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Determination of random schrödinger operators

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Determination of Random Schrödinger Operators

MA Shiqi

A thesis submitted in partial fulfilment of the requirements
for the degree of
Doctor of Philosophy

Principal Supervisors:
Prof. LIU Hongyu (Hong Kong Baptist University)
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July 2019

DECLARATION

I hereby declare that this thesis represents my own work which has been done after registration for the degree of PhD at Hong Kong Baptist University, and has not been previously included in a thesis or dissertation submitted to this or any other institution for a degree, diploma or other qualifications.

I have read the University's current research ethics guidelines, and accept responsibility for the conduct of the procedures in accordance with the University's Research Ethics Committee (REC). I have attempted to identify all the risks related to this research that may arise in conducting this research, obtained the relevant ethical and/or safety approval (where applicable), and acknowledged my obligations and the rights of the participants.

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Abstract

Inverse problems arise in many fields such as radar imaging, medical imaging and geophysics. It draws much attention in both mathematical communities and industrial members. Mathematically speaking, many inverse problems can be formulated by one or several physical equations and mathematical models. For example, the signal used in radar imaging is governed by Maxwell's equation, and most of geophysical studies can be formulated using elastic equation. Therefore, rigorous mathematical theories can be applied to study the inverse problems coming from this complex world.

Random inverse problem is a fascinating area studying how to extract useful statistical information from unknown object coming from real world. In this thesis, we focus on the study of inverse problem related to random Schrödinger operators. We are particularly interested in the case where both the source and the potential of the Schrödinger system are random.

In our first topic, we are concerned with the direct and inverse scattering problems associated with a time-harmonic random Schrödinger equation with unknown random source and unknown potential. The well-posedness of the direct scattering problem is first established. Three uniqueness results are then obtained for the corresponding inverse problems in determining the variance of the source, the potential and the expectation of the source, respectively, by the associated far-field measurements. First, a single realization of the passive scattering measurement can uniquely recover the variance of the source without the a priori knowledge of the other unknowns. Second, if active scattering measurement can be further obtained, a single realization can uniquely recover the potential function without knowing the source. Finally, both the potential and the first two statistic moments of the random source can be uniquely recovered with full measurement data.

Our second topic also focuses on the case where only the source is random. But in the second topic, the random model is different from our first topic. The second random model has received intensive study in recent years due to the reason that this random model has more flexibility fitting with different regularities. The recovering

framework is similar to our first topic, but we shall develop different asymptotic estimates of the higher order terms, which is more difficult than the first one.

Lastly, based on the previous two results, we study the case where both the source and the potential are random and unknown. The ergodicity is used to establish the single realization recovery. The asymptotic estimates of higher order terms are based on pseudodifferential operators and microlocal analysis.

Three major novelties of our works in this thesis are that, first, we studied the case where both the source and the potential are unknown; second, both passive and active scattering measurements are used for the recovery in different scenarios; finally, only a single realization of the random sample is required to establish the recovery of useful information.

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Chapter 1

Introduction

Inverse problems arise in many fields such as radar and sonar imaging, medical imaging, seismic inversion and geophysics. The inverse problems have drawn a lot of attention in both mathematical communities and industrial members. Mathematically speaking, many inverse problems can be formulated by one or several physical equations and mathematical models. Two examples are, the signal used in radar imaging is governed by Maxwell's equation, and most of geophysical study can be formulated using elastic equation. Due to this reason, rigorous mathematical theories and tools can be applied to study the inverse problems. In this thesis, we mainly focus on inverse problems associated with time-harmonic Schrödinger systems.

In some cases, a mathematical system is driven by certain source. Inverse source problems concern with recovering useful information of the unknown source from collected data [14, 20, 28, 29, 40, 47, 48]. In [23], the author considered the recovery of the convex hull of support of the source associated with a Helmholtz equation. The work [1] showed the non-uniqueness of the inverse source problems in some scenarios, and [4] obtained an uniqueness result by using multi-frequency data. Moreover, Bao et. al. [4] established the increasing stability of the inverse source problem under Helmholtz system.

Sometimes we are more interested in some properties of the medium that our system lies on. Recovering information of the medium is what inverse medium problems mainly cares about [10, 36–38].

There is abundant literature for the inverse scattering problems concerning the

Schrödinger system. Given an known potential, the recovery of an unknown source term by the corresponding passive measurement is referred to as the inverse source problem. We refer to [4–6, 12, 18, 24–26, 28, 48, 51] and the references therein for both theoretical uniqueness/stability results and computational methods for the inverse source problem in the deterministic setting.

1.1 Random inverse problems

Random inverse problem is a fascinating area studying how to extract useful statistical information from unknown object coming from real world. In this thesis, we focus on the study of inverse problems related to random Schrödinger operators. We are particularly interested in the case where both the source and the potential of the Schrödinger system are random. To that end, we first study the case that only the source is random. This is our first topic.

The determination of a random source by the corresponding passive measurements was also recently studied in [3, 39, 50], and the determination of a random potential by the corresponding active measurements was established in [11]. We also refer to [31] and the references therein for more relevant studies on the determination of random potentials. The simultaneous recovery of an unknown source and its surrounding potential was also investigated in the literature. In [27, 38], motivated by applications in thermo- and photo-acoustic tomography, the simultaneous recovery of an unknown source and its surrounding medium parameter was considered. The simultaneous recovery studies in [27, 38] were confined to the deterministic setting and were associated mainly with the passive measurements.

1.2 Three topics to be explored

In this thesis, we mainly focus on inverse problems associated with the following time-harmonic Schrödinger equation

$$(-\Delta - E + \text{potential})\, u(x) = \text{source}, \quad x \in \mathbb{R}^3, \tag{1.1}$$

where E is the energy level, and the source and the potential in (1.1) shall be specified in Chapters 3, 4 and 5. In the rest of this section, we give a brief introduction to the three topics that we will explore.

1.2.1 Gaussian white noise source

In our first topic, we are concerned with the direct and inverse scattering problems associated with the time-harmonic random Schrödinger equation with unknown source and potential terms. The well-posedness of the direct scattering problem is first established. Three uniqueness results are then obtained for the corresponding inverse problems in determining the variance of the source, the potential and the expectation of the source, respectively, by the associated far-field measurements. First, a single realization of the passive scattering measurements can uniquely recover the variance of the source without the a priori knowledge of the other unknowns. Second, if active scattering measurements can be further obtained, a single realization can uniquely recover the potential function without knowing the source. Finally, both the potential and the first two statistic moments of the random source can be uniquely recovered with full measurements data. The major novelty of our topic is that on the one hand, both the random source and the potential are unknown, and on the other hand, both passive and active scattering measurements are used for the recovery in different scenarios.

1.2.2 Rough source

Our second topic also focuses on the Schrödinger system where only the source is random. But in this topic, the random model is different from our first one. The second random model has received intensive study in recent years due to the reason that it has more flexibility to fit randomness with different regularities. The recovering framework is similar to our first topic, but we shall establish the asymptotic estimates of the higher order terms in a different way, which is more difficult than the corresponding estimates in the first work.

1.2.3 Random source and random potential

Lastly, base on the previous two results, we study the inverse scattering problem where both the potential and source are random. Statistical properties of both the unknown source and unknown medium are recovered. First, the mathematical analysis of the direct problem is given. Then, the asymptotics of higher order terms are established. Finally, two unique recovery results are presented in determining rough strength of the random source and the random potential, by using the corresponding far-field measurements. The first recovery result shows that a single realization of the passive scattering measurements can uniquely recover the rough strength of the random source without knowing the potential. The second shows that, with a single realization of the backscattering data, the rough strength of the random potential can be recovered, provided that the orders of singularities of the potential and the source are chosen particularly. The ergodicity is used to establish the single realization recovery. The asymptotic estimates of higher order terms are based on pseudodifferential operators and microlocal analysis.

Three major novelties of our work in this thesis are that, first, we studied the case where both the source and the potential are unknown; second, both passive and active scattering measurements are used for the recovery in different scenarios; finally, only a single realization of the random sample is required to establish the recovering of the information of interest.

The rest of this thesis is organized as follows. In Chapter 2, we present some useful definitions/notations and basic lemmas for the upcoming chapters. Chapter 3 focuses on studying the case where the source in (1.1) is Gaussian white noise. In Chapter 4, we study another random model which has more flexibility on its regularity. Based on the results from Chapters 3 and 4, Chapter 5 establishes the unique recoveries for the case where both the source and potential are unknown and random. We conclude this thesis in Chapter 6 with some comments.

Chapter 2

Preliminaries

In this chapter, we give some definitions and common notations which will be used throughout this thesis. Also, some basic lemmas is given in this chapter.

2.1 Definitions and notations

In what follows, the Fourier transform and inverse Fourier transform of the function φ are defined as

$$\mathcal{F}\varphi(\xi) = \widehat{\varphi}(\xi) := (2\pi)^{-n/2} \int e^{-ix\cdot\xi}\varphi(x)\,\mathrm{d}x,$$
$$\mathcal{F}^{-1}\varphi(\xi) := (2\pi)^{-n/2} \int e^{ix\cdot\xi}\varphi(x)\,\mathrm{d}x.$$

We set

$$\Phi(x,y) = \Phi_k(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \in \mathbb{R}^3\backslash\{y\}.$$

Φ_k is the outgoing fundamental solution, centered at y, to the differential operator $-\Delta - k^2$. Define the resolvent operator $\mathcal{R}_k$,

$$\mathcal{R}_k(\varphi)(x) = (\mathcal{R}_k\varphi)(x) := \int_{\operatorname{supp}\varphi} \Phi_k(x,y)\varphi(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^3, \tag{2.1}$$

where φ can be any measurable function on $\mathbb{R}^3$ as long as the (2.1) is well-defined for almost all x in $\mathbb{R}^3$.

Write $\langle x\rangle := (1+|x|^2)^{1/2}$ for $x \in \mathbb{R}^3$. We introduce the following weighted L^2-norm and the corresponding function space over $\mathbb{R}^3$ for any $s \in \mathbb{R}$,

$$\begin{cases} \|f\|_{L^2_s(\mathbb{R}^3)} := \|\langle\cdot\rangle^s f(\cdot)\|_{L^2(\mathbb{R}^3)} = \Big(\int_{\mathbb{R}^3} \langle x\rangle^{2s} |f|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}}, \\ \quad L^2_s(\mathbb{R}^3) := \big\{ f \in L^1_{loc}(\mathbb{R}^3); \|f\|_{L^2_s(\mathbb{R}^3)} < +\infty \big\} . \end{cases} \tag{2.2}$$

We also define $L^2_s(S)$ for any measurable subset S in $\mathbb{R}^3$ by replacing $\mathbb{R}^3$ in (2.2) with S. In what follows, we may denote $L^2_s(\mathbb{R}^3)$ as L^2_s for short if without ambiguities.

Define

$$H^{s,p}_\delta(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^n); (I - \Delta)^{s/2} f \in L^p_\delta(\mathbb{R}^n)\}.$$

The space $H^{s,2}_\delta(\mathbb{R}^n)$ is abbreviated as $H^s_\delta(\mathbb{R}^n)$, and $H^{s,p}_0(\mathbb{R}^n)$ is abbreviated as $H^{s,p}(\mathbb{R}^n)$. It can be verified that

$$\|f\|_{H^s_\delta(\mathbb{R}^n)} = \|\langle\cdot\rangle^s \hat{f}(\cdot)\|_{H^\delta(\mathbb{R}^n)}. \tag{2.3}$$

The space $L^2_s(\mathbb{R}^n)$ is used throughout this thesis, while the space $H^s_\delta(\mathbb{R}^n)$ is mainly used in Chapter 5.

For the notational convenience, we use "$\{K_j\} \in P(t)$" to mean that the sequence $\{K_j\}_{j\in\mathbb{N}^+}$ satisfies $K_j \geq C j^t$ $(j \in \mathbb{N}^+)$ for some fixed constant $C > 0$. Throughout the following context, γ stands for any fixed positive real number.

