\ =& \mathcal{B}_{q_1}(u_{g_1,q_1}, u_{g_2,q_2}) - \mathcal{B}_{q_2}(u_{g_2,q_2}, u_{g_1,q_1}).\end{aligned}$$

The proof is completed by using the definition (4.21) of the bilinear forms $\mathcal{B}_{q_j}$, $j = 1, 2$. □

4.4 An Important Property of $\mathcal{L}^s$

We introduce in this section the following important property for $\mathcal{L}^s$.

Theorem 4.2. *Let $\mathcal{O}$ be an open set in $\mathbb{R}^n$. Given any function $u \in H^s(\mathbb{R}^n)$ such that $u = \mathcal{L}^s u = 0$ in $\mathcal{O}$, one must have $u \equiv 0$ in $\mathbb{R}^n$.*

We remark here that the property stated in Theorem 4.2 does not hold for local elliptic operators. One can take any compactly supported function as a counter example.

Theorem 4.2 can be verified via a degenerate extension problem in $\mathbb{R}^{n+1}$ which was first introduced in [15] for $(-\Delta)^s$ and extend to the case for the more general fractional operator $\mathcal{L}^s$ in [126]. The proof of Theorem 4.2 will be given at the end of this section.

4.4.1 The Extension Problem

We shall introduce this section some results for the extension problem associated with $\mathcal{L}^s_\gamma$.

Notations in $\mathbb{R}^{n+1}$

A point in $\mathbb{R}^{n+1}$ is denoted as $\mathbf{x} = (x, \mathsf{y})$, with $x \in \mathbb{R}^n$ and $\mathsf{y} \in \mathbb{R}$. Denote $\mathbb{R}^{n+1}_+$ as the (open) upper half space of $\mathbb{R}^{n+1}$, namely, $\mathbb{R}^{n+1}_+ := \{\mathbf{x};\ x \in \mathbb{R}^n, \mathsf{y} > 0\}$, and denote $\partial\mathbb{R}^{n+1}_+ := \{(x, 0);\ x \in \mathbb{R}^n\}$ as its boundary. For any open domain $\mathfrak{O}$ in $\mathbb{R}^{n+1}$, denote $\mathfrak{O}_+ := \mathfrak{O} \cap \mathbb{R}^{n+1}_+$, $\mathfrak{O}_- := \mathfrak{O} \cap \mathbb{R}^{n+1}_-$ and $\mathfrak{O}^0 := \mathfrak{O} \cap \partial\mathbb{R}^{n+1}_+$. Given $\mathbf{x} \in \mathbb{R}^{n+1}$ and $R \in \mathbb{R}_+$, denote $\mathfrak{B}(\mathbf{x}; R)$ as the open ball in $\mathbb{R}^{N+1}$ centered at $\mathbf{x}$ with radius R, and denote $\mathfrak{B}(x; R) = \mathfrak{B}((x, 0); R)$ for $x \in \mathbb{R}^n$.

Given $\mathfrak{O}$ be a Lipschitz domain in $\mathbb{R}^{n+1}$ and w an A_2 Muckenhoupt weight function (cf. [50, 102]), denote $L^2(\mathfrak{O}, w)$ as the weighted Sobolev space containing all functions U which are defined a.e. in $\mathfrak{O}$ such that

$$\|U\|_{L^2(\mathfrak{O},w)} := \left(\int_{\mathfrak{O}} w|U|^2 d\mathbf{x}\right)^{1/2} < \infty.$$

Define

$$H^1(\mathfrak{O}, w) := \{U \in L^2(\mathfrak{O}, w);\ \nabla_{\mathbf{x}} U \in L^2(\mathfrak{O}, w)\},$$

where $\nabla_{\mathbf{x}} := (\nabla, \partial_{\mathsf{y}}) = (\nabla_x, \partial_{\mathsf{y}})$ is the gradient in $\mathbb{R}^{n+1}$. It is observed that $L^2(\mathfrak{O}, w)$ and $H^1(\mathfrak{O}, w)$ are Banach spaces with respect to the norms, respectively, $\|\cdot\|_{L^2(\mathfrak{O},w)}$ and

$$\|U\|_{H^1(\mathfrak{O},w)} := \left(\|U\|^2_{L^2(\mathfrak{O},w)} + \|\nabla_{\mathbf{x}} U\|^2_{L^2(\mathfrak{O},w)}\right)^{1/2}.$$

Denote $H^1_0(\mathfrak{O}, w)$ as the closure of $C_0^\infty(\mathfrak{O})$ in $H^1(\mathfrak{O}, w)$.

Based on the n-by-n matrix $\gamma = (\gamma_{i,j})$, we set

$$\tilde{\gamma}(x) = \begin{pmatrix} \gamma(x) & 0 \\ 0 & 1 \end{pmatrix}.$$

Given $\sigma \in (-1, 1)$, we define the following degenerate operator in $\mathbb{R}^{n+1}$,

$$\mathscr{L}_{\sigma,\tilde{\gamma}} := -\nabla_{\mathbf{x}} \cdot (|\mathsf{y}|^\sigma \tilde{\gamma}(x) \nabla_{\mathbf{x}}). \tag{4.39}$$

It is known for $\sigma \in (-1, 1)$ that $|\mathsf{y}|^\sigma$ is a A_2 Muckenhoupt weight (cf. [50, 102]) in $\mathbb{R}^{n+1}_+$. One can observe that

$$\mathscr{L}_{\sigma,\tilde{\gamma}} = y^\sigma \left(\mathcal{L}_\gamma - \sigma \mathsf{y}^{-1}\partial_{\mathsf{y}} - \partial^2_{\mathsf{y}}\right) \quad \text{in } \mathbb{R}^{n+1}_+.$$

Let $\mathfrak{O}$ be an open set in $\mathbb{R}^{n+1}$. We say that $U \in H^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$ (weakly) satisfies $\mathscr{L}_{\sigma,\tilde{\gamma}} U = 0$ in $\mathfrak{O}$ if

$$0 = \int_{\mathfrak{O}} |\mathsf{y}|^\sigma \left(\tilde{\gamma}\ \nabla_{\mathbf{x}} U\right) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x}, \quad \text{for all } \Phi \in C_0^\infty(\mathfrak{O}).$$

Remark 4.4. It is noticed that, away from $\partial\mathbb{R}^{n+1}_+$, the coefficients of the differential operator is C^∞ smooth. Hence by standard elliptic regularity results, we obtain that U is C^∞ in any open and relatively compact subset of $\mathfrak{O}_+ = \mathfrak{O} \cap \mathbb{R}^{n+1}_+$ or $\mathfrak{O}_- = \mathfrak{O} \cap \mathbb{R}^{n+1}_-$.

Given $u = u(x)$, we shall consider the following extension problem in $\mathbb{R}^{n+1}_+$ introduced by [126],

$$\begin{cases} \mathscr{L}_{\sigma,\tilde{\gamma}} U = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ U(\cdot, 0) = u & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{4.40}$$

The following lemma reveals the interesting fact that the Neumann data on $\partial\mathbb{R}^{n+1}_+$ of the problem (4.40) associated with the Dirichlet data u is exactly $\mathcal{L}^s_\gamma u$.

Lemma 4.6 ([126, 53])**.** *Given a function u defined in $\mathbb{R}^n$, let*

$$U(\mathbf{x}) := \int_{\mathbb{R}^n} P^s_{\mathsf{y}}(x,z) u(z) dz,$$

with the kernel P^s_{y} given by

$$P^s_{\mathsf{y}}(x,z) := \frac{y^{2s}}{4^s \Gamma(s)} \int_0^\infty e^{-\frac{y^2}{4t}} p_t(x,z) \frac{dt}{t^{1+s}}, \quad x, z \in \mathbb{R}^n,\ \mathsf{y} > 0.$$

Then U is a solution to (4.40) *and*

$$\lim_{y\to 0^+} \frac{U(\cdot,y) - U(\cdot,0)}{y^{2s}} = \frac{1}{2s} \lim_{y\to 0+} y^{1-2s} \partial_{\mathsf{y}} U(\cdot, y) = \frac{\Gamma(-s)}{4^s\Gamma(s)} \mathcal{L}^s u.$$

Proper Neumann Data on $\partial\mathbb{R}^{n+1}_+$

Let $\mathfrak{O}$ be an open set in $\mathbb{R}^{n+1}$ such that $\mathfrak{O}^0 = \mathfrak{O} \cap \partial\mathbb{R}^{n+1}_+ \neq \emptyset$. Let $U \in H^1(\mathfrak{O}_+, |\mathsf{y}|^\sigma)$ satisfy $\mathscr{L}_{\sigma,\tilde{\gamma}} U = 0$ in $\mathfrak{O}_+$. Then for any $\Phi \in C^\infty_0(\mathfrak{O})$ and any $\epsilon > 0$ we have that

$$\begin{aligned} &\int_{\mathfrak{O}_+} |\mathsf{y}|^\sigma \left(\widetilde{\gamma}\, \nabla_{\mathbf{x}} U\right) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x} \\ &= \int_{\mathfrak{O}_+ \cap \{\mathsf{y} \geq \epsilon\}} \mathsf{y}^\sigma \left(\widetilde{\gamma}\, \nabla_{\mathbf{x}} U\right) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x} + \int_{\mathfrak{O}_+ \cap \{\mathsf{y} < \epsilon\}} \mathsf{y}^\sigma \left(\widetilde{\gamma}\, \nabla_{\mathbf{x}} U\right) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x} \\ &= -\int_{\mathfrak{O}_+ \cap \{\mathsf{y} = \epsilon\}} \mathsf{y}^\sigma \partial_{\mathsf{y}} U \, \Phi \, dx + \int_{\mathfrak{O}_+ \cap \{\mathsf{y} < \epsilon\}} \mathsf{y}^\sigma \left(\widetilde{\gamma}\, \nabla_{\mathbf{x}} U\right) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x}. \end{aligned}$$

It is noticed that the second term of the RHS tends to zero when $\epsilon \to 0^+$, as long as $\Phi \in H^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$. Thus one has

$$\lim_{\epsilon\to 0^+} \int_{\mathfrak{O}_+ \cap \{\mathsf{y} = \epsilon\}} \mathsf{y}^\sigma \partial_{\mathsf{y}} U \, \Phi \, dx = -\int_{\mathfrak{O}_+} |\mathsf{y}|^\sigma \left(\widetilde{\gamma}\, \nabla_{\mathbf{x}} U\right) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x}, \tag{4.41}$$

for any $\Phi \in C_0^\infty(\mathfrak{O})$, or by density, any $\Phi \in H_0^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$.

