

DOCTORAL THESIS

Spectral collocation methods for the fractional PDEs in unbounded domain

Yuan, Huifang

Date of Award:
2018

[Link to publication](#)

General rights

Copyright and intellectual property rights for the publications made accessible in HKBU Scholars are retained by the authors and/or other copyright owners. In addition to the restrictions prescribed by the Copyright Ordinance of Hong Kong, all users and readers must also observe the following terms of use:

- Users may download and print one copy of any publication from HKBU Scholars for the purpose of private study or research
- Users cannot further distribute the material or use it for any profit-making activity or commercial gain
- To share publications in HKBU Scholars with others, users are welcome to freely distribute the permanent URL assigned to the publication

Abstract

This thesis is concerned with a particular numerical approach for solving the fractional partial differential equations (PDEs). In the last two decades, it has been observed that many practical systems are more accurately described by fractional differential equations (FDEs) rather than the traditional differential equation approaches. Consequently, it has become an important research area to study the theoretical and numerical aspects of various types of FDEs. This thesis will explore high order numerical methods for solving FDEs numerically. More precisely, spectral methods which exhibits exponential order of accuracy will be investigated. The method consists of expanding the solution with proper global basis functions and imposing collocation conditions on the Gauss quadrature points.

In this work, Hermite and modified rational functions are employed to serve as basis functions for solutions that decay exponentially and algebraically, respectively. The main emphasis of this thesis is to propose the spectral collocation method for FDEs posed in *unbounded domains*. Components of the differentiation matrix involving fractional Laplacian are derived which can then be computed recursively using the properties of confluent hypergeometric function or hypergeometric function.

The first part of the thesis introduces preliminaries useful for other parts of the thesis. Review of the relevant definitions and properties of special functions such as Hermite functions, Bessel functions, hypergeometric functions, Gegenbauer polynomials, mapped Jacobi polynomials, modified rational functions are presented. Fractional Sobolev space is introduced and some lemmas on interpolation approximation in the fractional Sobolev space are also included.

In the second part of the thesis, we present the spectral collocation method based on Hermite functions. Two bases are used, namely, the over-scaled Hermite function and generalized Hermite function, which are orthogonal functions on the whole line with appropriate weight functions. We will show that the fractional Laplacian of these two kinds of Hermite functions can be represented by confluent hypergeometric function. Behaviors of the condition numbers for the resulting spectral differentiation matrices with respect to the number of expansion terms are investigated.

Moreover, approximation in two-dimensional space using the tensorized bases, application to multi-term problems and use of scaling to match different decay rate are also considered. Convergence analysis for generalized Hermite function are derived and numerical errors for two bases are analyzed.

The third part of the thesis deals with the spectral collocation method based on modified rational functions. We first give a brief introduction for computation of the fractional Laplacian using modified rational functions, which is represented by hypergeometric functions. Then the differentiation matrix involving the fractional Laplace operator is given. Convergence analysis for modified Chebyshev rational functions and modified Legendre rational functions are derived and numerical experiments are carried out.

Keywords: Fractional PDEs, Hermite polynomials/functions, Gegenbauer polynomials, mapped Jacobi functions, modified rational function, unbounded domain, spectral collocation methods.

Table of Contents

Declaration	i
Abstract	ii
Acknowledgements	iv
Table of Contents	v
List of Figures	vii
List of Symbols	xi
List of Abbreviation	xii
Chapter 1 Introduction	1
1.1 Fractional differential equations	1
1.2 Numerical methods	3
1.3 Problem description	5
1.4 Plan of the thesis	9
Chapter 2 Preliminaries	10
2.1 Confluent hypergeometric functions	10
2.2 An integral involving parabolic cylinder function	11
2.3 Hermite polynomials/functions	11
2.4 Bessel functions	12
2.5 Hypergeometric function	14
2.6 Gegenbauer polynomials	15

2.7	Modified rational functions	16
2.8	Fractional Sobolev space	18
2.9	Useful lemmas for approximation error	19
2.10	Strang's first lemma	23
Chapter 3 Hermite spectral collocation methods		25
3.1	Over-scaled Hermite function $\{\tilde{H}_n(x)\}_n$	25
3.1.1	1D case	26
3.1.2	2D case	34
3.1.3	The use of the scaling factors	37
3.1.4	Applications to multi-term fractional PDEs	38
3.2	Generalized Hermite function $\{\hat{H}_n(x)\}_n$	39
3.2.1	The one dimensional case	39
3.2.2	2D case	40
3.3	Application to fractional differential equations	42
3.4	Convergence analysis	44
3.5	Numerical examples	46
3.5.1	The fractional Laplace equation	46
3.5.2	A linear fractional PDE	48
3.5.3	A two-dimensional example	50
3.5.4	A multi-term fractional model	50
3.5.5	A nonlinear example	51
3.5.6	An eigenvalue problem	51
3.6	Concluding remarks	53
Chapter 4 Modified rational spectral collocation methods		55
4.1	Computing fractional Laplacian with simple functions	56
4.2	Computing with modified rational functions	58
4.3	Application to fractional differential equations	60
4.4	Differentiation matrix of the spectral collocation method with Lagrange bases	61
4.5	Convergence analysis	62

4.6	Numerical examples	64
4.6.1	With exponential decay right hand side	65
4.6.2	With algebraic decay right hand side	67
4.6.3	A multi-term fractional model	67
4.6.4	A nonlinear example	70
4.6.5	An eigenvalue problem	70
4.7	Concluding remarks	72
Chapter 5 Summary and future work		73
Curriculum Vitae		84