In the sequel, we write $\mathcal{L}(\mathcal{A}, \mathcal{B})$ to denote the set of all the linear bounded mappings from a norm vector space $\mathcal{A}$ to a norm vector space $\mathcal{B}$. For any mapping $\mathcal{K} \in \mathcal{L}(\mathcal{A}, \mathcal{B})$, we denote its operator norm as $\|\mathcal{K}\|_{\mathcal{L}(\mathcal{A},\mathcal{B})}$. We write the identity operator as I. We also use notations C and its variants, such as C_D and $C_{D,f}$ to represent some generic constant(s) whose particular definition may change line by line. We use $\mathcal{A} \lesssim \mathcal{B}$ to signify $\mathcal{A} \leq C\mathcal{B}$ and $\mathcal{A} \simeq \mathcal{B}$ to signify $\mathcal{A} = C\mathcal{B}$, for some generic positive constant C. We denote "almost everywhere" as "a.e." and "almost surely" as "a.s." for short. We use $|\mathcal{S}|$ to denote the Lebesgue measure of any Lebesgue-measurable set $\mathcal{S}$.

2.2 Basic lemmas

Several important technical lemmas are presented here.

Lemma 2.2.1. *For any $\varphi \in L^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\varphi \subseteq D$ and any $\epsilon \in \mathbb{R}_+$, we have*

$$\mathcal{R}_k\varphi \in L^2_{-1/2-\epsilon}.$$

Proof of Lemma 2.2.1. Assume that φ belongs to $L^\infty(\mathbb{R}^3)$ with its support contained in D. Obviously we have that $\|\varphi\|_{L^2(D)} < +\infty$. Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}\|\mathcal{R}_k\varphi\|^2_{L^2_{-1/2-\epsilon}} &\lesssim \int_{\mathbb{R}^3} \langle x\rangle^{-1-2\epsilon}\Big(\int_D \frac{1}{|x-y|^2}\,\mathrm{d}y\Big)\cdot\Big(\int_D |\varphi(y)|^2\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &\lesssim \|\varphi\|^2_{L^2(D)}\Big[\int_{|x|\le 2M(0)}\Big(\int_D \frac{1}{|x-y|^2}\,\mathrm{d}y\Big)\,\mathrm{d}x \\ &\quad + \int_{|x|>2M(0)} \langle x\rangle^{-1-2\epsilon}\langle x\rangle^{-2}\,\mathrm{d}x\Big]. \end{aligned} \tag{2.4}$$

By the change of variable, the first term in the square brackets in (2.4) satisfies

$$\int_{|x|\le 2M(0)}\Big(\int_D \frac{1}{|x-y|^2}\,\mathrm{d}y\Big)\,\mathrm{d}x = \int_{|x|\le 2M(0)}\Big(\int_{z\in\{y-x\,;\,y\in D\}} \frac{1}{|z|^2}\,\mathrm{d}z\Big)\,\mathrm{d}x. \tag{2.5}$$

From (3.7), we can continue (2.5) as

$$\begin{aligned}\int_{|x|\le 2M(0)}\Big(\int_D \frac{1}{|x-y|^2}\,\mathrm{d}y\Big)\,\mathrm{d}x &\le \int_{|x|\le 2M(0)}\Big(\int_{\{z\,;\,|z|\le 3\operatorname{diam} D\}} \frac{1}{|z|^2}\,\mathrm{d}z\Big)\,\mathrm{d}x \\ &= \int_{|x|\le 2M(0)}\big(12\pi\operatorname{diam} D\big)\,\mathrm{d}x < +\infty. \end{aligned} \tag{2.6}$$

Meanwhile, the second term in the square brackets in (2.4) satisfies

$$\int_{|x|>2M(0)} \langle x\rangle^{-1-2\epsilon}\langle x\rangle^{-2}\,\mathrm{d}x \le \int_{\mathbb{R}^3} \langle x\rangle^{-3-2\epsilon}\,\mathrm{d}x < +\infty. \tag{2.7}$$

Note that (2.7) holds for every $\epsilon \in \mathbb{R}_+$. Combining (2.4), (2.6) and (2.7), we conclude

$$\|\mathcal{R}_k\varphi\|^2_{L^2_{-1/2-\epsilon}} < +\infty.$$

The proof is complete. □

Now we present a special version of Agmon's estimates for the convenience of our reader (cf. [16]). This special version will be used when proving Lemma 2.2.3.

Lemma 2.2.2 (Agmon's estimates [16])**.** *For any $\epsilon > 0$, there exists some $k_0 \geq 2$ such that for any $k > k_0$ we have*

$$\|\mathcal{R}_k \varphi\|_{L^2_{-1/2-\epsilon}} \leq C_\epsilon k^{-1} \|\varphi\|_{L^2_{1/2+\epsilon}}, \quad \forall \varphi \in L^2_{1/2+\epsilon} \tag{2.8}$$

where C_ϵ is independent of k and φ.

The proof of Lemma 2.2.2 can be found in [16]. The symbol k_0 appearing in Lemma 2.2.2 is preserved for future use.

Lemma 2.2.3. *For any fixed $\epsilon \geq 0$, when $k > k_0$, we have*

$$\|\mathcal{R}_k \circ V\|_{\mathcal{L}(L^2_{-1/2-\epsilon}, L^2_{-1/2-\epsilon})} \leq C_{\epsilon,D,V} k^{-1},$$

where the constant $C_{\epsilon,D,V}$ depends on ϵ, D and V but is independent of k.

Proof of Lemma 2.2.3. By Lemma 2.2.2, when $k > k_0$, we have the following estimate,

$$\|\mathcal{R}_k V u\|_{L^2_{-1/2-\epsilon}} = \|\mathcal{R}_k (V u)\|_{L^2_{-1/2-\epsilon}} \leq C_\epsilon k^{-1} \|V u\|_{L^2_{1/2+\epsilon}}.$$

Due to the boundedness of $\operatorname{supp} V$, there holds $\|V u\|_{L^2_{1/2+\epsilon}} \leq C_{D,V} \|u\|_{L^2_{-1/2-\epsilon}}$ for some constant $C_{D,V}$ depending on D and V but independent of u and ϵ. Thus, we have

$$\|\mathcal{R}_k V u\|_{L^2_{-1/2-\epsilon}} \leq C_{\epsilon,D,V} k^{-1} \|u\|_{L^2_{-1/2-\epsilon}}.$$

The proof is complete. □

In the rest of the chapter, we use k^* to represent the maximum between the quantity k_0 originated from Lemma 2.2.2 and the quantity

$$\sup_{k \in \mathbb{R}_+} \{k \,;\, \|\mathcal{R}_k V\|_{\mathcal{L}(L^2_{-1/2-\epsilon}, L^2_{-1/2-\epsilon})} \geq 1\} \;+\; 1.$$

This choice of k^* guarantees that if $k \geq k^*$, both the inequality (2.8) and the Neumann expansion $(I - \mathcal{R}_k V)^{-1} \;=\; \sum_{j \geq 0} (\mathcal{R}_k V)^j$ in $L^2_{-1/2-\epsilon}$ hold.

For the subsequent analysis we also need a local version of Lemma 2.2.3.

Lemma 2.2.4. *When $k > k_0$, we have*

$$\|\mathcal{R}_k V\|_{\mathcal{L}(L^2(D),L^2(D))} \leq C_{D,V} k^{-1}, \tag{2.9}$$

for some constant $C_{D,V}$ depending on D and V but independent of k. Moreover, for every $\varphi \in L^2(\mathbb{R}^3)$ with $\operatorname{supp}\varphi \subseteq D$, then

$$\|V\mathcal{R}_k\varphi\|_{L^2(D)} \leq C_{D,V} k^{-1}\|\varphi\|_{L^2(D)}, \tag{2.10}$$

for some constant $C_{D,V}$ depending on D and V but independent of φ and k.

Proof. For any $\varphi \in L^2(D)$, thanks to the boundedness of D we have

$$\|\mathcal{R}_k V\varphi\|_{L^2(D)} \leq C_D \|\mathcal{R}_k(V\varphi)\|_{L^2_{-1}}. \tag{2.11}$$

By Lemma 2.2.2 (letting the ϵ in Lemma 2.2.2 be $\frac{1}{2}$), we conclude that

$$\|\mathcal{R}_k(V\varphi)\|_{L^2_{-1}} \leq Ck^{-1}\|V\varphi\|_{L^2_1}. \tag{2.12}$$

By virtue of the boundedness of V, we have

$$\|V\varphi\|_{L^2_1} \leq C_{D,V}\|\varphi\|_{L^2(D)}. \tag{2.13}$$

Combining (2.11)-(2.13), we arrive at (2.9).

To prove (2.10), by Lemma 2.2.2, we have

$$\|\mathcal{R}_k\varphi\|_{L^2(D)} \leq C_D\|\mathcal{R}_k\varphi\|_{L^2_{-1}} \leq C_D k^{-1}\|\varphi\|_{L^2_1} \leq C_D k^{-1}\|\varphi\|_{L^2(D)}.$$

Therefore,

$$\|V\mathcal{R}_k\varphi\|_{L^2(D)} \leq \|V\|_{L^\infty(D)} \cdot \|\mathcal{R}_k\varphi\|_{L^2(D)} \leq C_{D,V} k^{-1}\|\varphi\|_{L^2(D)}.$$

The proof is complete. □

Lemma 2.2.5. *Assume X and X_n $(n = 1, 2, \cdots)$ be complex-valued random variables, then*

$$X_n \to X \ a.s. \quad \text{if and only if} \quad \lim_{K_0 \to +\infty} P\Big(\bigcup_{j \geq K_0} \{|X_j - X| \geq \epsilon\}\Big) = 0 \ \ \forall \epsilon > 0.$$

The proof of Lemma 2.2.5 can be found in [15, Lemma 9.2.4].

Lemma 2.2.6 (Isserlis' Theorem [42])**.** *Suppose $(X_1, \ldots, X_{2n})$ is a zero-mean multivariate normal random vector, then*

$$\mathbb{E}(X_1 X_2 \cdots X_{2n}) = \sum \prod \mathbb{E}(X_i X_j), \quad \mathbb{E}(X_1 X_2 \cdots X_{2n-1}) = 0.$$

Specially,

$$\mathbb{E}(\, X_1 X_2 X_3 X_4 \,) = \mathbb{E}(X_1 X_2)\, \mathbb{E}(X_3 X_4) + \mathbb{E}(X_1 X_3)\, \mathbb{E}(X_2 X_4) + \mathbb{E}(X_1 X_4)\, \mathbb{E}(X_2 X_3).$$

The proof of Lemma 2.2.6 can be found in [42]. Lemma 2.2.5 is the probabilistic foundation of our single realization recovery result, and Lemma 2.2.6 is called Isserlis' Theorem.

Chapter 3

Schrödinger operator with Gaussian white noise source

3.1 Introduction

In this chapter, we are mainly concerned with the following random Schrödinger system

$$
\begin{cases}
(-\Delta - E + V(x))u(x, E, d, \omega) = f(x) + \sigma(x)\dot{B}_x(\omega), \quad x \in \mathbb{R}^3, & \text{(3.1a)} \\
u(x, E, d, \omega) = \alpha e^{\mathrm{i}\sqrt{E}x\cdot d} + u^{sc}(x, E, d, \omega), & \text{(3.1b)} \\
\lim_{r\to\infty} r\left(\dfrac{\partial u^{sc}}{\partial r} - \mathrm{i}\sqrt{E}u^{sc}\right) = 0, \quad r := |x|, & \text{(3.1c)}
\end{cases}
$$

where $f(x)$ and $\sigma(x)$ in (3.1a) are the expectation and standard variance of the source term, $d \in \mathbb{S}^2 := \{x \in \mathbb{R}^3\,;\, |x| = 1\}$ signifies the impinging direction of the incident plane wave, and $E \in \mathbb{R}_+$ is the energy level. In (3.1b), α takes the value of either 0 or 1 to incur or suppress the presence of the incident wave, respectively. In the sequel, we follow the convention to replace E with k^2, namely $k := \sqrt{E} \in \mathbb{R}_+$, which can be understood as the wave number. The limit in (3.1c) is the Sommerfeld Radiation Condition (SRC) [13] that characterizes the outgoing nature of the scattered wave field u^{sc}. The random system (3.1) describes the quantum scattering associated with a potential V and a random active source (f, σ) at the energy level k^2.