Remark 4.5. One has for any $\Phi \in C_0^\infty(\mathfrak{O})$ that

$$\lim_{\epsilon\to 0^+} \int_{\mathfrak{O}_+\cap\{\mathsf{y}=\epsilon\}} \mathsf{y}^\sigma \partial_{\mathsf{y}} U \ (\Phi(x,\mathsf{y}) - \Phi(x,0)) \ dx$$
$$= \lim_{\epsilon\to 0^+} \epsilon \int_{\mathfrak{O}_+\cap\{\mathsf{y}=\epsilon\}} \mathsf{y}^\sigma \partial_{\mathsf{y}} U \ \frac{\Phi(x,\mathsf{y}) - \Phi(x,0)}{\mathsf{y}} \ dx = 0.$$

Therefore the LHS of (4.41) can be also written as

$$\lim_{\epsilon\to 0^+} \int_{\mathfrak{O}_+\cap\{\mathsf{y}=\epsilon\}} \mathsf{y}^\sigma \partial_{\mathsf{y}} U \, \Phi(x,0) \, dx. \tag{4.42}$$

We assume from now on that $\mathfrak{O}_+$ is Lipschitz. Recall that $\sigma \in (-1,1)$. It is known (cf. [128]) that there is a unique trace operator Tr continuously map from $H^1(\mathfrak{O}_+, |\mathsf{y}|^\sigma)$ to $H^s(\mathfrak{O}^0)$, with

$$s := (1-\sigma)/2 \in (0,1).$$

Moreover, a bounded linear right inverse $\mathrm{Tr}^{-1} : H^s(\mathfrak{O}^0) \to H^1(\mathfrak{O}_+, |\mathsf{y}|^\sigma)$ exists. By using a cutoff function on $\mathfrak{O}$, one can actually restrict the image of Tr^{-1} (with a little abuse of notations) as $H^1(\mathfrak{O}_+, |\mathsf{y}|^\sigma) \cap H_0^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$. Hence given any $\phi \in H^s(\mathfrak{O}^0)$, there exists $\Phi \in H_0^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$ such that $\Phi(x,0) = \phi(x)$ in the sense of trace and that,

$$c\|\phi\|_{H^s(\mathfrak{O}^0)} \le \|\Phi\|_{H_0^1(\mathfrak{O},|\mathsf{y}|^\sigma)} \le C\|\phi\|_{H^s(\mathfrak{O}^0)} \tag{4.43}$$

with positive constants c and C independent of ϕ.

For any $U \in H^1(\mathfrak{O}_+, \mathsf{y}^\sigma)$ satisfying $\mathscr{L}_{\sigma,\tilde{\gamma}} U = 0$ in $\mathfrak{O}_+$, we define the operator $\mathscr{N}_\sigma U$ mapping on $H^s(\mathfrak{O}^0)$ by

$$(\mathscr{N}_\sigma U)(\phi) := -\lim_{\epsilon\to 0^+} \int_{\mathfrak{O}_+\cap\{\mathsf{y}=\epsilon\}} \mathsf{y}^\sigma \partial_{\mathsf{y}} U \, \phi \, dx, \quad \phi \in H^s(\mathfrak{O}^0), \tag{4.44}$$

or formally,

$$\mathscr{N}_\sigma U = -\lim_{\mathsf{y}\to 0^+} \mathsf{y}^\sigma \partial_{\mathsf{y}} U.$$

In regard to (4.41) and (4.42) one has

$$(\mathscr{N}_\sigma U)(\phi) = \int_{\mathfrak{O}_+} |\mathsf{y}|^\sigma \, (\widetilde{\gamma} \, \nabla_{\mathbf{x}} U) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x}, \tag{4.45}$$

with $\Phi \in H^1(\mathfrak{O}_+, \mathsf{y}^\sigma)$ satisfying $\mathrm{Tr}\Phi = \phi$. Recall that one can always chose Φ such that (4.43) holds true. Then we obtain

$$
\begin{aligned}
|(\mathscr{N}_\sigma U)(\phi)| &\leq C\|\Phi\|_{H_0^1(\mathfrak{O},|\mathsf{y}|^\sigma)}\|U\|_{H^1(\mathfrak{O}_+;|\mathsf{y}|^\sigma)} \\
&\leq C\|\phi\|_{H^s(\mathfrak{O}^0)}\|U\|_{H^1(\mathfrak{O}_+;|\mathsf{y}|^\sigma)}, \quad \phi \in H^s(\mathfrak{O}^0).
\end{aligned}
$$

Therefore we have constructed a linear operator $\mathscr{N}_\sigma$ which maps continuously from $\{U \in H^1(\mathfrak{O}_+, \mathsf{y}^\sigma); \mathscr{L}_{\sigma,\tilde{\gamma}}U = 0 \text{ in } \mathfrak{O}_+\}$ into $(H^s(\mathfrak{O}^0))^* = \widetilde{H}^{-s}(\mathfrak{O}^0)$, by assuming that $\mathfrak{O}^0$ is Lipschitz in $\mathbb{R}^n$.

The Regularity

We have the following interior regularity result for a (weak) solution $U \in H^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$ to $\mathscr{L}_{\sigma,\tilde{\gamma}}U = 0$.

Lemma 4.7. *Let $U \in H^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$ be a (weak) solution of $\mathscr{L}_{\sigma,\tilde{\gamma}}U = 0$. Then $\nabla_x U \in H^1_{loc}(\mathfrak{O}, |\mathsf{y}|^\sigma)$, with*

$$
\|\nabla_x U\|_{H^1(\mathfrak{O}',|\mathsf{y}|^\sigma)} \leq C(1 + c')\|U\|_{H^1(\mathfrak{O},|\mathsf{y}|^\sigma)}
$$

for any $\mathfrak{O}' \subset \mathfrak{O}$, where $c' := 1/\operatorname{dist}(\mathfrak{O}', \partial\mathfrak{O})$.

Proof. Notice that the weight $|\mathsf{y}|^\sigma$ is independent of x. Then with the help of Lemmas 4.9 and 4.11, the proof can be done by the classical difference quotient arguments (cf. [56, Theorem 8.8]). □

Remark 4.6. Unlike the interior regularity results for differential operators with non-singular coefficients, we are lack of the property for higher smoothness for the $(n+1)$-th derivative $\partial_{\mathsf{y}}U$. This is due to the appearance of the singular weight $|\mathsf{y}|^\sigma$. In particular, one can still improve locally the regularity of $\partial_{\mathsf{y}}U$ alway from $\partial\mathbb{R}^{n+1}_+$, but not across it. For example $U(x, \mathsf{y}) = \mathsf{y}|\mathsf{y}|^{-\sigma}$ is a solution of $\mathscr{L}_{\sigma,\tilde{\gamma}}U = 0$, and one can check that $U \in H^1_{loc}(\mathbb{R}^{n+1}, |\mathsf{y}|^\sigma)$, with $\partial_{\mathsf{y}}U = (1-\sigma)|\mathsf{y}|^{-\sigma}$. However when $\mathfrak{O}$ intersects $\partial\mathbb{R}^{n+1}_+$, there is no weak derivative on the y-direction of $\partial_{\mathsf{y}}U$ which belongs to the space $L^2(\mathfrak{O}, |\mathsf{y}|^\sigma)$.

Lemma 4.8 ([59])**.** *$C^1(\mathfrak{O}) \cap H^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$ is dense in $H^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$.*

Lemma 4.9. *Let $U \in H^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$. For any $\mathfrak{O}' \subset \mathfrak{O}$ such that $|h| <$ $\operatorname{dist}(\mathfrak{O}', \partial\mathfrak{O})$, one has $\Delta^h_{x_i} U \in L^2(\mathfrak{O}', |\mathsf{y}|^\sigma)$ and,*

$$\|\Delta^h_{x_i} U\|_{L^2(\mathfrak{O}', |\mathsf{y}|^\sigma)} \leq \|\partial_{x_i} U\|_{L^2(\mathfrak{O}, |\mathsf{y}|^\sigma)}.$$

Proof. The first result is classical, since the weight $|\mathsf{y}|^\sigma$ does not affect difference quotient on x. The corresponding proof can be found in [56, Chapter 7]. □

Lemma 4.10. *$L^2(\mathfrak{O}, |\mathsf{y}|^\sigma)$ is a separable reflexive Banach space, that is, any bounded sequence in $L^2(\mathfrak{O}, |\mathsf{y}|^\sigma)$ contains a weakly convergent subsequence.*

Lemma 4.11. *Let $U \in L^2(\mathfrak{O}, |\mathsf{y}|^\sigma)$, and $i \in \{1, \dots, n+1\}$. Suppose there exists a constant C such that $\|\Delta^h_{\mathsf{x}_i} U\|_{L^2(\mathfrak{O}', |\mathsf{y}|^\sigma)} \leq C$ for any $h > 0$ and any $\mathfrak{O}'$ $\subset \mathfrak{O}$ such that $h < \operatorname{dist}(\mathfrak{O}', \partial\mathfrak{O})$. Then the weak derivative $\partial_{\mathsf{x}_i} U$ exists and $\|\partial_{\mathsf{x}_i} U\|_{L^2(\mathfrak{O}, |\mathsf{y}|^\sigma)} \leq C$.*

Proof. With the help of Lemma 4.10, this is actually the classical result, see for instance, [56, Chapter 7]. □

Odd and Even Reflections at $\partial\mathbb{R}^{n+1}_+$

In order to extend the problem (4.40) across $\partial\mathbb{R}^{n+1}_+$, we give some results on even and odd reflections of a solution U to (4.40).