In the system (3.1), the random parameter ω belongs to Ω with $(\Omega, \mathcal{F}, \mathbb{P})$ signifying a complete probability space. The term $\dot{B}_x(\omega)$ denotes the three-dimensional spatial Gaussian white noise [15]. The random part $\sigma(x)\dot{B}_x(\omega)$ within the source term in (3.1a) is an ideal mathematical model for noises arising from real world applications [15]. We note that the $\sigma^2(x)$ gives the intensity of the randomness of the source at the point x, and can be understood as the variance of $\sigma(x)\dot{B}(x,\omega)$. In what follows, we call $\sigma^2(x)$ the variance function. The statistical information of a single zero-mean Gaussian white noise is encoded in its variance function [44]. In this chapter, we are mainly concerned with the recovery of the variance and expectation of the random source as well as the potential function in (3.1) by the associated scattering measurements as described in what follows.

In order to study the corresponding inverse problems, one needs to have a thorough understanding of the direct scattering problem. In the deterministic case with $\sigma \equiv 0$, the scattering system (3.1) is well understood; see, e.g., [13,19]. There exists a unique solution $u^{sc} \in H^1_{loc}(\mathbb{R}^3)$, and moreover there holds the following asymptotic expansion as $|x| \to \infty$,

$$u^{sc}(x) = \frac{e^{\mathrm{i}kr}}{r} u^\infty(\hat{x}, k, d) + \mathcal{O}\left(\frac{1}{r^2}\right),$$

where $\hat{x} := x/|x| \in \mathbb{S}^2$. The term u^∞ is referred to as the far-field pattern, which encodes the information of the potential V and the source f. In principle, we shall show that the random scattering system (3.1) is also well-posed in a proper sense and possesses a far-field pattern. To that end, throughout the rest of the chapter, we assume that σ^2, V, f belong to $L^\infty(\mathbb{R}^3; \mathbb{R})$, respectively, and that they are compactly supported in a fixed bounded domain $D \subset \mathbb{R}^3$ containing the origin. Under the aforementioned regularity assumption, we establish that the following mapping of the direct problem (**DP**) is well-posed in a proper sense,

$$\mathbf{DP} \;\; : \;\; (\sigma, V, f) \to \{u^{sc}(\hat{x}, k, d, \omega), u^\infty(\hat{x}, k, d, \omega)\,;\, \omega \in \Omega,\, \hat{x} \in \mathbb{S}^2, k > 0,\, d \in \mathbb{S}^2\}. \tag{3.2}$$

The well-posedness of the direct scattering problem paves the way for our further study of the inverse problem (**IP**). In **IP**, we are concerned with the recoveries of the three unknowns σ^2, V, f in a *sequential* way, by knowledge of the associated far-field pattern measurements $u^\infty(\hat{x}, k, d, \omega)$. By sequential, we mean the σ^2, V, f

are recovered by the corresponding data sets one-by-one. In addition to this, in the recovery procedure, both the *passive* and *active* measurements are utilized. When $\alpha = 0$, the incident wave is suppressed and the scattering is solely generated by the unknown source. The corresponding far-field pattern is thus referred to as the passive measurement. In this case, the far-field pattern is independent of the incident direction d, and we denote it as $u^\infty(\hat{x}, k, \omega)$. When $\alpha = 1$, the scattering is generated by both the active source and the incident wave, and the far-field pattern is referred to as the active measurement, denoted as $u^\infty(\hat{x}, k, d, \omega)$. Under these settings, we formulate our **IP** as

$$\mathbf{IP} \;:\; \begin{cases} \mathcal{M}_1(\omega) := \{u^\infty(\hat{x}, k, \omega)\,;\, \forall \hat{x} \in \mathbb{S}^2,\, \forall k \in \mathbb{R}_+\} & \to \; \sigma^2, \\ \mathcal{M}_2(\omega) := \{u^\infty(\hat{x}, k, d, \omega)\,;\, \forall \hat{x} \in \mathbb{S}^2,\, \forall k \in \mathbb{R}_+,\, \forall d \in \mathbb{S}^2\} & \to \; V, \\ \mathcal{M}_3 := \{u^\infty(\hat{x}, k, d, \omega)\,;\, \forall \hat{x} \in \mathbb{S}^2,\, \forall k \in \mathbb{R}_+,\, d \text{ fixed},\, \forall \omega \in \Omega\,\} & \to \; f. \end{cases} \tag{3.3}$$

The data set $\mathcal{M}_1(\omega)$ (abbr. $\mathcal{M}_1$) corresponds to the passive measurement ($\alpha = 0$), while the data sets $\mathcal{M}_2(\omega)$ (abbr. $\mathcal{M}_2$) and $\mathcal{M}_3$ correspond to the active measurements ($\alpha = 1$). Different random sample ω gives different data sets $\mathcal{M}_1$ and $\mathcal{M}_2$. All of the σ^2, V, f in the **IP** are assumed to be unknown, and our study shows that the data sets $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ can recover σ^2, V, f, respectively. The mathematical arguments of our study are constructive and we derive explicitly recovery formulas, which can be employed for numerical reconstruction in future work.

In the aforementioned **IP**, we are particularly interested in the case with a single realization, namely the sample ω is fixed in the recovery of σ^2 and V in (3.3). Intuitively, a particular realization of $\dot{B}_x$ provides little information about the statistical properties of the random source. However, our study indicates that a *single realization* of the far-field measurement can be used to uniquely recover the variance function and the potential in certain scenarios. A crucial assumption to make the single realization recovery possible is that the randomness is independent of the wave number k. Indeed, there are assorted applications in which the randomness changes slowly or is independent of time [11, 32], and by Fourier transforming into the frequency domain, they actually correspond to the aforementioned situation. The single realization recovery has been studied in the literature; see, e.g., [11, 31, 32]. The idea of this chapter is mainly motivated by [11].

There are abundant literatures for the inverse scattering problem associated with either the passive or active measurements. Given an known potential, the recovery of an unknown source term by the corresponding passive measurement is referred to as the inverse source problem. We refer to [4–6, 12, 18, 24–26, 28, 48, 51] and the references therein for both theoretical uniqueness/stability results and computational methods for the inverse source problem in the deterministic setting, namely $\sigma \equiv 0$. The authors are also aware of some study on the inverse source problem concerning the recovery of a random source [33, 34]. In [33], the homogeneous Helmholtz system with a random source is studied. Compared with [33], our system (3.1) comprises of both unknown source and unknown potential, which make the corresponding study radically more challenging.

The determination of a random source by the corresponding passive measurement was also recently studied in [3, 39, 50], and the determination of a random potential by the corresponding active measurement was established in [11]. We also refer to [31] and the references therein for more relevant studies on the determination of a random potential. The simultaneous recovery of an unknown source and its surrounding potential was also investigated in the literature. In [27, 38], motivated by applications in thermo- and photo-acoustic tomography, the simultaneous recovery of an unknown source and its surrounding medium parameter was considered. The simultaneous recovery study in [27, 38] was confined to the deterministic setting and associated mainly with the passive measurement.

In this chapter, we consider the recovery of an unknown random source and an unknown potential term associated with the Schrödinger system (3.1). The major novelty of our unique recovery results compared to those existing ones in the literature is that on the one hand, both the random source and the potential are unknown, and on the other hand, we use both passive and active measurements for the unique recovery. We established three unique recovery results.

Theorem 3.1.1. *Without knowing V and f in system* (3.1)*, the data set $\mathcal{M}_1$ can recover σ^2 almost surely.*

Remark 3.1.1. Theorem 3.1.1 implies that the variance function can be uniquely recovered without *a priori* knowledge of f or V. Moreover, since the passive measurement $\mathcal{M}_1$ is used, Theorem 3.1.1 indicates that the variance function can be uniquely recovered by a single realization of the passive scattering measurement. Moreover, for the sake of simplicity, we set the wave number k in the definition of $\mathcal{M}_1$ to be running over all positive real numbers. But in practice, it is enough to let k be greater than any fixed positive number. This remark equally applies to Theorem 3.1.2.

Theorem 3.1.2. *Without knowing σ and f in system* (3.1)*, the data set $\mathcal{M}_2$ uniquely recovers the potential V.*

Remark 3.1.2. Theorem 3.1.2 shows that the potential V can be uniquely recovered without knowing the random source, namely σ and f. Moreover, we only make use of a single realization of the active scattering measurement.

Theorem 3.1.3. *In system* (3.1)*, suppose that σ is unknown and the potential V is known in advance. Then there exists a positive constant C that depends only on D such that if $\|V\|_{L^\infty(\mathbb{R}^3)} < C$, the data set $\mathcal{M}_3$ can uniquely recover the expectation f.*

The rest of the chapter is outlined as follows. In Section 3.2, we present the mathematical analysis of the forward scattering problem given in (3.1). Section 3.3 establishes some asymptotic estimates, which are of key importance in the recovery of the variance function. In Section 3.4, we prove the first recovery result of the variance function with a single realization of the passive scattering measurement. Section 3.5 is devoted to the second and third recovery results of the potential and the random source.

3.2 Mathematical analysis of the direct problem

In this section, the uniqueness and existence of a *mild solution* is established for the system (3.1). Before analyzing the direct problem, some preparations are made in the beginning. In Section 3.2.1, we introduce some preliminaries which are used throughout the rest of the chapter. Some technical lemmas that are necessary for the analysis of both the direct and inverse problems are presented in Section 3.2.2. In Section 3.2.3, we give the well-posedness of the direct problem.

3.2.1 Preliminaries

Let us first introduce the generalized Gaussian white noise $\dot{B}_x(\omega)$ [30]. To give a brief introduction, we write $\dot{B}_x(\omega)$ temporarily as $\dot{B}(x,\omega)$. It is known that $\dot{B}(\cdot,\omega) \in H_{loc}^{-3/2-\epsilon}(\mathbb{R}^3)$ almost surely for any $\epsilon \in \mathbb{R}_+$ [30]. Then $\dot{B}\colon \omega \in \Omega \mapsto \dot{B}(\cdot,\omega) \in \mathscr{D}'(D)$ defines a map from the probability space to the space of the generalized functions. Here, $\mathscr{D}(D)$ signifies the space consisting of smooth functions that are compactly supported in D, and $\mathscr{D}'(D)$ signifies its dual space. For any $\varphi \in \mathscr{D}(D)$, $\dot{B}\colon \omega \in \Omega \mapsto \langle \dot{B}(x,\omega), \varphi(x)\rangle \in \mathbb{R}$ is assumed to be a Gaussian random variable with zero-mean and $\int_D |\varphi(x)|^2 \,\mathrm{d}x$ as its variance. We also recall that a function ψ in $L^1_{loc}(\mathbb{R}^n)$ defines a distribution through $\langle \psi, \varphi\rangle = \int_{\mathbb{R}^n} \psi(x)\varphi(x)\,\mathrm{d}x$ [11]. Then $\dot{B}(x,\omega)$ satisfies:

$$\langle \dot{B}(\cdot,\omega), \varphi(\cdot)\rangle \sim \mathcal{N}(0, \|\varphi\|^2_{L^2(D)}), \quad \forall \varphi \in \mathscr{D}(D).$$

Moreover, the covariance of the $\dot{B}(x,\omega)$ is assumed to satisfy the following property. For every φ, ψ in $\mathscr{D}(D)$, the covariance between $\langle \dot{B}(\cdot,\omega), \varphi\rangle$ and $\langle \dot{B}(\cdot,\omega), \psi\rangle$ is defined as $\int_D \varphi(x)\psi(x)\,\mathrm{d}x$:

$$\mathbb{E}\big(\langle \dot{B}(\cdot,\omega), \varphi\rangle \langle \dot{B}(\cdot,\omega), \psi\rangle\big) := \int_D \varphi(x)\psi(x)\,\mathrm{d}x. \tag{3.4}$$

These aforementioned definitions can be generalized to the case where $\varphi, \psi \in L^2(D)$ by the density arguments. The $\delta(x)\dot{B}(x,\omega)$ is defined as

$$\delta(x)\dot{B}(x,\omega)\colon \varphi \in L^2(D) \mapsto \langle \dot{B}(\cdot,\omega), \delta(\cdot)\varphi(\cdot)\rangle \in \mathbb{R}. \tag{3.5}$$

Similar to the (2.1), we define $\mathcal{R}_k(\delta\dot{B}_x)(\omega)$ as

$$\mathcal{R}_k(\delta\dot{B}_x)(\omega) := \langle \dot{B}(\cdot,\omega), \delta(\cdot)\Phi(x,\cdot)\rangle, \tag{3.6}$$

for any $\delta \in L^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\delta \subseteq D$. We write $\mathcal{R}_k(\delta\dot{B}_x)(\omega)$ as $\mathcal{R}_k(\delta\dot{B}_x)$ for short. We may also write $\mathcal{R}_k(\delta\dot{B}_x)$ as $\int_{\mathbb{R}^3} \Phi_k(x,y)\delta(y)\dot{B}_y\,\mathrm{d}y$ or $\int_{\mathbb{R}^3} \Phi_k(x,y)\delta(y)\,\mathrm{d}B_y$. We may omit the subscript x in $\mathcal{R}_k(\delta\dot{B}_x)$ if it is clear in the context.

Define $M(x) = \sup_{y\in D} |x-y|$, and $\operatorname{diam} D := \sup_{x,y\in D} |x-y|$, where D is the bounded domain containing $\operatorname{supp}\sigma$, $\operatorname{supp}V$, $\operatorname{supp}f$ and the origin. Thus we have $M(0) \leq \operatorname{diam} D < \infty$. It can be verified that

$$\{y - x \in \mathbb{R}^3; |x| \leq 2M(0), y \in D\} \subseteq \{z \in \mathbb{R}^3; |z| \leq 3 \operatorname{diam} D\}. \tag{3.7}$$

This is because $|y - x| \leq |y| + |x| \leq \operatorname{diam} D + 2M(0) \leq 3\operatorname{diam} D$.

3.2.2 Several technical lemmas

Lemma 3.2.1 shows some basic properties of $\mathcal{R}_k(\sigma \dot{B}_x)$ defined in (3.6).

Lemma 3.2.1. *We have*

$$\mathcal{R}_k(\sigma \dot{B}_x) \in L^2_{-1/2-\epsilon} \qquad a.s. \ .$$

Moreover, we have

$$\mathbb{E}\|\mathcal{R}_k(\sigma \dot{B}_x)\|_{L^2(D)} < C < +\infty$$

for some constant C independent of k.

Proof. From (3.6), (3.5) and (3.4), one can compute,

$$\begin{aligned}
\mathbb{E}(\|\mathcal{R}_k(\sigma \dot{B}_x)\|^2_{L^2_{-1/2-\epsilon}}) &= \int_{\mathbb{R}^3} \langle x \rangle^{-1-2\epsilon} \mathbb{E}\big(\langle \dot{B}(\cdot,\omega), \sigma(\cdot)\Phi(x,\cdot)\rangle \langle \dot{B}(\cdot,\omega), \sigma(\cdot)\overline{\Phi}(x,\cdot)\rangle\big) \,\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \langle x \rangle^{-1-2\epsilon} \int_D \sigma^2(y) \frac{1}{16\pi^2 |x-y|^2} \,\mathrm{d}y \,\mathrm{d}x \\
&\leq C\|\sigma\|^2_{L^\infty(D)} \int_{\mathbb{R}^3} \langle x \rangle^{-1-2\epsilon} \int_D |x-y|^{-2} \,\mathrm{d}y \,\mathrm{d}x.
\end{aligned}$$

By arguments similar to the ones used in the proof of Lemma 2.2.1 we arrive at

$$\mathbb{E}(\|\mathcal{R}_k(\sigma \dot{B}_x)\|^2_{L^2_{-1/2-\epsilon}}) \leq C_D < +\infty, \tag{3.8}$$

for some constant C_D depending on D but not on k. By the Hölder inequality applied to the probability measure, (3.8) gives

$$\mathbb{E}(\|\mathcal{R}_k(\sigma \dot{B}_x)\|_{L^2_{-1/2-\epsilon}}) \leq [\mathbb{E}(\|\mathcal{R}_k(\sigma \dot{B}_x)\|^2_{L^2_{-1/2-\epsilon}})]^{1/2} \leq C_D^{1/2} < +\infty, \tag{3.9}$$

for some constant C_D independent of k. The inequality (3.9) gives

$$\mathcal{R}_k(\sigma \dot{B}_x) \in L^2_{-1/2-\epsilon} \quad \text{a.s. .}$$

By replacing $\mathbb{R}^3$ with D and deleting all the terms $\langle x \rangle^{-1-2\epsilon}$ in the derivations above, one arrives at $\mathbb{E}\|\mathcal{R}_k(\sigma \dot{B}_x)\|_{L^2(D)} < +\infty$. The proof is done. □

3.2.3 The well-posedness of the direct problem

For a particular realization of the random sample $\omega \in \Omega$, the term $\dot{B}_x(\omega)$, treated as a function of the spatial argument x, could be very rough. The roughness of this term could make these classical second-order elliptic PDEs theories invalid to (3.1). Due to this reason, the notion of the *mild solution* is introduced for random PDEs (cf. [3]). In what follows, we adopt the mild solution in our problem setting, and we show that this mild solution and the corresponding far-field pattern are well-posed in a proper sense.

Reformulating (3.1) into the Lippmann-Schwinger equation formally (cf. [13]), we have

$$(I - \mathcal{R}_k V)u = \alpha \cdot u^i - \mathcal{R}_k f - \mathcal{R}_k(\sigma \dot{B}_x), \tag{3.10}$$

where the term $\mathcal{R}_k(\sigma \dot{B}_x)$ is defined by (3.6). Recall that $u^{sc} = u - \alpha \cdot u^i$. From (3.10) we have

$$(I - \mathcal{R}_k V)u^{sc} = \alpha \mathcal{R}_k V u^i - \mathcal{R}_k f - \mathcal{R}_k(\sigma \dot{B}_x). \tag{3.11}$$

Theorem 3.2.1. *When $k > k^*$, there exists a unique stochastic process $u^{sc}(\cdot, \omega) \colon \mathbb{R}^3 \to \mathbb{C}$ such that $u^{sc}(x)$ satisfies (3.11) a.s. , and $u^{sc}(\cdot, \omega) \in L^2_{-1/2-\epsilon}$ a.s. for any $\epsilon \in \mathbb{R}_+$. Moreover, we have*

$$\|u^{sc}(\cdot, \omega)\|_{L^2_{-1/2-\epsilon}} \lesssim \|\alpha V u^i\|_{L^2_{1/2+\epsilon}} + \|f\|_{L^2_{1/2+\epsilon}} + \|\mathcal{R}_k(\sigma \dot{B}_x)\|_{L^2_{-1/2-\epsilon}}. \tag{3.12}$$

Then we call $u(x) := u^{sc} + \alpha \cdot u^i(x)$ the mild solution *to the random scattering system (3.1).*

Proof. By Lemmas 2.2.1, 2.2.3 and 3.2.1, we see

$$F := \alpha \mathcal{R}_k V u^i - \mathcal{R}_k f - \mathcal{R}_k(\sigma \dot{B}_x) \in L^2_{-1/2-\epsilon}.$$

Note that $k > k^*$, so the term $\sum_{j=0}^{\infty}(\mathcal{R}_k V)^j$ is well-defined, thus the term $\sum_{j=0}^{\infty}(\mathcal{R}_k V)^j F$ belongs to $L^2_{-1/2-\epsilon}$. Because $\sum_{j=0}^{\infty}(\mathcal{R}_k V)^j = (I - \mathcal{R}_k V)^{-1}$, we see $(I - \mathcal{R}_k V)^{-1} F \in L^2_{-1/2-\epsilon}$. Let $u^{sc} := (I - \mathcal{R}_k V)^{-1} F \in L^2_{-1/2-\epsilon}$, then u^{sc} is the unique solution of (3.11). That is, the existence of the mild solution is proved The uniqueness of the mild solution follows from the invertibility of the operator $(I - \mathcal{R}_k V)^{-1}$.

From (3.11) and Lemmas 2.2.2-2.2.3, we have

$$\begin{aligned}
\|u^{sc}(\cdot, \omega)\|_{L^2_{-1/2-\epsilon}} &= \|(I - \mathcal{R}_k V)^{-1}(\alpha \mathcal{R}_k V u^i - \mathcal{R}_k f - \mathcal{R}_k(\sigma \dot{B}_x))\|_{L^2_{-1/2-\epsilon}} \\
&\leq \sum_{j \geq 0} \|\mathcal{R}_k V\|^j_{\mathcal{L}(L^2_{-1/2-\epsilon}, L^2_{-1/2-\epsilon})} \cdot \|\alpha \mathcal{R}_k V u^i - \mathcal{R}_k f - \mathcal{R}_k(\sigma \dot{B}_x)\|_{L^2_{-1/2-\epsilon}} \\
&\leq C(\|\alpha \mathcal{R}_k V u^i\|_{L^2_{-1/2-\epsilon}} + \|\mathcal{R}_k f\|_{L^2_{-1/2-\epsilon}} + \|\mathcal{R}_k(\sigma \dot{B}_x)\|_{L^2_{-1/2-\epsilon}}) \\
&\leq C(\|\alpha V u^i\|_{L^2_{1/2+\epsilon}} + \|f\|_{L^2_{1/2+\epsilon}} + \|\mathcal{R}_k(\sigma \dot{B}_x)\|_{L^2_{-1/2-\epsilon}}).
\end{aligned}$$

Therefore (3.12) is proved. The proof is complete. □

Next we show that the far-field pattern is well-defined in the L^2 sense. From (3.11) we derive that

$$\begin{aligned}
u^{sc} &= (I - \mathcal{R}_k V)^{-1}\big(\alpha \mathcal{R}_k V u^i - \mathcal{R}_k f - \mathcal{R}_k(\sigma \dot{B}_x)\big) \\
&= \mathcal{R}_k (I - V\mathcal{R}_k)^{-1}(\alpha V u^i - f - \sigma \dot{B}_x).
\end{aligned}$$

Therefore, we define the far-field pattern of the scattered wave $u^{sc}(x, k, d, \omega)$ formally in the following manner,

$$u^{\infty}(\hat{x}, k, d, \omega) := \frac{1}{4\pi} \int_D e^{-ik\hat{x}\cdot y}(I - V\mathcal{R}_k)^{-1}(\alpha V u^i - f - \sigma \dot{B}_y)\, \mathrm{d}y, \quad \hat{x} \in \mathbb{S}^2. \quad (3.13)$$

The another result concerning the **DP** is Theorem 3.2.2, showing that $u^{\infty}(\hat{x}, k, d, \omega)$ is well-defined.