Lemma 4.12. *Given $x_0 \in \mathbb{R}^n$ and $R > 0$, denote $\mathfrak{B} = \mathfrak{B}(x_0; R)$ for the time being. Let $U \in H^1(\mathfrak{B}_+, \mathsf{y}^\sigma)$ satisfy*

$$\begin{cases} \mathscr{L}_{\sigma,\tilde{\gamma}} U = 0 & \text{in } \mathfrak{B}_+, \\ \mathscr{N}_\sigma U = 0 & \text{on } \mathfrak{B}^0. \end{cases} \tag{4.46}$$

Extend the data of U to the whole ball $\mathfrak{B}$ by defining the even reflection

$$U_e(x, \mathsf{y}) := \begin{cases} U(x, \mathsf{y}) & \text{if } \mathsf{y} \geq 0, \\ U(x, -\mathsf{y}) & \text{if } \mathsf{y} < 0. \end{cases} \tag{4.47}$$

Then $U_e \in H^1(\mathfrak{B}, |\mathsf{y}|^\sigma)$ and satisfies $\mathscr{L}_{\sigma,\tilde{\gamma}} U_e = 0$ in $\mathfrak{B}$.

Proof. We claim that $(\nabla_x U_e)(x,\mathsf{y}) = (\nabla_x U)(x,|\mathsf{y}|)$ and

$$\partial_{\mathsf{y}} U_e(x,\mathsf{y}) := \begin{cases} \partial_{\mathsf{y}} U(x,\mathsf{y}) & \text{if } \mathsf{y} \geq 0, \\ -\left(\partial_{\mathsf{y}} U\right)(x,-\mathsf{y}) & \text{if } \mathsf{y} < 0, \end{cases} \tag{4.48}$$

in the distributional sense. If it is true, then $U \in H^1(\mathfrak{B}, |\mathsf{y}|^\sigma)$ is a direct consequence of the fact that $U_e \in H^1(\mathfrak{B}_+, \mathsf{y}^\sigma)$.

Now we prove the claim. For any $\Phi \in C_0^\infty(\mathfrak{B}) \subset H_0^1(\mathfrak{B}_+, \mathsf{y}^{-\sigma})$, one has

$$\begin{aligned} \int_{\mathfrak{B}_+} \nabla_{\mathbf{x}} U \, \Phi \, d\mathbf{x} &= \lim_{\epsilon\to 0^+} \int_{\mathfrak{B}_+ \cap \{\mathsf{y}>\epsilon\}} \nabla_{\mathbf{x}} U \, \Phi \, d\mathbf{x} \\ &= \lim_{\epsilon\to 0^+} \left(- \int_{\mathfrak{B}_+ \cap \{\mathsf{y}>\epsilon\}} U \, \nabla_{\mathbf{x}} \Phi \, d\mathbf{x} - \int_{\mathfrak{B}_+ \cap \{\mathsf{y}=\epsilon\}} e_{n+1} U \, \Phi \, dx \right) \\ &= - \int_{\mathfrak{B}_+} U_e \, \nabla_{\mathbf{x}} \Phi \, d\mathbf{x} - \lim_{\epsilon\to 0^+} \int_{\mathfrak{B}_+ \cap \{\mathsf{y}=\epsilon\}} e_{n+1} U \, \Phi \, dx. \end{aligned}$$

Furthermore,

$$\begin{aligned} \int_{\mathfrak{B}_-} \nabla_x U(x,|\mathsf{y}|) \, \Phi \, d\mathbf{x} &= \int_{\mathfrak{B}_+} \nabla_x U \, \Phi(x,-\mathsf{y}) \, d\mathbf{x} \\ &= - \int_{\mathfrak{B}_+} U \, \nabla_x \Phi_e \, d\mathbf{x} = - \int_{\mathfrak{B}_-} U_e \, \nabla_x \Phi \, d\mathbf{x}, \end{aligned}$$

and

$$\begin{aligned} \int_{\mathfrak{B}_-} -\left(\partial_{\mathsf{y}} U\right)(x,-\mathsf{y}) \, \Phi \, d\mathbf{x} &= \int_{\mathfrak{B}_+} -\left(\partial_{\mathsf{y}} U\right) \, \Phi(x,-\mathsf{y}) \, d\mathbf{x} \\ &= - \int_{\mathfrak{B}_-} U_e \, \partial_{\mathsf{y}} \Phi \, d\mathbf{x} + \lim_{\epsilon\to 0^+} \int_{\mathfrak{B}_+ \cap \{\mathsf{y}=\epsilon\}} U \, \Phi(x,-\mathsf{y}) \, dx. \end{aligned}$$

Notice that

$$\begin{aligned} &\lim_{\epsilon\to 0^+} \int_{\mathfrak{B}_+ \cap \{\mathsf{y}=\epsilon\}} U \, \left(\Phi(x,\mathsf{y}) - \Phi(x,-\mathsf{y})\right) dx \\ &= \lim_{\epsilon\to 0^+} 2\epsilon \int_{\mathfrak{B}_+ \cap \{\mathsf{y}=\epsilon\}} U \, \frac{\Phi(x,\mathsf{y}) - \Phi(x,-\mathsf{y})}{2\mathsf{y}} dx = 0. \end{aligned}$$

Thus we have proved the claim, and hence the assertion that $U_e \in H^1(\mathfrak{B}, |\mathsf{y}|^\sigma)$.

Next we show that U_e satisfies the differential equation in $\mathfrak{B}$. It is observed from the definition of U_e and the regularity of U away from $\partial\mathbb{R}^{n+1}_+$ that, for any $\epsilon > 0$,

$$\mathscr{L}_{\sigma,\tilde{\gamma}} U_e(x,\mathsf{y}) = \left(\mathscr{L}_{\sigma,\tilde{\gamma}} U\right)(x,|\mathsf{y}|) = 0, \quad \text{for all } (x,\mathsf{y}) \in \mathfrak{B} \cap \{|\mathsf{y}| > \epsilon\},$$

holds in the classical sense. As a consequence one has,

$$
\begin{aligned}
&\int_{\mathfrak{B}\cap\{|\mathsf{y}|>\epsilon\}} |\mathsf{y}|^{\sigma}\,(\widetilde{\gamma}\,\nabla_{\mathbf{x}}\,U_e)\cdot\nabla_{\mathbf{x}}\,\Phi\,d\mathbf{x}\\
&=-\int_{\mathfrak{B}\cap\{\mathsf{y}=\epsilon\}} |\mathsf{y}|^{\sigma}\partial_{\mathsf{y}}U_e\,\Phi\,dx+\int_{\mathfrak{B}\cap\{\mathsf{y}=-\epsilon\}} |\mathsf{y}|^{\sigma}\partial_{\mathsf{y}}U_e\,\Phi\,dx\\
&=-\int_{\mathfrak{B}\cap\{\mathsf{y}=\epsilon\}} |\mathsf{y}|^{\sigma}\partial_{\mathsf{y}}U\,\left(\Phi(x,\mathsf{y})+\Phi(x,-\mathsf{y})\right)\,dx,
\end{aligned}
$$

for any $\Phi\in C_0^\infty(\mathfrak{B})$. Therefore by using $U_e\in H^1(\mathfrak{B},|\mathsf{y}|^{\sigma})$ and the condition that $\mathscr{N}_{\sigma}U=0$, one obtains

$$
\int_{\mathfrak{B}} |\mathsf{y}|^{\sigma}\,(\widetilde{\gamma}\,\nabla_{\mathbf{x}}\,U_e)\cdot\nabla_{\mathbf{x}}\,\Phi\,d\mathbf{x}=\lim_{\epsilon\to0^+}\int_{\mathfrak{B}\cap\{|\mathsf{y}|>\epsilon\}} |\mathsf{y}|^{\sigma}\,(\widetilde{\gamma}\,\nabla_{\mathbf{x}}\,U_e)\cdot\nabla_{\mathbf{x}}\,\Phi\,d\mathbf{x}=0,
$$

for all $\Phi\in C_0^\infty(\mathfrak{B})$.

The proof is complete. □

Similar to (4.53), we define

$$
W_o:=|\mathsf{y}|^{\sigma}\partial_{\mathsf{y}}U_e. \tag{4.49}
$$

It is easy to check that W_o is actually the odd reflection of $W=\mathsf{y}^{\sigma}\partial_{\mathsf{y}}U$, namely,

$$
W_o(x,\mathsf{y})=\begin{cases} W(x,\mathsf{y}) & \text{if } \mathsf{y}\geq 0,\\ -W(x,-\mathsf{y}) & \text{if } \mathsf{y}<0.\end{cases} \tag{4.50}
$$

Lemma 4.13. *Let $\mathfrak{B}=\mathfrak{B}(x_0,R)$ and U be the same as in Lemma 4.12. Then $W_o\in H^1(\mathfrak{B}';|y|^{-\sigma})$ and satisfies $\mathscr{L}_{-\sigma,\widetilde{\gamma}}W_o=0$ in $\mathfrak{B}'=\mathfrak{B}'(x_0,r)$ with any $r<R$.*

Proof. This is a direct consequence of Lemma 4.12 and Theorem 4.3. □

By similar arguments for (4.56) we have that, for any $\Phi\in C_0^\infty(\mathfrak{B})$, or any $\Phi\in H_0^1(\mathfrak{B},|\mathsf{y}|^{\sigma})$ by density,

$$
\begin{aligned}
\int_{\mathfrak{B}'} |\mathsf{y}|^{-\sigma}\,(\widetilde{\gamma}\,\nabla_{\mathbf{x}}\,W)\cdot\nabla_{\mathbf{x}}\,\Phi\,d\mathbf{x}&=-\lim_{\epsilon\to0^+}\int_{\mathfrak{B}'\cap\{\mathsf{y}=\epsilon\}}\mathcal{L}_xU\,\Phi\,dx\\
&=-\int_{\mathfrak{B}'\cap\{\mathsf{y}=0\}}\mathcal{L}_xU(x,0)\,\Phi(x,0)\,dx.
\end{aligned}
$$