Theorem 3.2.2. *Define the far-field pattern of the mild solution as in* (3.13)*. When* $k > k^*$*, there is a subset* $\Omega_0 \subset \Omega$ *with zero measure* $\mathbb{P}(\Omega_0) = 0$*, such that there holds*

$$u^\infty(\hat{x}, k, d, \omega) \in L^2(\mathbb{S}^2), \quad \forall\, \omega \in \Omega \backslash \Omega_0.$$

Proof of Theorem 3.2.2. By Lemma 2.2.4,

$$\|V\mathcal{R}_k\|_{\mathcal{L}(L^2(D),L^2(D))} \leq Ck^{-1} < 1$$

when k is sufficiently large. Therefore we have,

$$\begin{aligned} |u^\infty(\hat{x})|^2 \lesssim |D|^2 \cdot & \int_D |\sum_{j\geq 0} (V\mathcal{R}_k)^j (\alpha V u^i - f)|^2 \,\mathrm{d}y \\ & + \big|\int_D e^{-ik\hat{x}\cdot y} \sum_{j\geq 1} (V\mathcal{R}_k)^j (\sigma \dot{B}_y) \,\mathrm{d}y\big|^2 \\ & + \big|\int_D e^{-ik\hat{x}\cdot y} \sigma \dot{B}_y \,\mathrm{d}y\big|^2 \\ =:\, & f_1(\hat{x}, k) + f_2(\hat{x}, k, \omega) + f_3(\hat{x}, k, \omega). \end{aligned} \tag{3.14}$$

We next derive estimates on each term f_j $(j = 1, 2, 3)$ defined in (3.14). For f_1, we have

$$f_1(\hat{x}, k) \leq C|D|^2 \cdot (\sum_{j\geq 0} k^{-j} \|\alpha V u^i - f\|_{L^2(D)})^2 \leq C|D|^2 (\|V\|_{L^2(D)} + \|f\|_{L^2(D)})^2. \tag{3.15}$$

For f_2, by utilizing (2.10), one can compute

$$f_2(\hat{x}, k, \omega) \leq C \int_D |\sum_{j\geq 0} (V\mathcal{R}_k)^j V\mathcal{R}_k(\sigma \dot{B}_y)|^2 \,\mathrm{d}y \leq C\big(\sum_{j\geq 0} k^{-j} \|V\mathcal{R}_k(\sigma \dot{B}_y)\|_{L^2(D)}\big)^2. \tag{3.16}$$

By virtue of the boundedness of the support of V, we can continue (3.16) as

$$f_2(\hat{x}, k, \omega) \leq C\big(\sum_{j\geq 0} k^{-j} \|V\mathcal{R}_k(\sigma \dot{B}_y)\|_{L^2_{-1/2-\epsilon}}\big)^2 \leq C_V \|\mathcal{R}_k(\sigma \dot{B}_y)\|^2_{L^2_{-1/2-\epsilon}}. \tag{3.17}$$

By (3.4), the expectation of $f_3(\hat{x}, k, \omega)$ is

$$\mathbb{E}f_3(\hat{x}, k, \omega) = \mathbb{E}|\langle \dot{B}_y, e^{-ik\hat{x}\cdot y}\sigma(y)\rangle|^2 = \int_D |\sigma(y)|^2 \,\mathrm{d}y. \tag{3.18}$$

Combining (3.8), (3.14)-(3.15) and (3.17)-(3.18), we arrive at

$$\begin{aligned}\mathbb{E}|u^\infty(\hat{x})|^2 \leq C|D|^2(\|V\|_{L^2(D)} + \|f\|_{L^2(D)})^2 + C_V\mathbb{E}(\|\mathcal{R}_k(\sigma\dot{B}_y)\|^2_{L^2_{-1/2-\epsilon}}) \\ + \int_D |\sigma(y)|^2 \,\mathrm{d}y \\ \leq C < +\infty \end{aligned} \tag{3.19}$$

for some positive constant C. From (3.19) we arrive at

$$\mathbb{E}\int_{\mathbb{S}^2} |u^\infty(\hat{x})|^2 \,\mathrm{d}S \leq C < +\infty. \tag{3.20}$$

Our conclusion follows from (3.20) immediately. □

3.3 Some asymptotic estimates

This section is devoted to some preparations of the recovery of the variance function. To recovery $\sigma^2(x)$, only the passive far-field patterns are utilized. Therefore, throughout this section, the α in (3.1) is set to be 0. Motivated by [11], our recovery formula of the variance function is of the form

$$\frac{1}{K}\int_K^{2K} \overline{u^\infty(\hat{x}, k, \omega)} \cdot u^\infty(\hat{x}, k+\tau, \omega)\,\mathrm{d}k. \tag{3.21}$$

After expanding $u^\infty(\hat{x}, k, \omega)$ in the form of Neumann series, there will be several crossover terms in (3.21) which decay in different rates in terms of K. In this section, we focus on the asymptotic estimates of these terms, which pave the way to the recovery of $\sigma^2(x)$. The recovery of $\sigma^2(x)$ is presented in the next section.

To start out, we write

$$u_1^\infty(\hat{x}, k, \omega) := u^\infty(\hat{x}, k, \omega) - \mathbb{E}u^\infty(\hat{x}, k). \tag{3.22}$$

Note that u_1^∞ is independent of the incident direction d. Assume that $k > k^*$, then the operator $(I - \mathcal{R}_k V)^{-1}$ has the Neumann expansion $\sum_{j=0}^{+\infty} (\mathcal{R}_k V)^j$. By (3.13) and (3.22) we have

$$\begin{aligned} u_1^\infty(\hat{x}, k, \omega) &= \frac{-1}{4\pi} \sum_{j=0}^{+\infty} \int_D e^{-ik\hat{x}\cdot y} (\mathcal{R}_k V)^j (\sigma \dot{B}_y) \, \mathrm{d}y, \quad \hat{x} \in \mathbb{S}^2 \\ &:= \frac{-1}{4\pi} \big[F_0(k, \hat{x}) + F_1(k, \hat{x}) \big], \end{aligned} \tag{3.23}$$

where

$$\begin{cases} F_0(k, \hat{x}, \omega) := \displaystyle\int_D e^{-ik\hat{x}\cdot y} (\sigma \dot{B}_y) \, \mathrm{d}y, \\ F_1(k, \hat{x}, \omega) := \displaystyle\sum_{j \geq 1} \int_D e^{-ik\hat{x}\cdot y} (V \mathcal{R}_k)^j (\sigma \dot{B}_y) \, \mathrm{d}y. \end{cases} \tag{3.24}$$

Meanwhile, the expectation of the far-field pattern $\mathbb{E}u^\infty$ is

$$\mathbb{E}u^\infty(\hat{x}, k) = \frac{-1}{4\pi} \int_D e^{-ik\hat{x}\cdot y} (I - V\mathcal{R}_k)^{-1}(f) \, \mathrm{d}y, \quad \hat{x} \in \mathbb{S}^2. \tag{3.25}$$

Lemma 3.3.1. *We have*

$$\lim_{k \to +\infty} |\mathbb{E}u^\infty(\hat{x}, k)| = 0 \quad \textit{uniformly in } \hat{x} \in \mathbb{S}^2.$$

Proof of Lemma 3.3.1. Due to the fact that $f \in L^\infty(D) \subset L^2(D)$, we know

$$\forall \, \epsilon > 0, \, \exists \, \varphi_\epsilon \in \mathscr{D}(D), \text{ s.t. } \|f - \varphi_\epsilon\|_{L^2(D)} < \epsilon / (2|D|^{\frac{1}{2}}). \tag{3.26}$$

Recall that $k > k^*$, so $(I - V\mathcal{R}_k)^{-1}$ equals to $I + \sum_{j=1}^{+\infty} (V\mathcal{R}_k)^j$. By (3.26) and Lemma 2.2.4 and utilizing the stationary phase lemma, one can deduce as follows,

$$\begin{aligned} |\mathbb{E}u^\infty(\hat{x}, k)| &\lesssim \Big| \int_D e^{-ik\hat{x}\cdot y} \varphi_\epsilon(y) \, \mathrm{d}y \Big| + \Big| \int_D e^{-ik\hat{x}\cdot y} \big[f(y) - \varphi_\epsilon(y) + \big(\sum_{j \geq 1} (V\mathcal{R}_k)^j f \big)(y) \big] \, \mathrm{d}y \Big| \\ &\lesssim \Big| k^{-2} \int_D e^{-ik\hat{x}\cdot y} \cdot \Delta\varphi_\epsilon(y) \, \mathrm{d}y \Big| + |D|^{\frac{1}{2}} \cdot \|f - \varphi_\epsilon + \sum_{j \geq 1} (V\mathcal{R}_k)^j f\|_{L^2(D)} \\ &\leq k^{-2} \cdot |D|^{\frac{1}{2}} \cdot \|\Delta\varphi_\epsilon\|_{L^2(D)} + |D|^{\frac{1}{2}} \cdot \big(\epsilon / (2|D|^{\frac{1}{2}}) + C \sum_{j \geq 1} k^{-j} \|f\|_{L^2(D)} \big) \\ &= k^{-2} \cdot |D|^{\frac{1}{2}} \|\Delta\varphi_\epsilon\|_{L^2(D)} + \epsilon/2 + C(k-1)^{-1} \cdot \|f\|_{L^2(D)}. \end{aligned} \tag{3.27}$$

Write $\mathcal{K} := \max\{K_0, \frac{2}{\sqrt{\epsilon}}|D|^{\frac{1}{4}}\|\Delta\varphi_\epsilon\|^{\frac{1}{2}}_{L^2(D)}, 1 + \frac{4C}{\epsilon}\|f\|_{L^2(D)}\}$. From (3.27) we have

$$\forall\, k > \mathcal{K}, \quad |\mathbb{E}u^\infty(\hat{x}, k)| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon, \quad \text{uniformly for } \forall \hat{x} \in \mathbb{S}^2.$$

Since the ϵ is taken arbitrarily, the conclusion follows. □

By substituting (3.22)-(3.25) into (3.21), we obtain several crossover terms among F_0, F_1 and $\mathbb{E}u^\infty$. The asymptotic estimates of these crossover terms are the main purpose of Sections 3.3.1 and 3.3.2. Section 3.3.1 focuses on the estimate of the leading order term while the estimates of the higher order terms are presented in Section 3.3.2.

3.3.1 Asymptotic estimates of the leading order term

Lemma 3.3.2 below is the asymptotic estimate of the crossover leading order term. By utilizing the ergodicity, the result of Lemma 3.3.2 is also statistically stable. To prove Lemma 3.3.2, we need Lemmas 2.2.5, 2.2.6 and 3.3.3. In order to keep our arguments flowing, we postpone Lemma 3.3.3 until we finish Lemma 3.3.2.

Recall the notation $\{K_j\} \in P(t)$ defined in Section 2.1. Lemma 3.3.2 gives the asymptotic estimates of the crossover leading order term.

Lemma 3.3.2. *Write*

$$X_{0,0}(K, \tau, \hat{x}, \omega) = \frac{1}{K}\int_K^{2K} \overline{F_0(k, \hat{x}, \omega)} \cdot F_0(k + \tau, \hat{x}, \omega)\,\mathrm{d}k.$$

Assume $\{K_j\} \in P(2 + \gamma)$, then for any $\tau > 0$, we have

$$\lim_{j \to +\infty} X_{0,0}(K_j, \tau, \hat{x}, \omega) = (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x}) \qquad a.s.\ .$$

We may denote $X_{0,0}(K, \tau, \hat{x}, \omega)$ as $X_{0,0}$ for short if it is clear in the context.

Proof of Lemma 3.3.2. We have

$$\mathbb{E}\big(\overline{F_0(k, \hat{x}, \omega)}F_0(k + \tau, \hat{x}, \omega)\big)$$

$$= \mathbb{E}\Big(\int_{D_y} e^{ik\hat{x}\cdot y}\sigma(y)\,\mathrm{d}B_y \cdot \int_{D_z} e^{-i(k+\tau)\hat{x}\cdot z}\sigma(z)\,\mathrm{d}B_z \Big)$$

$$= \int_D e^{ik\hat{x}\cdot y}e^{-i(k+\tau)\hat{x}\cdot y}\sigma(y)\sigma(y)\,\mathrm{d}y = (2\pi)^{3/2}\,\widehat{\sigma^2}(\tau\hat{x}). \tag{3.28}$$

From (3.28) we conclude that

$$\mathbb{E}(X_{0,0}) = \frac{1}{K}\int_K^{2K} \mathbb{E}\big(\overline{F_0(k,\hat{x},\omega)}F_0(k+\tau,\hat{x},\omega)\big)\,\mathrm{d}k = (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x}).$$

By Isserlis' Theorem and (3.28), and note that $\overline{F_j(k,\hat{x},\omega)} = F_j(-k,\hat{x},\omega)$, one can compute

$$\begin{aligned}
&\mathbb{E}\big(|X_{0,0} - (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x})|^2\big) \\
=& \frac{1}{K^2}\int_K^{2K}\int_K^{2K} \mathbb{E}\Big(\overline{F_0(k_1,\hat{x},\omega)}F_0(k_1+\tau,\hat{x},\omega)F_0(k_2,\hat{x},\omega)\overline{F_0(k_2+\tau,\hat{x},\omega)}\Big)\,\mathrm{d}k_1\,\mathrm{d}k_2 \\
&- (2\pi)^3|\widehat{\sigma^2}(\tau\hat{x})|^2 - (2\pi)^3|\widehat{\sigma^2}(\tau\hat{x})|^2 + (2\pi)^3|\widehat{\sigma^2}(\tau\hat{x})|^2 \qquad \text{(by (3.28))} \\
=& \frac{1}{K^2}\int_K^{2K}\int_K^{2K} \mathbb{E}\big(\overline{F_0(k_1,\hat{x},\omega)}F_0(k_1+\tau,\hat{x},\omega)\big)\cdot\mathbb{E}\big(F_0(k_2,\hat{x},\omega)\overline{F_0(k_2+\tau,\hat{x},\omega)}\big) \\
&+ \mathbb{E}\big(\overline{F_0(k_1,\hat{x},\omega)}F_0(k_2,\hat{x},\omega)\big)\cdot\mathbb{E}\big(F_0(k_1+\tau,\hat{x},\omega)\overline{F_0(k_2+\tau,\hat{x},\omega)}\big) \\
&+ \mathbb{E}\big(\overline{F_0(k_1,\hat{x},\omega)}F_0(-k_2-\tau,\hat{x},\omega)\big)\cdot\mathbb{E}\big(\overline{F_0(-k_1-\tau,\hat{x},\omega)}F_0(k_2,\hat{x},\omega)\big)\,\mathrm{d}k_1\,\mathrm{d}k_2 \\
&- (2\pi)^3|\widehat{\sigma^2}(\tau\hat{x})|^2 \\
=& \frac{(2\pi)^3}{K^2}\int_K^{2K}\int_K^{2K} \big(|\widehat{\sigma^2}((k_2-k_1)\hat{x})|^2 + |\widehat{\sigma^2}((k_1+k_2+\tau)\hat{x})|^2\big)\,\mathrm{d}k_1\,\mathrm{d}k_2.
\end{aligned} \tag{3.29}$$

Note that $|\widehat{\sigma^2}\big((k_1-k_2)\hat{x}\big)| = |\widehat{\sigma^2}\big(-(k_1-k_2)\hat{x}\big)|$. Combining (3.29) and Lemma 3.3.3, we have

$$\mathbb{E}\big(|X_{0,0} - (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x})|^2\big) = \mathcal{O}(K^{-1/2}), \quad K \to +\infty. \tag{3.30}$$

For any integer $K_0 > 0$, by Chebyshev's inequality and (3.30) we have

$$\begin{aligned}
&P\Big(\bigcup_{j\geq K_0}\{|X_{0,0}(K_j) - (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x})| \geq \epsilon\}\Big) \leq \frac{1}{\epsilon^2}\sum_{j\geq K_0}\mathbb{E}\big(|X_{0,0}(K_j) - (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x})|^2\big) \\
\lesssim& \frac{1}{\epsilon^2}\sum_{j\geq K_0}K_j^{-1/2} = \frac{1}{\epsilon^2}\sum_{j\geq K_0}j^{-1-\gamma/2} = \frac{2}{\epsilon^2\gamma}(K_0-1)^{-\gamma/2}.
\end{aligned} \tag{3.31}$$

Here $X_{0,0}(K_j)$ stands for $X_{0,0}(K_j,\tau,\hat{x},\omega)$. By Lemma 2.2.5, formula (3.31) implies that for any fixed $\tau \geq 0$ and fixed $\hat{x} \in \mathbb{S}^2$, we have

$$X_{0,0}(K_j, \tau, \hat{x}, \omega) \to (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x}) \quad \text{a.s. .}$$

The proof is done. □

Lemma 3.3.3 plays a critical role in the estimates of the leading order term.

Lemma 3.3.3. *Assume that $\tau \geq 0$ is fixed, then $\exists K_0 > \tau$, and K_0 is independent of $\hat{x}$, such that for all $K > K_0$, we have the following estimates:*

$$\frac{(2\pi)^3}{K^2}\int_K^{2K}\int_K^{2K} \left|\widehat{\sigma^2}((k_1 - k_2)\hat{x})\right|^2 \mathrm{d}k_1\,\mathrm{d}k_2 \leq CK^{-1/2}, \tag{3.32}$$

$$\frac{(2\pi)^3}{K^2}\int_K^{2K}\int_K^{2K} \left|\widehat{\sigma^2}((k_1 + k_2 + \tau)\hat{x})\right|^2 \mathrm{d}k_1\,\mathrm{d}k_2 \leq CK^{-1/2}, \tag{3.33}$$

for some constant C independent of τ and $\hat{x}$.

Proof of Lemma 3.3.3. Note that for every $x \in \mathbb{R}^3$, we have

$$|\widehat{\sigma^2}(x)|^2 \simeq \Big|\int_{\mathbb{R}^3} e^{-ix\cdot\xi}\sigma^2(\xi)\,\mathrm{d}\xi\Big|^2 \leq \Big(\int_{\mathbb{R}^3} |\sigma^2(\xi)|\,\mathrm{d}\xi\Big)^2 \leq \|\sigma\|^4_{L^\infty(D)} \cdot |D|^2.$$

To conclude (3.32), we make a change of variable,

$$\begin{cases} s = k_1 - k_2, \\ t = k_2. \end{cases}$$

Write $Q = \{(s,t) \in \mathbb{R}^2 \,\big|\, K \leq s + t \leq 2K,\ K \leq t \leq 2K\}$. Q is illustrated as in Figure 3.1.

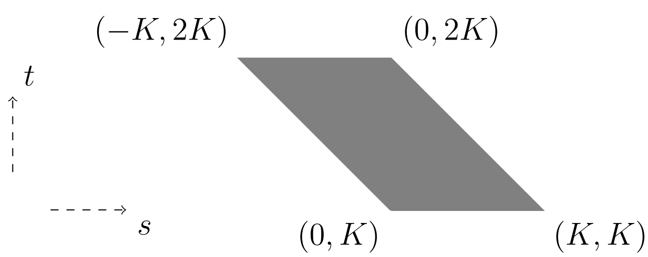

Figure 3.1: Illustration of Q

Recall that $\operatorname{supp}\sigma \subseteq D$, so we have

$$
\begin{aligned}
&\frac{1}{K^2}\int_K^{2K}\int_K^{2K} |\widehat{\sigma^2}((k_1-k_2)\hat{x})|^2 \,\mathrm{d}k_1\,\mathrm{d}k_2 = \frac{1}{K^2}\iint_Q \left|\widehat{\sigma^2}(s\hat{x})\right|^2 \mathrm{d}s\,\mathrm{d}t \\
=&\frac{1}{K^2}\int_{-K}^{0}(K+s)|\widehat{\sigma^2}(s\hat{x})|^2\,\mathrm{d}s + \frac{1}{K^2}\int_0^K (K-s)|\widehat{\sigma^2}(s\hat{x})|^2\,\mathrm{d}s \\
\simeq&\int_0^1 \Big(\int_D e^{-iKs\hat{x}\cdot y}\sigma^2(y)\,\mathrm{d}y \cdot \int_D e^{iKs\hat{x}\cdot z}\sigma^2(z)\,\mathrm{d}z\Big)\,\mathrm{d}s \\
=&\int_{(D\times D)\backslash E_\epsilon}\Big(\int_0^1 e^{iK(\hat{x}\cdot z-\hat{x}\cdot y)s}\,\mathrm{d}s\Big)\sigma^2(y)\sigma^2(z)\,\mathrm{d}y\,\mathrm{d}z \\
&\quad + \int_{E_\epsilon}\Big(\int_0^1 e^{iK(\hat{x}\cdot z-\hat{x}\cdot y)s}\,\mathrm{d}s\Big)\sigma^2(y)\sigma^2(z)\,\mathrm{d}y\,\mathrm{d}z \\
=:&A_1 + A_2,
\end{aligned}
\tag{3.34}
$$

where $E_\epsilon := \{(y,z)\in D\times D; |\hat{x}\cdot z - \hat{x}\cdot y| < \epsilon\}$. We first estimate A_1,

$$
\begin{aligned}
|A_1| &= \Big|\int_{(D\times D)\backslash E_\epsilon}\Big(\int_0^1 e^{iK(\hat{x}\cdot z-\hat{x}\cdot y)s}\,\mathrm{d}s\Big)\sigma^2(y)\sigma^2(z)\,\mathrm{d}y\,\mathrm{d}z\Big| \\
&\le \int_{(D\times D)\backslash E_\epsilon}\Big|\frac{e^{iK(\hat{x}\cdot z-\hat{x}\cdot y)}-1}{iK(\hat{x}\cdot z-\hat{x}\cdot y)}\sigma^2(y)\sigma^2(z)\Big|\,\mathrm{d}y\,\mathrm{d}z \\
&\le \frac{2}{K\epsilon}\|\sigma\|^4_{L^\infty(D)}\int_{D\times D} 1\,\mathrm{d}y\,\mathrm{d}z = \frac{2|D|^2}{K\epsilon}\|\sigma\|^4_{L^\infty(D)}.
\end{aligned}
\tag{3.35}
$$

Recall that $\operatorname{diam} D < +\infty$ and that the problem setting is in $\mathbb{R}^3$. We can estimate A_2 as

$$
\begin{aligned}
|A_2| &\le \|\sigma\|^4_{L^\infty(D)}\int_{E_\epsilon} 1\,\mathrm{d}y\,\mathrm{d}z \\
&= \|\sigma\|^4_{L^\infty(D)}\int_D\Big(\int_{y\in D\,,\,|\hat{x}\cdot z-\hat{x}\cdot y|<\epsilon} 1\,\mathrm{d}y\Big)\,\mathrm{d}z \\
&\le \|\sigma\|^4_{L^\infty(D)}\int_D 2\epsilon(\operatorname{Diam} D)^2\,\mathrm{d}z \\
&\le 2\|\sigma\|^4_{L^\infty(D)}(\operatorname{Diam} D)^2|D|\cdot\epsilon.
\end{aligned}
\tag{3.36}
$$

Set $\epsilon = K^{-1/2}$. By (3.34)-(3.36), we arrive at

$$
\frac{1}{K^2}\int_K^{2K}\int_K^{2K} |\widehat{\sigma^2}((k_1-k_2)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2 \le CK^{-1/2},
$$

for some constant C independent of $\hat{x}$.

Now we prove (3.33). Similarly, we make a change of variable:

$$\begin{cases} s = k_1 + k_2 + \tau, \\ t = k_2. \end{cases}$$

Write $Q' = \{(s,t) \in \mathbb{R}^2 \,\big|\, K \leq s - t - \tau \leq 2K,\ K \leq t \leq 2K\}$. One can compute

$$\begin{aligned} &\frac{1}{K^2} \int_K^{2K} \int_K^{2K} |\widehat{\sigma^2}((k_1 + k_2 + \tau)\hat{x})|^2 \,\mathrm{d}k_1 \,\mathrm{d}k_2 = \frac{1}{K^2} \iint_{Q'} |\widehat{\sigma^2}(s\hat{x})|^2 \,\mathrm{d}s \,\mathrm{d}s \\ =&\frac{1}{K^2} \int_{2K+\tau}^{3K+\tau} (s - 2K - \tau)|\widehat{\sigma^2}(s\hat{x})|^2 \,\mathrm{d}s + \frac{1}{K^2} \int_{3K+\tau}^{4K+\tau} (4K + \tau - s)|\widehat{\sigma^2}(s\hat{x})|^2 \,\mathrm{d}s \\ \leq&\frac{2}{K} \int_{2K-\tau}^{2K+\tau} |\widehat{\sigma^2}(s\hat{x})|^2 \,\mathrm{d}s = 2 \int_{2+\tau/K}^{4+\tau/K} |\widehat{\sigma^2}(Ks\hat{x})|^2 \,\mathrm{d}s. \end{aligned}$$

Thus when $K > \tau$,

$$\frac{1}{K^2} \int_K^{2K} \int_K^{2K} |\widehat{\sigma^2}((k_1 + k_2 + \tau)\hat{x})|^2 \,\mathrm{d}k_1 \,\mathrm{d}k_2 \leq 2 \int_2^5 |\widehat{\sigma^2}(Ks\hat{x})|^2 \,\mathrm{d}s. \tag{3.37}$$

Following the same manner as in (3.34)-(3.36), from (3.37) we arrive at (3.33). The proof is done. □

3.3.2 Asymptotic estimates of higher order terms

The asymptotic estimates of the higher order terms are presented in Lemma 3.3.4.