Recall from (4.54) that

$$
\mathcal{L}_xU=\mathsf{y}^{-\sigma}\partial_{\mathsf{y}}W. \tag{4.51}
$$

Hence in parallel to (4.44), we can define the operator $\mathscr{N}_{-\sigma}$ on W such that

$$\mathscr{N}_{-\sigma} W = \int_{\mathfrak{B}'} |\mathsf{y}|^{-\sigma} \, (\widetilde{\gamma} \, \nabla_{\mathbf{x}} W) \cdot \nabla_{\mathbf{x}} \, \Phi \, d\mathbf{x}. \tag{4.52}$$

Lemma 4.14. *Let $\mathfrak{B}, \mathfrak{B}'$ and U be the same as in Lemma 4.13. If in addition $U = 0$ on $\mathfrak{B}^0$, then $\mathscr{N}_{-\sigma} W = \mathsf{y}^{-\sigma} \partial_{\mathsf{y}} W = 0$ on $\mathfrak{B}'^0$.*

Proof. This is a direct consequence of (4.51). □

The Conjugation

Let $U \in H^1(\mathfrak{D}, |\mathsf{y}|^\sigma)$ be a (weak) solution of $\mathscr{L}_{\sigma,\tilde{\gamma}} U = 0$ in $\mathfrak{D}$. Define

$$W := |\mathsf{y}|^\sigma \partial_{\mathsf{y}} U. \tag{4.53}$$

Then for $U \in H^1(\mathfrak{D}, |\mathsf{y}|^\sigma)$, one observes that $W \in L^2(\mathfrak{D}; |\mathsf{y}|^{-\sigma})$. One also has from Theorem 4.7 that $\nabla_x W \in L^2(\mathfrak{D}'; |\mathsf{y}|^{-\sigma})$ for any $\mathfrak{D}' \subset \mathfrak{D}$. Moreover, we have the following result.

Theorem 4.3. *Let $U \in H^1(\mathfrak{D}, |\mathsf{y}|^\sigma)$ be a (weak) solution of $\mathscr{L}_{\sigma,\tilde{\gamma}} U = 0$ in $\mathfrak{D}$. Given any $\mathfrak{D}' \subset \mathfrak{D}$, denote $c' := 1/\operatorname{dist}(\mathfrak{D}', \partial\mathfrak{D})$. Then one has $W \in H^1(\mathfrak{D}'; |\mathsf{y}|^{-\sigma})$ with*

$$\|W\|_{H^1(\mathfrak{D}', |\mathsf{y}|^{-\sigma})} \leq C(1 + c') \|U\|_{H^1(\mathfrak{D}, |\mathsf{y}|^\sigma)}$$

and satisfies (weakly) $\mathscr{L}_{-\sigma,\tilde{\gamma}} W = 0$ in $\mathfrak{D}'$.

Proof. We know by Theorem 4.7 that $\nabla_x W = |\mathsf{y}|^\sigma \partial_{\mathsf{y}} \nabla_x U \in L^2(\mathfrak{D}'; |\mathsf{y}|^{-\sigma})$. Now we prove that there exists $V \in L^2(\mathfrak{D}'; |\mathsf{y}|^{-\sigma})$ such that

$$\int_{\mathfrak{D}'} V \, \Phi \, d\mathbf{x} = - \int_{\mathfrak{D}'} W \, \partial_{\mathsf{y}} \Phi \, d\mathbf{x}, \quad \text{for any } \Phi \in C_0^\infty(\mathfrak{D}').$$

Given any $\Phi \in C_0^\infty(\mathfrak{D}')$ we have that

$$\begin{aligned} 0 &= \int_{\mathfrak{D}'} |\mathsf{y}|^\sigma \, (\widetilde{\gamma} \, \nabla_{\mathbf{x}} U) \cdot \nabla_{\mathbf{x}} \, \Phi \, d\mathbf{x} \\ &= \int_{\mathfrak{D}'} |\mathsf{y}|^\sigma \, (\gamma \, \nabla_x U) \cdot \nabla_x \Phi \, d\mathbf{x} + \int_{\mathfrak{D}'} W \, \partial_{\mathsf{y}} \Phi \, d\mathbf{x}. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} \int_{\mathfrak{D}'} W \, \partial_{\mathsf{y}} \Phi \, d\mathbf{x} &= - \int_{\mathfrak{D}'} |\mathsf{y}|^\sigma \, (\gamma \, \nabla_x U) \cdot \nabla_x \Phi \, d\mathbf{x} \\ &= \int_{\mathfrak{D}'} |\mathsf{y}|^\sigma \, \nabla_x \cdot (\gamma \, \nabla_x U) \, \Phi \, d\mathbf{x}. \end{aligned}$$

Recall from Theorem 4.7 that $\partial_{x_i x_j} U \in L^2(\mathfrak{O}', |\mathsf{y}|^\sigma)$ for all $1 \le i,j \le n$. Then one can verify that $V(x,\mathsf{y}) := -|\mathsf{y}|^\sigma \nabla_x \cdot (\gamma \nabla_x U)$ belongs to $L^2(\mathfrak{O}', |\mathsf{y}|^{-\sigma})$. Therefore, we have verified that $W \in H^1(\mathfrak{O}'; |\mathsf{y}|^{-\sigma})$ with

$$\partial_{\mathsf{y}} W = -|\mathsf{y}|^\sigma \, \nabla_x \cdot (\gamma \, \nabla_x U) \,. \tag{4.54}$$

It is left to show that

$$0 = \int_{\mathfrak{O}'} |\mathsf{y}|^{-\sigma} \, (\widetilde{\gamma} \; \nabla_{\mathbf{x}} W) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x}, \quad \text{for all } \Phi \in C_0^\infty(\mathfrak{O}'). \tag{4.55}$$

Given any $\epsilon > 0$, recall from Remark 4.4, the regularity of solutions away from $\partial \mathbb{R}^{n+1}_+$, that U satisfies the equation

$$\mathscr{L}_{\sigma,\tilde{\gamma}} U = |\mathsf{y}|^\sigma \mathcal{L}_x U - \partial_{\mathsf{y}} \left(|\mathsf{y}|^\sigma \partial_{\mathsf{y}} U \right) = 0$$

in the classical sense in $\mathfrak{O}' \cap \{|\mathsf{y}| > \epsilon\}$. Then for any $\Phi \in C_0^\infty(\mathfrak{O}')$, integrating by parts one obtains

$$\begin{aligned}
&\int_{\mathfrak{O}' \cap \{|\mathsf{y}|>\epsilon\}} |\mathsf{y}|^{-\sigma} \, (\widetilde{\gamma} \; \nabla_{\mathbf{x}} W) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x} \\
&= \int_{\mathfrak{O}' \cap \{|\mathsf{y}|>\epsilon\}} \partial_{\mathsf{y}} \left(\gamma \, \nabla_x U \right) \cdot \nabla_x \Phi \, d\mathbf{x} + \int_{\mathfrak{O}' \cap \{|\mathsf{y}|>\epsilon\}} \partial_{\mathsf{y}} \left(|\mathsf{y}|^\sigma \partial_{\mathsf{y}} U \right) \, |\mathsf{y}|^{-\sigma} \partial_{\mathsf{y}} \Phi \, d\mathbf{x} \\
&= \int_{\mathfrak{O}' \cap \{|\mathsf{y}|>\epsilon\}} \nabla_x \cdot \left(\gamma \, \nabla_x U \right) \partial_{\mathsf{y}} \Phi \, d\mathbf{x} + \int_{\mathfrak{O}' \cap \{|\mathsf{y}|>\epsilon\}} \partial_{\mathsf{y}} \left(|\mathsf{y}|^\sigma \partial_{\mathsf{y}} U \right) \, |\mathsf{y}|^{-\sigma} \partial_{\mathsf{y}} \Phi \, d\mathbf{x} \\
&\quad + \int_{\mathfrak{O}' \cap \{\mathsf{y}=\epsilon\}} \nabla_x \cdot \left(\gamma \, \nabla_x U \right) \, \Phi \, dx - \int_{\mathfrak{O}' \cap \{\mathsf{y}=-\epsilon\}} \nabla_x \cdot \left(\gamma \, \nabla_x U \right) \, \Phi \, dx \\
&= - \int_{\mathfrak{O}' \cap \{\mathsf{y}=\epsilon\}} \mathcal{L}_x U \, \Phi \, dx + \int_{\mathfrak{O}' \cap \{\mathsf{y}=-\epsilon\}} \mathcal{L}_x U \, \Phi \, dx \\
&= - \int_{\mathfrak{O}' \cap \{|\mathsf{y}|<\epsilon\}} \mathcal{L}_x \partial_{\mathsf{y}} U \, \Phi + \mathcal{L}_x U \, \partial_{\mathsf{y}} \Phi \, d\mathbf{x}.
\end{aligned} \tag{4.56}$$

Recall, again from Theorem 4.7 that $\mathcal{L}_x U$, $\mathcal{L}_x \partial_{\mathsf{y}} U \in L^2(\mathfrak{O}', |\mathsf{y}|^\sigma)$. Then the RHS of (4.56) tends to 0 as $\epsilon \to 0^+$, since $\Phi \in C_0^\infty(\mathfrak{O}') \subset H_0^1(\mathfrak{O}', |\mathsf{y}|^{-\sigma})$. Finally, one has

$$\lim_{\epsilon \to 0^+} \int_{\mathfrak{O}' \cap \{|\mathsf{y}| \le \epsilon\}} |\mathsf{y}|^{-\sigma} \, (\widetilde{\gamma} \; \nabla_{\mathbf{x}} W) \cdot \nabla_{\mathbf{x}} \Phi \, d\mathbf{x} = 0,$$

since $W \in H^1(\mathfrak{O}', |\mathsf{y}|^{-\sigma})$. Thus we have verified (4.55).