Lemma 3.3.4. *For every $\hat{x}_1,\ \hat{x}_2 \in \mathbb{S}^2$ and every $k_1,\ k_2 \geq k$, we have the following estimates ($j = 0, 1$) as $k \to +\infty$,*

$$\big|\mathbb{E}\big(\overline{F_j(k_1, \hat{x}_1, \omega)} \cdot F_1(k_2, \hat{x}_2, \omega)\big)\big| = \mathcal{O}(k^{-1}), \tag{3.38}$$

$$\big|\mathbb{E}\big(F_j(k_1, \hat{x}_1, \omega) \cdot F_1(k_2, \hat{x}_2, \omega)\big)\big| = \mathcal{O}(k^{-1}). \tag{3.39}$$

Proof of Lemma 3.3.4. The proof of formulas (3.39) is similar to that of (3.38), so we only present the proof of (3.38). In this proof, we may drop the arguments k, $\hat{x}$

or ω from F_j if it is clear in the context. For the notational convenience, we write

$$G_j(k,\hat{x},\omega) := \int_D e^{-ik\hat{x}\cdot y}(V\mathcal{R}_k)^j(\sigma\dot{B}_y)\,\mathrm{d}y,$$
$$r_j(k,\hat{x},\omega) := \sum_{s\geq j} G_s(k,\hat{x},\omega),$$

for $j=0,1,\cdots$. To prove (3.38) for the case where $j=0$, we first show that

$$\mathbb{E}\big(\overline{G_0(k_1,\hat{x}_1,\omega)}\cdot G_j(k_2,\hat{x}_2,\omega)\big) = \int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^j\big(e^{ik_1\hat{x}_1\cdot(\cdot)}\sigma^2\big)\,\mathrm{d}z, \quad j\geq 1. \tag{3.40}$$

This can be seen from the following computation

$$\begin{aligned}
&\mathbb{E}\big(\overline{G_0(k_1,\hat{x}_1,\omega)}\cdot G_j(k_2,\hat{x}_2,\omega)\big)\\
&= \mathbb{E}\Big(\int_D e^{ik_1\hat{x}_1\cdot y}\sigma(y)\,\mathrm{d}B_y\cdot\int_D\big[e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}(V(\cdot)\int_{D_s}\Phi(\cdot,s)\sigma(s)\,\mathrm{d}B_s)\big]\,\mathrm{d}z\Big)\\
&= \int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\Big\{V(\cdot)\,\mathbb{E}\big[\int_{D_y} e^{ik_1\hat{x}_1\cdot y}\sigma(y)\,\mathrm{d}B_y\cdot\int_{D_s}\Phi(\cdot,s)\sigma(s)\,\mathrm{d}B_s\big]\Big\}\,\mathrm{d}z\\
&= \int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\big(V(\cdot)\mathcal{R}_{k_2}(e^{ik_1\hat{x}_1\cdot(\cdot)}\sigma^2)\big)\,\mathrm{d}z\\
&= \int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^j(e^{ik_1\hat{x}_1\cdot(\cdot)}\sigma^2)\,\mathrm{d}z.
\end{aligned} \tag{3.41}$$

From (3.41), equality (3.40) is proved. Using (3.40) and Lemma 2.2.4, we have

$$\begin{aligned}
&\big|\mathbb{E}\big(\overline{F_0(k_1,\hat{x}_1,\omega)}\cdot F_1(k_2,\hat{x}_2,\omega)\big)\big|\\
\leq\ & \sum_{j\geq 1}\big|\mathbb{E}\big(G_0(k_1,\hat{x}_1,\omega)\cdot\overline{G_j(k_2,\hat{x}_2,\omega)}\big)\big|\\
=\ & \sum_{j\geq 1}\Big|\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^j\big(e^{ik_1\hat{x}_1\cdot(\cdot)}\sigma^2\big)\,\mathrm{d}z\Big|\\
\leq\ & |D|^{1/2}\cdot\sum_{j\geq 1}\|(V\mathcal{R}_{k_2})^j\big(e^{ik_1\hat{x}_1\cdot(\cdot)}\sigma^2\big)\|_{L^2(D)}\\
\leq\ & C|D|^{1/2}\cdot\sum_{j\geq 1}k_2^{-j}\|e^{ik_1\hat{x}_1\cdot(\cdot)}\sigma^2\|_{L^2(D)} = \mathcal{O}(k_2^{-1}), \quad k\to+\infty.
\end{aligned}$$

To prove (3.38) for the case where $j=1$, we split $\mathbb{E}(\overline{F_1}F_1)$ into four terms,

$$\mathbb{E}(\overline{F_1}F_1) = \mathbb{E}(\overline{G_1}G_1) + \mathbb{E}(\overline{r_1}r_2) - \mathbb{E}(\overline{r_2}r_2) + \mathbb{E}(\overline{r_2}r_1). \tag{3.42}$$

We estimate these four terms on the right-hand side of (3.42) one by one. First, we estimate

$$
\begin{aligned}
&\big|\mathbb{E}\big(\overline{G_1(k_1,\hat{x}_1,\omega)}\cdot G_1(k_2,\hat{x}_2,\omega)\big)\big| \\
&= \Big|\iint e^{-ik_1\hat{x}_1\cdot y}e^{ik_2\hat{x}_2\cdot z}V(y)\overline{V}(z)\cdot\mathbb{E}\big[\int_{D_s}\Phi(y,s)\sigma(s)\,\mathrm{d}B_s\cdot\int_{D_t}\overline{\Phi}(z,t)\sigma(t)\,\mathrm{d}B_t\big]\,\mathrm{d}y\,\mathrm{d}z\Big| \\
&= \Big|\iint_{D_y\times D_z} e^{-ik_1\hat{x}_1\cdot y}e^{ik_2\hat{x}_2\cdot z}V(y)\overline{V}(z)\cdot\big[\int_{D_s}\Phi(y,s)\sigma(s)\overline{\Phi}(z,s)\sigma(s)\,\mathrm{d}s\big]\,\mathrm{d}y\,\mathrm{d}z\Big| \\
&= \Big|\int_D \sigma^2(s)\cdot\mathcal{R}_{k_1}V(e^{-ik_1\hat{x}_1\cdot(\cdot)})(s)\cdot\overline{\mathcal{R}_{k_2}V(e^{-ik_2\hat{x}_2\cdot(\cdot)})(s)}\,\mathrm{d}s\Big| \\
&\leq Ck_1^{-1}k_2^{-1}\|\sigma\|^2_{L^\infty(D)}\quad\big(\text{Lemma 2.2.4}\big) \\
&= \mathcal{O}(k_1^{-1}k_2^{-1}),\quad k\to+\infty.
\end{aligned}
\tag{3.43}
$$

Then we estimate

$$
\begin{aligned}
&\big|\mathbb{E}\big(\overline{r_1(k_1,\hat{x}_1,\omega)}\cdot r_2(k_2,\hat{x}_2,\omega)\big)\big| \leq \mathbb{E}\Big(\sum_{j\geq1}\big|G_j(k_1,\hat{x}_1,\omega)\big|\times\sum_{\ell\geq2}\big|G_\ell(k_2,\hat{x}_2,\omega)\big|\Big) \\
&= \mathbb{E}\Big(\sum_{j\geq1}\Big|\int_D e^{-ik_1\hat{x}_1\cdot y}(V\mathcal{R}_{k_1})^j(\sigma\dot{B}_y)\,\mathrm{d}y\Big|\cdot\sum_{\ell\geq2}\Big|\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^\ell(\sigma\dot{B}_z)\,\mathrm{d}z\Big|\Big) \\
&= \|V\|^2_{L^\infty(D)}|D|\cdot\mathbb{E}\Big(\sum_{j\geq0}\|(\mathcal{R}_{k_1}V)^j[\mathcal{R}_{k_1}(\sigma\dot{B})]\|_{L^2(D)}\cdot\sum_{\ell\geq1}\|(\mathcal{R}_{k_2}V)^\ell[\mathcal{R}_{k_2}(\sigma\dot{B})]\|_{L^2(D)}\Big) \\
&\leq C\|V\|^2_{L^\infty(D)}|D|\cdot\mathbb{E}\Big(\sum_{j\geq0}\big(k_1^{-j}\|\mathcal{R}_{k_1}(\sigma\dot{B})\|_{L^2(D)}\big)\cdot\sum_{\ell\geq1}\big(k_2^{-\ell}\|\mathcal{R}_{k_2}(\sigma\dot{B})\|_{L^2(D)}\big)\Big) \\
&\leq \|V\|^2_{L^\infty(D)}|D|\cdot\frac{k_1}{k_1-1}\cdot\frac{1}{k_2-1}\cdot\frac{1}{2}\mathbb{E}\big(\|\mathcal{R}_{k_1}(\sigma\dot{B})\|^2_{L^2(D)}+\|\mathcal{R}_{k_2}(\sigma\dot{B})\|^2_{L^2(D)}\big).
\end{aligned}
\tag{3.44}
$$

Utilizing (3.8), we obtain

$$
\mathbb{E}\big(\|\mathcal{R}_k(\sigma\dot{B})\|^2_{L^2(D)}\big)\leq C\mathbb{E}\big(\|\mathcal{R}_k(\sigma\dot{B})\|^2_{L^2_{-1/2-\epsilon}}\big)\leq C_D<+\infty. \tag{3.45}
$$

From (3.44)-(3.45) we arrive at

$$
\big|\mathbb{E}\big(\overline{r_1(k_1,\hat{x}_1,\omega)}\cdot r_2(k_2,\hat{x}_2,\omega)\big)\big|\leq\mathcal{O}(k_2^{-1}),\quad k\to+\infty. \tag{3.46}
$$

Mimicking (3.44)-(3.45), one can obtain

$$\left|\mathbb{E}\left(\overline{r_2(k_1,\hat{x}_1,\omega)}\cdot r_1(k_2,\hat{x}_2,\omega)\right)\right| \leq \mathcal{O}(k_1^{-1}), \quad k\to+\infty. \tag{3.47}$$

By modify $\sum_{j\geq 0} k_1^{-j}$ to $\sum_{j\geq 1} k_1^{-j}$ in (3.44), one can conclude

$$\left|\mathbb{E}\left(\overline{r_2(k_1,\hat{x}_1,\omega)}\cdot r_2(k_2,\hat{x}_2,\omega)\right)\right| \leq \mathcal{O}(k_1^{-1}k_2^{-1}), \quad k\to+\infty. \tag{3.48}$$

Combining (3.42)-(3.43) and (3.46)-(3.48), we arrive at (3.38) for the case where $j=1$. The proof is complete. □

Lemma 3.3.5 is the ergodic version of Lemma 3.3.4.

Lemma 3.3.5. *Write*

$$X_{p,q}(K,\tau,\hat{x},\omega) = \frac{1}{K}\int_K^{2K} \overline{F_p(k,\hat{x},\omega)}\cdot F_q(k+\tau,\hat{x},\omega)\,\mathrm{d}k,$$

where $(p,q)\in\{(0,1),(1,0),(1,1)\}$. Then for any $\hat{x}\in\mathbb{S}^2$ and any $\tau\geq 0$, we have the following estimates as $K\to+\infty$,

$$\left|\mathbb{E}(X_{p,q}(K,\tau,\hat{x},\omega))\right| = \mathcal{O}(K^{-1}),\ \mathbb{E}(|X_{p,q}(K,\tau,\hat{x},\omega)|^2) = \mathcal{O}(K^{-5/4}), \tag{3.49}$$

$$\left|\mathbb{E}(X_{1,1}(K,\tau,\hat{x},\omega))\right| = \mathcal{O}(K^{-1}),\ \mathbb{E}(|X_{1,1}(K,\tau,\hat{x},\omega)|^2) = \mathcal{O}(K^{-2}), \tag{3.50}$$

for $(p,q)\in\{(0,1),(1,0)\}$. Let $\{K_j\}\in P(4/5+\gamma)$. Then for any $\tau\geq 0$, we have

$$\lim_{j\to+\infty} X_{p,q}(K_j,\tau,\hat{x},\omega) = 0 \qquad a.s.\ , \tag{3.51}$$

for every $(p,q)\in\{(0,1),(1,0),(1,1)\}$.

We may denote $X_{p,q}(K,\tau,\hat{x},\omega)$ as $X_{p,q}$ for short if it is clear in the context.