The proof is complete. □

Now, let $\mathfrak{O}$ be an open domain in $\mathbb{R}^{n+1}$. For any given $h > 0$, denote $\mathfrak{O}_h$ as the open set defined by

$$\mathfrak{O}_h := \{\mathbf{x} \in \mathbb{R}^{n+1}; \operatorname{dist}(x, \mathfrak{O}) < h\}.$$

Let $U \in H^1(\mathfrak{O}_1, |\mathsf{y}|^\sigma)$ be a (weak) solution of $\mathscr{L}_{\sigma,\tilde{\gamma}}U = 0$ in $\mathfrak{O}_1$. Define

$$W = |\mathsf{y}|^\sigma \partial_\mathsf{y} U.$$

Then $W \in L^2(\mathfrak{O}_1, |\mathsf{y}|^{-\sigma}) \cap H^1(\mathfrak{O}_{3/4}, |\mathsf{y}|^{-\sigma})$ and solves $\mathscr{L}_{-\sigma,\tilde{\gamma}}W = 0$ in $\mathfrak{O}_{3/4}$. Hence we can further define

$$U_1 = |\mathsf{y}|^{-\sigma} \partial_\mathsf{y} W.$$

It turns out by the same arguments that $U_1 \in L^2(\mathfrak{O}_{3/4}, |\mathsf{y}|^\sigma) \cap H^1(\mathfrak{O}_{1/2}, |\mathsf{y}|^\sigma)$ and solves $\mathscr{L}_{\sigma,\tilde{\gamma}}U_1 = 0$ in $\mathfrak{O}_{1/2}$. Then again, it is possible to define

$$W_1 = |\mathsf{y}|^\sigma \partial_\mathsf{y} U_1.$$

By iteration, one can define, for each $k \in \mathbb{N}$,

$$U_k = |\mathsf{y}|^{-\sigma} \partial_\mathsf{y} W_{k-1} \quad \text{and} \quad W_k = |\mathsf{y}|^\sigma \partial_\mathsf{y} U_k.$$

Moreover, $U_k \in L^2(\mathfrak{O}_{1/2^k+1/4}, |\mathsf{y}|^\sigma) \cap H^1(\mathfrak{O}_{1/2^k}, |\mathsf{y}|^\sigma)$ solves $\mathscr{L}_{\sigma,\tilde{\gamma}}U_k = 0$ in $\mathfrak{O}_{1/2^k}$, and $W_k \in L^2(\mathfrak{O}_{1/2^k}, |\mathsf{y}|^{-\sigma}) \cap H^1(\mathfrak{O}_{1/2^k-1/4}, |\mathsf{y}|^{-\sigma})$ solves $\mathscr{L}_{-\sigma,\tilde{\gamma}}W_k = 0$ in $\mathfrak{O}_{1/2^k-1/4}$. Notice that $\mathfrak{O}_h \Supset \mathfrak{O}$ for any $h > 0$. Then we have $U_k \in H^1(\mathfrak{O}, |\mathsf{y}|^\sigma)$ solves $\mathscr{L}_{\sigma,\tilde{\gamma}}U_k = 0$ and $W_k \in H^1(\mathfrak{O}, |\mathsf{y}|^{-\sigma})$ solves $\mathscr{L}_{-\sigma,\tilde{\gamma}}W_k = 0$ in $\mathfrak{O}$ for every $k \in \mathbb{N}$.

It is observed that

$$\mathscr{L}_{\sigma,\tilde{\gamma}}U_k = |\mathsf{y}|^\sigma \mathcal{L}_x U_k - \partial_\mathsf{y} W_k.$$

Hence $\mathscr{L}_{\sigma,\tilde{\gamma}}U_k = 0$ can be also understood as in the space $L^2(\mathfrak{O}, |\mathsf{y}|^{-\sigma})$, or,

$$\mathcal{L}_x U_k = U_{k+1} \quad \text{in } L^2(\mathfrak{O}, |\mathsf{y}|^\sigma).$$

Similarly,

$$\mathscr{L}_{-\sigma,\tilde{\gamma}}W_k = |\mathsf{y}|^{-\sigma} \mathcal{L}_x W_k - \partial_\mathsf{y} U_{k+1} = 0 \quad \text{in } L^2(\mathfrak{O}, |\mathsf{y}|^\sigma),$$

or,

$$\partial_{\mathsf{y}} \mathcal{L}_x U_k = \partial_{\mathsf{y}} U_{k+1} \quad \text{in } L^2(\mathfrak{O}, |\mathsf{y}|^\sigma).$$

Therefore one has

$$\mathcal{L}_x U_k = U_{k+1} \quad \text{in } H^1(\mathfrak{O}, |\mathsf{y}|^\sigma).$$

With the above reflection and conjugation strategies, we are able to derive the following result for the degenerate problem (4.40). For a detailed proof, please see [53].

Lemma 4.15. *Let $U \in H^1(\mathbb{R}^{n+1}_+, \mathsf{y}^{1-2s})$ be the solution to* (4.40) *with the Dirichlet data $u \in H^s(\mathbb{R}^n)$. If $u = \mathcal{L}^s u = 0$ in $B(x_0, 2r)$ with given $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, then one has*

$$\lim_{r \to 0} r^{-m} \int_{\mathfrak{B}(x_0, r)} |\mathsf{y}|^{1-2s} U_e^2(\mathbf{x}) \, d\mathbf{x} = 0, \quad \textit{for any } m \in \mathbb{N},$$

with U_e the even reflection (and extension) of U introduced in (4.47).

4.4.2 Proof of Theorem 4.2

Now we are able to prove Theorem 4.2.

Proof of Theorem 4.2. Since $\mathcal{O}$ is an open set in $\mathbb{R}^n$, we can find $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$ satisfying $u = \mathcal{L}^s u = 0$ in $B(x_0, 2r)$. Then combining Lemma 4.15 with the result in [133, Theorem 1.2] or that in [53, Proposition 5.4], we obtain that the solution U to the degenerate exterior problem in $\mathbb{R}^{n+1}_+$ vanishes identically. Therefore, we have that u, as the Dirichlet data of U on $\partial\mathbb{R}^{n+1}_+$, vanishes identically in $\mathbb{R}^n$. □

4.5 The Calderón Problem for $\mathcal{L}^s + q$

We establish in this section our final result for the inverse problem of recovering q from information of the DtN map Λ^s_q. The result is valid for very general anisotropic operators under a very general partial data setting.

4.5.1 Density of Solutions

The following result on the density of solutions is crucial in proving the uniqueness of recovering q from the DtN map Λ_q. It is an important consequence of Theorem 4.2 as stated in the previous section.

Theorem 4.4. *Denote u_g as the solution to* (4.33) *with the Dirichlet data g and the zero RHS. Given any open subset $\mathcal{O}$ of Ω_e, the solution space*

$$\{u_g|_\Omega;\ g \in C_c^\infty(\mathcal{O})\} \tag{4.57}$$

is dense in $L^2(\Omega)$.

Proof. Let $f \in L^2(\Omega)$ satisfy

$$(f, u_g)_{L^2(\Omega)} = 0, \quad \text{for any } g \in C_c^\infty(\mathcal{O}). \tag{4.58}$$

We shall verify the triviality of f.

Denote $\phi \in \widetilde{H}^s(\Omega)$ as the solution to the nonlocal problem (4.23) with the RHS function f. We claim

$$\mathcal{B}_q(\phi, g) = -(f, u_g)_{L^2(\Omega)}, \quad \text{for any } g \in C_c^\infty(\mathcal{O}). \tag{4.59}$$

If (4.59) holds true, then by noticing (4.58) and recalling $g = 0$ in Ω, we obtain that

$$\langle \mathcal{L}^s \phi, g \rangle = 0, \quad \text{for any } g \in C_c^\infty(\mathcal{O}),$$

which yields

$$\mathcal{L}^s \phi|_{\mathcal{O}} = 0.$$

Since $\phi \in \widetilde{H}^s(\Omega)$ we have that $\phi = 0$ in $\mathcal{O}$. Therefore, Theorem 4.2 implies that $\phi \equiv 0$ in $\mathbb{R}^n$, and as a consequence, $f \equiv 0$ in $\mathbb{R}^n$.

We are left to verify (4.59). Recalling that u_g is the solution to (4.33) with a trivial RHS, and that $\phi \in \widetilde{H}^s(\Omega)$, one has

$$\mathcal{B}_q(\phi, u_g) = \mathcal{B}_q(u_g, \phi) = 0, \quad \text{for any } g \in C_c^\infty(\mathcal{O}).$$

Furthermore, for any $g \in C_c^\infty(\mathcal{O})$, noticing $u_g - g \in \widetilde{H}^s(\Omega)$ and the fact that ϕ is the solution to (4.23), one has

$$-\mathcal{B}_q(\phi, g) = \mathcal{B}_q(\phi, u_g - g) = (f, u_g - g)_{L^2(\Omega)} = (f, u_g)_{L^2(\Omega)}.$$

The proof is complete. □

Theorem 4.4, in other words, states that any L^2 function can be approximated by solutions to the nonlocal problem (4.33). This is closely related to the Runge approximation for local and classical differential equations. For relevant study on approaching a L^2 by solutions of $\mathcal{L}^s$ related problems, please see [55, 39, 54].

4.5.2 Uniqueness in the Anisotropic Calderón Problem with Partial Data

Finally, we establish our main result in the inverse problem of recovering q from Dirichlet and Neumann data Λ_q^s associated with the nonlocal operator $\mathcal{L}^s + q$.

Theorem 4.5. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, and q_1 and q_2 be two $L^\infty(\Omega)$ potentials. Let $\mathcal{L}^s = \mathcal{L}^s_\gamma$ satisfy Assumptions A and B as introduced in Sections 4.2 and 4.3.2. Denote Λ_{q_j} as the DtN map associated with the nonlocal operator $\mathcal{L}^s + q_j$ and the domain Ω (see, Section 4.3.2). Assume*

$$\Lambda_{q_1} g|_{\mathcal{O}_2} = \Lambda_{q_2} g|_{\mathcal{O}_2}, \quad \text{for any } g \in C_c^\infty(\mathcal{O}_1), \tag{4.60}$$

where $\mathcal{O}_1, \mathcal{O}_2$ are arbitrarily fixed open subsets of Ω_e. Then there must have

$$q_1 = q_2. \tag{4.61}$$

Theorem 4.5 is in fact a consequence of the following much stronger result.

Theorem 4.6 (A Single Measurement)**.** *Under the same notations of Theorem 4.5, assume that there exits a nontrivial function $g_0 \in C_c^\infty(\mathcal{O}_1)$ such that*

$$\Lambda_{q_1} g_0|_{\mathcal{O}_2} = \Lambda_{q_2} g_0|_{\mathcal{O}_2}. \tag{4.62}$$

Then the set where q_1 and q_2 are not the same cannot contain any open set of $\mathbb{R}^n$.

Corollary 4.4. *Under the conditions of Theorem 4.6, assume further that the following Conjecture A holds true. Then one must have* (4.61).

Conjecture A. Let $u \in H^s(\mathbb{R}^n)$ satisfy $(\mathcal{L}^s + q)\, u = 0$ in Ω. If u is zero in a set of nonzero measure in Ω, then u must vanish identically.

Remark 4.7. Conjecture A has been verified for the fractional Laplacian $(-\Delta)^s$ in [54].

Proof of Theorem 4.6 and Corollary 4.4. If (4.62) holds true, then one has from Lemma 4.5 that

$$0 = \left\langle (\Lambda_{q_1} - \Lambda_{q_2}) g_0, \phi \right\rangle = \int_\Omega (q_1 - q_2) u_{g_0,q_1} u_{\phi,q_2} dx, \tag{4.63}$$

for any $\phi \in C_c^\infty(\mathcal{O}_2)$. Recall from Theorem 4.4 that the set of solutions

$$\{u_{\phi,q_2}|_\Omega;\ \phi \in C_c^\infty(\mathcal{O}_2)\} \tag{4.64}$$

is dense in $L^2(\Omega)$. Under the setting of Theorem 4.5, the Dirichlet data g_0 in (4.63) can be taken as any function in $C_c^\infty(\mathcal{O}_2)$. Then (4.63) along with the density of (4.64) already implies (4.61). If (4.62) holds instead of (4.60), then (4.63) yields

$$(q_1 - q_2) u_{g_0,q_1} = 0 \quad \text{in } L^2(\Omega),$$

which implies that q_1 and q_2 are the same wherever u_{g_0,q_1} does not vanish. By Theorem 4.2, u cannot vanish on any open set in Ω, since g_0 is nontrivial, which completes the proof of Theorem 4.6. If one further has Conjecture A, then the set where u vanishes is of zero measure in $\mathbb{R}^n$. We have hence verified the conclusion of Corollary 4.4. □

4.6 Future Work

Unique determination of the potential q from the DtN map of the associated nonlocal operator $\mathcal{L}_\gamma^s + q$ has been established in this study. However, nothing has been shown for the recovery of γ. This is essentially different to the case of recovering q. In particular, instead of the identity (4.38) which reveals a linear relation of the difference between two potentials, the counterpart integral identity for different coefficients γ_j, $j = 1, 2$, will be

$$\left\langle \mathcal{L}_{\gamma_1}^s v_1 - \mathcal{L}_{\gamma_2}^s v_1, v_2 \right\rangle = 0, \tag{4.65}$$

for any v_j satisfies $\mathcal{L}_{\gamma_j}^s v_j = 0$ in Ω, $j = 1, 2$.

The Isotropic Case

For the standard Calderón problem with $s = 1$, this has been solved for L^∞ conductivities in two dimensions and Lipschitz ones in dimension $n \geq 3$ (see the review paper [131]).

However, unlike the case for standard Calderón problems where the recovery of a isotropic conductivity γ can be reduced to the determination of a potential q, the parallel problem for the fractional operator $\mathcal{L}_\gamma^s$ cannot be reduced to the current study.

The first step to establish uniqueness for the Calderón problem, is to assume that there are two conductivities γ_j, $j = 1, 2$, such that the corresponding DtN maps are the same. By doing this for the fractional operator case, one obtains the identity Compared the current study of determining the potential q, where the analogous identity is simply $\int_\Omega q v_1 v_2 = 0$, the integral in (4.65) is much more complicated. One does not even have an explicit representation of the conductivities γ_j, $j = 1, 2$, in the identity (4.65).

The Anisotropic Case

The Calderón problem of determining an anisotropic conductivity or geometry arises from many real cases like muscle tissues and the structure of the Earth. It is known (cf. [84, 87]) that one can identify, at most up to a diffeomorphism, an anisotropic conductivity from the DtN map on the boundary. Therefore one asks whether the diffeomorphism invariance is the only obstruction of uniqueness. This has been done for L^∞ conductivities in dimension two by using isothermal coordinates and reducing to the isotropic case. In dimension $n \geq 3$ this has been shown only for real analytic conductivities (see [131] for a review).

However, for the fractional Calderón problem, we are able to deal with the very general anisotropic case in recovering q, which is still an open problem for the classical and local Calderón problem in dimension three or higher. Therefore, another possibility for future work is to consider the problem of determining the metric γ from the DtN map for the fractional operator $\mathcal{L}_\gamma^s$ with $0 < s < 1$. Besides its own interest as an inverse problem for nonlocal operators, it might also be possible to give some light for the classical Calderón problem for anisotropic conductivities. The first question to ask, analogous to the standard anisotropic Calderón problem, is that whether there is also diffeomorphism invariance. This is not clear for the fractional and nonlocal operator $\mathcal{L}_\gamma^s$ yet, since it is not a classical differential operator and the reflection between γ and $\mathcal{L}_\gamma^s$ is not explicit. Two possible alternative ways into it are to inspect, from the heat kernel which is used to define the kernel of $\mathcal{L}_\gamma^s$, and from the extension problem in one more dimension introduced in [126].

Bibliography

[1] Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, 2nd ed., Pure and Applied Mathematics, vol. 140, Elsevier, 2003.

[2] Giovanni Alessandrini and Michele Di Cristo, *Stable determination of an inclusion by boundary measurements*, SIAM J. Math. Anal. **37** (2005), no. 1, 200–217. MR 2176930

[3] Giovanni Alessandrini and Luca Rondi, *Determining a sound-soft polyhedral scatterer by a single far-field measurement*, Proceedings of the American Mathematical Society **133** (2005), no. 6, 1685–1691.

[4] Carlos J. S. Alves and Rainer Kress, *On the far-field operator in elastic obstacle scattering*, IMA Journal of Applied Mathematics **67** (2002), no. 1, 1–21.

[5] Habib Ammari, Elie Bretin, Josselin Garnier, Hyeonbae Kang, Hyundae Lee, and Abdul Wahab, *Mathematical methods in elasticity imaging*, Princeton University Press, 2015.

[6] Kari Astala, Matti Lassas, and Lassi Päivärinta, *Calderón's inverse problem for anisotropic conductivity in the plane*, Communications in Partial Differential Equations **30** (2005), no. 1-3, 207–224.

[7] Peter W. Bates and Adam Chmaj, *An integrodifferential model for phase transitions: stationary solutions in higher space dimensions*, Journal of Statistical Physics **95** (1999), no. 5-6, 1119–1139.

[8] Jöran Bergh and Jörgen Löfström, *Interpolation spaces: An introduction*, Grundlehren der mathematischen Wissenschaften, vol. 223, Springer-Verlag Berlin Heidelberg, 1976.

[9] Eemeli Blåsten and Hongyu Liu, *On corners scattering stably and stable shape determination by a single far-field pattern*, arXiv preprint, arXiv:1611.03647 (2016).

[10] Eemeli Blåsten, Lassi Päivärinta, and John Sylvester, *Corners always scatter*, Communications in Mathematical Physics **331** (2014), no. 2, 725–753.

[11] Liliana Borcea, *Electrical impedance tomography*, Inverse Problems **18** (2002), no. 6, R99–R136.

[12] Antonio Buades, Bartomeu Coll, and Jean-Michel Morel, *Image denoising methods. A new nonlocal principle*, SIAM Review **52** (2010), no. 1, 113–147, Reprint of "A review of image denoising algorithms, with a new one".

[13] Xavier Cabré and Yannick Sire, *Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates*, Annales de l'Institut Henri Poincaré. Analyse Non Linéaire **31** (2014), no. 1, 23–53.

[14] Luis Caffarelli, Xavier Ros-Oton, and Joaquim Serra, *Obstacle problems for integro-differential operators: regularity of solutions and free boundaries*, Inventiones Mathematicae **208** (2017), no. 3, 1155–1211.

[15] Luis Caffarelli and Luis Silvestre, *An extension problem related to the fractional Laplacian*, Communications in Partial Differential Equations **32** (2007), no. 7-9, 1245–1260.

[16] Wenshan Cai, Uday K. Chettiar, Alexander V. Kildishev, and Vladimir M. Shalaev, *Optical cloaking with metamaterials*, Nature photonics **1** (2007), no. 4, 224.

[17] Fioralba Cakoni and David Colton, *A qualitative approach to inverse scattering theory*, Applied Mathematical Sciences, vol. 188, Springer, New York, 2014.

[18] Fioralba Cakoni, David Colton, and Drossos Gintides, *The interior transmission eigenvalue problem*, SIAM Journal on Mathematical Analysis **42** (2010), no. 6, 2912–2921.

[19] Fioralba Cakoni, David Colton, and Houssem Haddar, *Inverse scattering theory and transmission eigenvalues*, CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 88, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016.

[20] Fioralba Cakoni and Houssem Haddar, *Transmission eigenvalues*, Inverse Problems **29** (2013), no. 10, 100201.

[21] ———, *Transmission eigenvalues in inverse scattering theory*, Inverse problems and applications: inside out. II, Math. Sci. Res. Inst. Publ., vol. 60, Cambridge Univ. Press, Cambridge, 2013, pp. 529–580.

[22] Alberto-P. Calderón, *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), Soc. Brasil. Mat., Rio de Janeiro, 1980, pp. 65–73.

[23] Pedro Caro, Petri Ola, and Mikko Salo, *Inverse boundary value problem for Maxwell equations with local data*, Communications in Partial Differential Equations **34** (2009), no. 11, 1425–1464.

[24] Pedro Caro and Keith M. Rogers, *Global uniqueness for the Calderón problem with Lipschitz conductivities*, Forum of Mathematics. Pi **4** (2016), e2, 28.

[25] Margaret Cheney, David Isaacson, and Jonathan C. Newell, *Electrical impedance tomography*, SIAM Review **41** (1999), no. 1, 85–101.

[26] Jin Cheng and Masahiro Yamamoto, *Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves*, Inverse Problems **19** (2003), no. 6, 1361.

[27] ______, *Corrigendum: "Uniqueness in an inverse scattering problem within non-trapping polygonal obstacles with at most two incoming waves" [Inverse Problems* **19** *(2003), no. 6, 1361–1384; mr2036535]*, Inverse Problems **21** (2005), no. 3, 1193.

[28] Philippe G. Ciarlet, *Mathematical elasticity. Vol. I*, Studies in Mathematics and its Applications, vol. 20, North-Holland Publishing Co., Amsterdam, 1988, Three-dimensional elasticity.

[29] David Colton and Andreas Kirsch, *A simple method for solving inverse scattering problems in the resonance region*, Inverse Problems **12** (1996), no. 4, 383–393.

[30] David Colton and Rainer Kress, *Inverse acoustic and electromagnetic scattering theory*, third ed., Applied Mathematical Sciences, vol. 93, Springer, New York, 2013.

[31] David Colton and Peter Monk, *The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium*, The Quarterly Journal of Mechanics and Applied Mathematics **41** (1988), no. 1, 97–125.

[32] David Colton and Lassi Päivärinta, *The uniqueness of a solution to an inverse scattering problem for electromagnetic waves*, Archive for Rational Mechanics and Analysis **119** (1992), no. 1, 59–70.

[33] David Colton, Lassi Päivärinta, and John Sylvester, *The interior transmission problem*, Inverse Problems and Imaging **1** (2007), no. 1, 13–28.

[34] Anne-Laure Dalibard and David Gérard-Varet, *On shape optimization problems involving the fractional Laplacian*, ESAIM. Control, Optimisation and Calculus of Variations **19** (2013), no. 4, 976–1013.

[35] Edward B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92, Cambridge University Press, Cambridge, 1989.

[36] Marta D'Elia and Max Gunzburger, *The fractional Laplacian operator on bounded domains as a special case of the nonlocal diffusion operator*, Computers & Mathematics with Applications **66** (2013), no. 7, 1245–1260.

[37] Youjun Deng, Hongyu Liu, and Gunther Uhlmann, *Full and partial cloaking in electromagnetic scattering*, Archive for Rational Mechanics and Analysis **223** (2017), no. 1, 265–299.

[38] ______, *On regularized full- and partial-cloaks in acoustic scattering*, Communications in Partial Differential Equations **42** (2017), no. 6, 821–851.

[39] Serena Dipierro, Ovidiu Savin, and Enrico Valdinoci, *All functions are locally s-harmonic up to a small error*, Journal of the European Mathematical Society (JEMS) **19** (2017), no. 4, 957–966.

[40] Serena Dipierro, Nicola Soave, and Enrico Valdinoci, *On fractional elliptic equations in Lipschitz sets and epigraphs: regularity, monotonicity and rigidity results*, Mathematische Annalen **369** (2017), no. 3-4, 1283–1326.

[41] David Dos Santos Ferreira, Carlos E. Kenig, Mikko Salo, and Gunther Uhlmann, *Limiting Carleman weights and anisotropic inverse problems*, Inventiones Mathematicae **178** (2009), no. 1, 119–171.

[42] Qiang Du, Max Gunzburger, R. B. Lehoucq, and Kun Zhou, *Analysis and approximation of nonlocal diffusion problems with volume constraints*, SIAM Review **54** (2012), no. 4, 667–696.

[43] Bartł omiej Dyda, Alexey Kuznetsov, and Mateusz Kwaśnicki, *Eigenvalues of the fractional Laplace operator in the unit ball*, Journal of the London Mathematical Society **95** (2017), no. 2, 500–518.

[44] ______, *Fractional Laplace operator and Meijer G-function*, Constructive Approximation **45** (2017), no. 3, 427–448.

[45] Johannes Elschner and Guanghui Hu, *Corners and edges always scatter*, Inverse Problems **31** (2015), no. 1, 015003.

[46] ______, *Acoustic scattering from corners, edges and circular cones*, Archive for Rational Mechanics and Analysis **228** (2018), no. 2, 653–690.

[47] Johannes Elschner and Masahiro Yamamoto, *Uniqueness in determining polygonal sound-hard obstacles with a single incoming wave*, Inverse Problems **22** (2006), no. 1, 355.

[48] ______, *Uniqueness in determining polyhedral sound-hard obstacles with a single incoming wave*, Inverse Problems **24** (2008), no. 3, 035004.

[49] ______, *Uniqueness in inverse elastic scattering with finitely many incident waves*, Inverse Problems **26** (2010), no. 4, 045005.

[50] Eugene B. Fabes, Carlos E. Kenig, and Raul P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Communications in Partial Differential Equations **7** (1982), no. 1, 77–116.

[51] Mouhamed Moustapha Fall and Sven Jarohs, *Overdetermined problems with fractional Laplacian*, ESAIM. Control, Optimisation and Calculus of Variations **21** (2015), no. 4, 924–938.

[52] Matthieu Felsinger, Moritz Kassmann, and Paul Voigt, *The Dirichlet problem for nonlocal operators*, Mathematische Zeitschrift **279** (2015), no. 3-4, 779–809.

[53] Tuhin Ghosh, Yi-Hsuan Lin, and Jingni Xiao, *The Calderón problem for variable coefficients nonlocal elliptic operators*, Communications in Partial Differential Equations **42** (2017), no. 12, 1923–1961.

[54] Tuhin Ghosh, Angkana Rüland, Mikko Salo, and Gunther Uhlmann, *Uniqueness and reconstruction for the fractional calderón problem with a single measurement*, arXiv preprint arXiv:1801.04449 (2018).

[55] Tuhin Ghosh, Mikko Salo, and Gunther Uhlmann, *The Calderón problem for the fractional Schrödinger equation*, arXiv preprint arXiv:1609.09248 (2016).

[56] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1998 edition.

[57] Guy Gilboa and Stanley Osher, *Nonlocal operators with applications to image processing*, Multiscale Modeling & Simulation **7** (2008), no. 3, 1005–1028.

[58] Drossos Gintides and Mourad Sini, *Identification of obstacles using only the scattered P-waves or the scattered S-waves*, Inverse Problems and Imaging **6** (2012), no. 1, 39–55.

[59] Vladimir Gol'dshtein and Alexander Ukhlov, *Weighted Sobolev spaces and embedding theorems*, Transactions of the American Mathematical Society **361** (2009), no. 7, 3829–3850.

[60] Allan Greenleaf, Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann, *Cloaking devices, electromagnetic wormholes, and transformation optics*, SIAM review **51** (2009), no. 1, 3–33.

[61] ———, *Invisibility and inverse problems*, American Mathematical Society. Bulletin. New Series **46** (2009), no. 1, 55–97.

[62] Allan Greenleaf, Matti Lassas, and Gunther Uhlmann, *On nonuniqueness for Calderón's inverse problem*, Mathematical Research Letters **10** (2003), no. 5, 685–693.

[63] Alexander Grigor'yan, *Heat kernel and analysis on manifolds*, AMS/IP Studies in Advanced Mathematics, vol. 47, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009.

[64] Boaz Haberman, *Uniqueness in Calderón's problem for conductivities with unbounded gradient*, Communications in Mathematical Physics **340** (2015), no. 2, 639–659.

[65] Peter Hähner and George C. Hsiao, *Uniqueness theorems in inverse obstacle scattering of elastic waves*, Inverse Problems **9** (1993), no. 5, 525–534.

[66] Michael Hitrik, Katsiaryna Krupchyk, Petri Ola, and Lassi Päivärinta, *The interior transmission problem and bounds on transmission eigenvalues*, Mathematical Research Letters **18** (2011), no. 2, 279–293.

[67] Naofumi Honda, Gen Nakamura, and Mourad Sini, *Analytic extension and reconstruction of obstacles from few measurements for elliptic second order operators*, Mathematische Annalen **355** (2013), no. 2, 401–427.

[68] Guanghui Hu, Andreas Kirsch, and Mourad Sini, *Some inverse problems arising from elastic scattering by rigid obstacles*, Inverse Problems **29** (2013), no. 1, 015009.

[69] Guanghui Hu, Jingzhi Li, and Hongyu Liu, *Recovering complex elastic scatterers by a single far-field pattern*, Journal of Differential Equations **257** (2014), no. 2, 469–489.

[70] Guanghui Hu and Hongyu Liu, *Nearly cloaking the elastic wave fields*, Journal de Mathématiques Pures et Appliquées **104** (2015), no. 6, 1045–1074.

[71] Guanghui Hu and Xiaodong Liu, *Unique determination of balls and polyhedral scatterers with a single point source wave*, Inverse Problems **30** (2014), no. 6, 065010, 14.

[72] Guanghui Hu, Mikko Salo, and Esa V. Vesalainen, *Shape identification in inverse medium scattering problems with a single far-field pattern*, SIAM Journal on Mathematical Analysis **48** (2016), no. 1, 152–165.

[73] Masaru Ikehata, *The enclosure method for inverse obstacle scattering using a single electromagnetic wave in time domain*, Inverse Problems and Imaging **10** (2016), no. 1, 131–163.

[74] Victor Isakov, *Stability estimates for obstacles in inverse scattering*, Journal of Computational and Applied Mathematics **42** (1992), no. 1, 79 – 88.

[75] Victor Isakov, *New stability results for soft obstacles in inverse scattering*, Inverse Problems **9** (1993), no. 5, 535.

[76] ______, *Inverse problems for partial differential equations*, second ed., Applied Mathematical Sciences, vol. 127, Springer, New York, 2006.

[77] Carlos E. Kenig and Mikko Salo, *Recent progress in the Calderón problem with partial data*, Inverse problems and applications, Contemp. Math., vol. 615, Amer. Math. Soc., Providence, RI, 2014, pp. 193–222.

[78] Carlos E. Kenig, Mikko Salo, and Gunther Uhlmann, *Inverse problems for the anisotropic Maxwell equations*, Duke Mathematical Journal **157** (2011), no. 2, 369–419.

[79] Stefan Kindermann, Stanley Osher, and Peter W. Jones, *Deblurring and denoising of images by nonlocal functionals*, Multiscale Modeling & Simulation **4** (2005), no. 4, 1091–1115.

[80] Andreas Kirsch, *On the existence of transmission eigenvalues*, Inverse Problems and Imaging **3** (2009), no. 2, 155–172.

[81] Robin John Knops and Lawrence Edward Payne, *Uniqueness theorems in linear elasticity*, Springer Tracts in Natural Philosophy, vol. 19, Springer-Verlag Berlin Heidelberg, 1971.

[82] Robert V. Kohn, Daniel Onofrei, Michael S. Vogelius, and Michael I. Weinstein, *Cloaking via change of variables for the Helmholtz equation*, Communications on Pure and Applied Mathematics **63** (2010), no. 8, 973–1016.

[83] Robert V. Kohn, Haiping Shen, Michael S. Vogelius, and Michael I. Weinstein, *Cloaking via change of variables in electric impedance tomography*, Inverse Problems **24** (2008), no. 1, 015016, 21.

[84] Robert V. Kohn and Michael Vogelius, *Identification of an unknown conductivity by means of measurements at the boundary*, Inverse problems (New York, 1983), SIAM-AMS Proc., vol. 14, Amer. Math. Soc., Providence, RI, 1984, pp. 113–123.

[85] Viktor D. Kupradze (ed.), *Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity*, North-Holland Series in Applied Mathematics and Mechanics, vol. 25, North-Holland Publishing Company, 1979.

[86] Mateusz Kwaśnicki, *Ten equivalent definitions of the fractional Laplace operator*, Fractional Calculus and Applied Analysis **20** (2017), no. 1, 7–51.

[87] John M. Lee and Gunther Uhlmann, *Determining anisotropic real-analytic conductivities by boundary measurements*, Communications on Pure and Applied Mathematics **42** (1989), no. 8, 1097–1112.

[88] Ulf Leonhardt, *Optical conformal mapping*, Science **312** (2006), no. 5781, 1777–1780.

[89] Hongyu Liu, *A global uniqueness for formally determined inverse electromagnetic obstacle scattering*, Inverse Problems **24** (2008), no. 3, 035018.

[90] Hongyu Liu, Michele Petrini, Luca Rondi, and Jingni Xiao, *Stable determination of sound-hard polyhedral scatterers by a minimal number of scattering measurements*, Journal of Differential Equations **262** (2017), no. 3, 1631–1670.

[91] Hongyu Liu, Luca Rondi, and Jingni Xiao, *Mosco convergence for $H(curl)$ spaces, higher integrability for Maxwell's equations, and stability in direct and inverse EM scattering problems*, To appear, Journal of the European Mathematical Society, arXiv:1603.07555.

[92] Hongyu Liu and Jingni Xiao, *Decoupling elastic waves and its applications*, Journal of Differential Equations **263** (2017), no. 8, 4442–4480.

[93] ———, *On electromagnetic scattering from a penetrable corner*, SIAM Journal on Mathematical Analysis **49** (2017), no. 6, 5207–5241.

[94] Hongyu Liu, Masahiro Yamamoto, and Jun Zou, *Reflection principle for the Maxwell equations and its application to inverse electromagnetic scattering*, Inverse Problems **23** (2007), no. 6, 2357–2366.

[95] Hongyu Liu, Hai Zhang, and Jun Zou, *Recovery of polyhedral scatterers by a single electromagnetic far-field measurement*, Journal of Mathematical Physics **50** (2009), no. 12, 123506.

[96] Hongyu Liu and Jun Zou, *Uniqueness in an inverse acoustic obstacle scattering problem for both sound-hard and sound-soft polyhedral scatterers*, Inverse Problems **22** (2006), no. 2, 515–524.

[97] ———, *On unique determination of partially coated polyhedral scatterers with far field measurements*, Inverse Problems **23** (2007), no. 1, 297–308.

[98] Annalisa Massaccesi and Enrico Valdinoci, *Is a nonlocal diffusion strategy convenient for biological populations in competition?*, Journal of Mathematical Biology **74** (2017), no. 1-2, 113–147.

[99] William McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.

[100] Ralf Metzler and Joseph Klafter, *The random walk's guide to anomalous diffusion: a fractional dynamics approach*, Physics Reports. A Review Section of Physics Letters **339** (2000), no. 1, 77.

[101] ———, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, Journal of Physics. A. Mathematical and General **37** (2004), no. 31, R161–R208.

[102] Benjamin Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Transactions of the American Mathematical Society **165** (1972), 207–226.

[103] Claus Müller, *Foundations of the mathematical theory of electromagnetic waves*, Revised and enlarged translation from the German. Die Grundlehren der mathematischen Wissenschaften, Band 155, Springer-Verlag, New York-Heidelberg, 1969.

[104] Adrian I. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem*, Annals of Mathematics **143** (1996), no. 1, 71–96.

[105] Gen Nakamura and Gunther Uhlmann, *Global uniqueness for an inverse boundary value problem arising in elasticity*, Inventiones Mathematicae **152** (2003), no. 1, 205–207.

[106] Jean-Claude Nédélec, *Acoustic and electromagnetic equations: Integral representations for harmonic problems*, Applied Mathematical Sciences, vol. 144, Springer-Verlag New York, 2001.

[107] Hoai-Minh Nguyen and Michael S. Vogelius, *Approximate cloaking for the full wave equation via change of variables: the Drude-Lorentz model*, Journal de Mathématiques Pures et Appliquées. Neuvième Série **106** (2016), no. 5, 797–836.

[108] Petri Ola, Lassi Päivärinta, and Erkki Somersalo, *An inverse boundary value problem in electrodynamics*, Duke Mathematical Journal **70** (1993), no. 3, 617–653.

[109] Petri Ola and Erkki Somersalo, *Electromagnetic inverse problems and generalized Sommerfeld potentials*, SIAM Journal on Applied Mathematics **56** (1996), no. 4, 1129–1145.

[110] Lassi Päivärinta, Mikko Salo, and Esa V. Vesalainen, *Strictly convex corners scatter*, Revista Matemática Iberoamericana **33** (2017), no. 4, 1369–1396.

[111] Lassi Päivärinta and John Sylvester, *Transmission eigenvalues*, SIAM Journal on Mathematical Analysis **40** (2008), no. 2, 738–753.

[112] John B Pendry, David Schurig, and David R Smith, *Controlling electromagnetic fields*, science **312** (2006), no. 5781, 1780–1782.

[113] Roland Potthast, *Stability estimates and reconstructions in inverse acoustic scattering using singular sources*, Journal of Computational and Applied Mathematics **114** (2000), no. 2, 247–274.

[114] Luca Rondi, *Stable determination of sound-soft polyhedral scatterers by a single measurement*, Indiana University Mathematics Journal **57** (2008), no. 3, 1377–1408.

[115] Luca Rondi and Mourad Sini, *Stable determination of a scattered wave from its far-field pattern: the high frequency asymptotics*, Archive for Rational Mechanics and Analysis **218** (2015), no. 1, 1–54.

[116] Xavier Ros-Oton, *Nonlocal elliptic equations in bounded domains: a survey*, Publicacions Matemàtiques **60** (2016), no. 1, 3–26.

[117] Xavier Ros-Oton and Enrico Valdinoci, *The Dirichlet problem for nonlocal operators with singular kernels: convex and nonconvex domains*, Advances in Mathematics **288** (2016), 732–790.

[118] Lorenzo Rosasco, Mikhail Belkin, and Ernesto De Vito, *On learning with integral operators*, Journal of Machine Learning Research **11** (2010), 905–934.

[119] Angkana Rüland, *Unique continuation for fractional Schrödinger equations with rough potentials*, Communications in Partial Differential Equations **40** (2015), no. 1, 77–114.

[120] Mikko Salo, *The fractional Calderón problem*, arXiv preprint arXiv:1711.06103 (2017).

[121] Otmar Scherzer (ed.), *Handbook of mathematical methods in imaging. Vol. 1, 2, 3*, second ed., Springer, New York, 2015.

[122] David Schurig, J. J. Mock, B. J. Justice, Steven A. Cummer, John B. Pendry, A. F. Starr, and D. R. Smith, *Metamaterial electromagnetic cloak at microwave frequencies*, Science **314** (2006), no. 5801, 977–980.

[123] Eva Sincich and Mourad Sini, *Local stability for soft obstacles by a single measurement*, Inverse Problems and Imaging **2** (2008), no. 2, 301–315.

[124] Igor M. Sokolov and Joseph Klafter, *From diffusion to anomalous diffusion: a century after Einstein's Brownian motion*, Chaos **15** (2005), no. 2, 026103, 7.

[125] Plamen Stefanov, *Stability of the inverse problem in potential scattering at fixed energy*, Université de Grenoble. Annales de l'Institut Fourier **40** (1990), no. 4, 867–884 (1991).

[126] Pablo Raúl Stinga and José Luis Torrea, *Extension problem and Harnack's inequality for some fractional operators*, Communications in Partial Differential Equations **35** (2010), no. 11, 2092–2122.

[127] John Sylvester and Gunther Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Annals of Mathematics **125** (1987), no. 1, 153–169.

[128] Hans Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland Mathematical Library, vol. 18, North-Holland Publishing Co., Amsterdam-New York, 1978.

[129] Gunther Uhlmann, *Electrical impedance tomography and Calderón's problem*, Inverse Problems **25** (2009), no. 12, 123011, 39.

[130] ———, *Visibility and invisibility*, ICIAM 07—6th International Congress on Industrial and Applied Mathematics, Eur. Math. Soc., Zürich, 2009, pp. 381–408.

[131] ———, *Inverse problems: seeing the unseen*, Bulletin of Mathematical Sciences **4** (2014), no. 2, 209–279.

[132] Esa V. Vesalainen, *On non-scattering energies*, Ph.D. thesis, University of Helsinki, Helsinki, 6 2014.

[133] Hui Yu, *Unique continuation for fractional orders of elliptic equations*, Annals of PDE **3** (2017), no. 2, Art. 16, 21.

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