Proof of Lemma 3.3.5. According to Lemma 3.3.4, we have

$$\begin{aligned}\mathbb{E}\left(X_{0,1}\right) &= \frac{1}{K}\int_K^{2K} \mathbb{E}\left(\overline{F_0(k,\hat{x},\omega)}\cdot F_1(k+\tau,\hat{x},\omega)\right)\mathrm{d}k \\ &= \mathcal{O}(K^{-1}), \quad K\to+\infty. \end{aligned}\tag{3.52}$$

By (3.28), Isserlis' Theorem and Lemma 3.3.3, we compute the secondary moment of $X_{0,1}$ as

$$\begin{aligned}
&\mathbb{E}\big(|X_{0,1}|^2\big)\\
&= \frac{1}{K^2}\int_K^{2K}\int_K^{2K} \mathbb{E}\big(F_0(k_1,\hat{x},\omega)\overline{F_1(k_1+\tau,\hat{x},\omega)}\big)\cdot\mathbb{E}\big(\overline{F_0(k_2,\hat{x},\omega)}F_1(k_2+\tau,\hat{x},\omega)\big)\\
&\quad + \mathbb{E}\big(F_0(k_1,\hat{x},\omega)\overline{F_0(k_2,\hat{x},\omega)}\big)\cdot\mathbb{E}\big(\overline{F_1(k_1+\tau,\hat{x},\omega)}F_1(k_2+\tau,\hat{x},\omega)\big)\\
&\quad + \mathbb{E}\big(F_0(k_1,\hat{x},\omega)F_1(k_2+\tau,\hat{x},\omega)\big)\cdot\mathbb{E}\big(\overline{F_1(k_1+\tau,\hat{x},\omega)}\,\overline{F_0(k_2,\hat{x},\omega)}\big)\,\mathrm{d}k_1\,\mathrm{d}k_2\\
&= \frac{1}{K^2}\int_K^{2K}\int_K^{2K} \mathcal{O}(K^{-2}) + (2\pi)^{3/2}\widehat{\sigma^2}((k_1-k_2)\hat{x})\cdot\mathcal{O}(K^{-1}) + \mathcal{O}(K^{-2})\,\mathrm{d}k_1\,\mathrm{d}k_2\\
&= \mathcal{O}(K^{-1/4})\cdot\mathcal{O}(K^{-1}) + \mathcal{O}(K^{-2}) \quad \text{(Hölder ineq. and Lemma 3.3.3)}\\
&= \mathcal{O}(K^{-5/4}), \quad K\to+\infty. \qquad (3.53)
\end{aligned}$$

From (3.52)-(3.53) we obtain (3.49) for the case where $(p,q)=(0,1)$. Using similar arguments, formula (3.49) for $(p,q)=(1,0)$ can be proved and we skip the details.

By Chebyshev's inequality and (3.53), for any $\epsilon>0$, we have

$$\begin{aligned}
&P\Big(\bigcup_{j\geq K_0}\{|X_{0,1}(K_j,\tau,\hat{x},\omega)-0|\geq\epsilon\}\Big) \leq \frac{C}{\epsilon^2}\sum_{j\geq K_0}K_j^{-5/4} \leq \frac{C}{\epsilon^2}\sum_{j\geq K_0} j^{-1-5\gamma/4}\\
&\leq \frac{C}{\epsilon^2}\int_{K_0}^{+\infty}(t-1)^{-1-5\gamma/4}\,\mathrm{d}t = \frac{C}{\epsilon^2\gamma}(K_0-1)^{-5\gamma/4}\to 0, \quad K_0\to+\infty. \qquad (3.54)
\end{aligned}$$

According to Lemma 2.2.5, inequality (3.54) implies (3.51) for the case where $(p,q)=(0,1)$. Similarly, formula (3.51) can be proved for the case where $(p,q)=(1,0)$.

We now prove (3.50). We have

$$\mathbb{E}\big(X_{1,1}\big) = \frac{1}{K}\int_K^{2K}\mathbb{E}\big(\overline{F_1(k,\hat{x},\omega)}\cdot F_1(k+\tau,\hat{x},\omega)\big)\,\mathrm{d}k = \mathcal{O}(K^{-1}). \qquad (3.55)$$

Similar to (3.53), we compute the secondary moment of $X_{1,1}$ as

$$\begin{aligned}
&\mathbb{E}\big(|X_{1,1}|^2\big)\\
&= \mathbb{E}\Big(\frac{1}{K}\int_K^{2K} F_1(k_1,\hat{x},\omega)\cdot\overline{F_1(k_1+\tau,\hat{x},\omega)}\,\mathrm{d}k_1\\
&\quad\cdot\frac{1}{K}\int_K^{2K}\overline{F_1(k_2,\hat{x},\omega)}\cdot F_1(k_2+\tau,\hat{x},\omega)\,\mathrm{d}k_2\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{K^2}\int_K^{2K}\int_K^{2K} \mathcal{O}(K^{-1})\cdot\mathcal{O}(K^{-1})\,\mathrm{d}k_1\,\mathrm{d}k_2 \quad \text{(Lemma 3.3.4)}\\
&= \mathcal{O}(K^{-2}), \quad K\to+\infty. \qquad (3.56)
\end{aligned}
$$

Formulae (3.55) and (3.56) give (3.50).

By Chebyshev's inequality and (3.56), for any $\epsilon > 0$, we have

$$
\begin{aligned}
&P\Big(\bigcup_{j\geq K_0}\{|X_{1,1}-0|\geq\epsilon\}\Big) \leq \frac{C}{\epsilon^2}\sum_{j\geq K_0} K_j^{-2} \leq \frac{C}{\epsilon^2}\sum_{j\geq K_0} j^{-8/5-2\gamma}\\
\leq&\frac{C}{\epsilon^2}\int_{K_0}^{+\infty}(t-1)^{-8/5-2\gamma}\,\mathrm{d}t = \frac{C(K_0-1)^{-3/5-2\gamma}}{\epsilon^2(3+10\gamma)} \to 0, \quad K_0\to+\infty. \qquad (3.57)
\end{aligned}
$$

Lemma 2.2.5 together with (3.57) implies (3.51) for the case that $(p,q)=(1,1)$. The proof is thus completed. □

3.4 The recovery of the variance function

In this section we focus on the recovery of the variance function. We employ only a single passive scattering measurement. Namely, there is no incident plane wave sent and the random sample ω is fixed. Throughout this section, α is set to be 0. The data set $\mathcal{M}_1$ is utilized to achieve the unique recovery result. We present the main results of recovering the variance function in Section 3.4.1, and put the corresponding proofs in Section 3.4.2.

3.4.1 Main unique recovery results

To make it clearer, we use three lemmas, i.e., Lemmas 3.4.1, 3.4.2 and 3.4.3, to illustrate our recovering scheme of the variance function. The first main result is as follows.

Lemma 3.4.1. *We have the following asymptotic identity,*

$$
\begin{aligned}
&4\sqrt{2\pi}\lim_{k\to+\infty}\mathbb{E}\Big(\big[\overline{u^\infty(\hat{x},k,\omega)}-\overline{\mathbb{E}u^\infty(\hat{x},k)}\,\big]\cdot\big[u^\infty(\hat{x},k+\tau,\omega)-\mathbb{E}u^\infty(\hat{x},k+\tau)\big]\Big)\\
&=\widehat{\sigma^2}(\tau\hat{x}), \qquad (3.58)
\end{aligned}
$$

where $\tau \geq 0,\ \hat{x} \in \mathbb{S}^2$.

Lemma 3.4.1 clearly yields a recovery formula for the variance function. However, it requires many realizations. The result in Lemma 3.4.1 can be improved by using the ergodicity. See, e.g., [11, 21, 32].

Lemma 3.4.2. *Assume* $\{K_j\} \in P(2+\gamma)$. *Then* $\exists\, \Omega_0 \subset \Omega\colon \mathbb{P}(\Omega_0) = 0$, Ω_0 *depends only on* $\{K_j\}_{j\in\mathbb{N}^+}$*, such that for any* $\omega \in \Omega\backslash\Omega_0$*, there exists* $S_\omega \subset \mathbb{R}^3\colon m(S_\omega) = 0$*, such that for* $\forall x \in \mathbb{R}^3\backslash S_\omega$,

$$4\sqrt{2\pi}\lim_{j\to+\infty}\frac{1}{K_j}\int_{K_j}^{2K_j}\left[\overline{u^\infty(\hat{x},k,\omega)} - \overline{\mathbb{E}u^\infty(\hat{x},k)}\right]\cdot\left[u^\infty(\hat{x},k+\tau,\omega) - \mathbb{E}u^\infty(\hat{x},k+\tau)\right]\mathrm{d}k$$
$$= \widehat{\sigma^2}(x), \tag{3.59}$$

where $\tau = |x|$ *and* $\hat{x} := x/|x|$.

The recovering formula (3.59) holds for any $\hat{x} \in \mathbb{S}^2$ when $x = 0$. The recovery formulae presented in Lemma 3.4.2 still involves every realization of the random sample ω. To recover the variance function by only one realization, the term $\mathbb{E}u^\infty(\hat{x},k)$ should be further relaxed in Lemma 3.4.2, and this is achieved by Lemma 3.4.3.

Lemma 3.4.3. *Under the same condition as in Lemma 3.4.2, we have*

$$4\sqrt{2\pi}\lim_{j\to+\infty}\frac{1}{K_j}\int_{K_j}^{2K_j}\overline{u^\infty(\hat{x},k,\omega)}\cdot u^\infty(\hat{x},k+\tau,\omega)\,\mathrm{d}k = \widehat{\sigma^2}(x), \qquad a.s.\ . \tag{3.60}$$

Remark 3.4.1. In Lemma 3.4.3, it should be noted that the left-hand-side of (3.60) contains the random sample ω, while the right-hand side does not. This means that the limit in (3.60) is statistically stable.

Now Theorem 3.1.1 becomes a direct consequence of Lemma 3.4.3.

Proof of Theorem 3.1.1. Lemma 3.4.3 provides a recovery formula for the variance function σ^2 by the data set $\mathcal{M}_1$. □

3.4.2 Proofs of the main results

In this subsection, we present proofs of Lemmas 3.4.1, 3.4.2 and 3.4.3.

Proof of Lemma 3.4.1. Write $u_1^\infty(\hat{x},k,\omega) = u^\infty(\hat{x},k,\omega) - \mathbb{E}u^\infty(\hat{x},k)$ as in (3.22). Therefore $4\pi u_1^\infty(\hat{x},k,\omega)$ equals to $(-1)\sum_{j=0}^{+\infty}\int_D e^{-ik\hat{x}\cdot y}(V\mathcal{R}_k)^j(\sigma\dot{B}_y)\,\mathrm{d}y$. Recall the definition of $F_j(k,\hat{x},\omega)$ $(j=0,1)$ in (3.24). Let $k_1,k_2>k>k^*$. One can compute

$$\begin{aligned}16\pi^2\mathbb{E}\big(\overline{u_1^\infty(\hat{x},k_1,\omega)}u_1^\infty(\hat{x},k_2,\omega)\big) &= \sum_{j,\ell=0,1}\mathbb{E}\big(\overline{F_j(k_1,\hat{x},\omega)}F_\ell(k_2,\hat{x},\omega)\big)\\ &=: I_0+I_1+I_2+I_3. \end{aligned}\tag{3.61}$$

From Lemma 3.3.4, we have I_1, I_2, I_3 are all of order k^{-1}, hence

$$16\pi^2\mathbb{E}\big(\overline{u_1^\infty(\hat{x},k_1,\omega)}u_1^\infty(\hat{x},k_2,\omega)\big) = I_0 + \mathcal{O}(k^{-1}), \quad k\to+\infty. \tag{3.62}$$

By (3.28), (3.61) and (3.62), we have

$$16\pi^2\lim_{k\to+\infty}\mathbb{E}\big(\overline{u_1^\infty(\hat{x},k_1,\omega)}u_1^\infty(\hat{x},k_2,\omega)\big) = (2\pi)^{3/2}\,\widehat{\sigma^2}((k_2-k_1)\hat{x}),$$

which implies (3.58). □

Proof of Lemma 3.4.2. Our proof is divided into two steps. In the first step we give a basic result, i.e., the conclusion (3.66), and in the second step the logical order between y and ω in (3.66) is exchanged.

Step 1: give a basic result.

We denote by $\mathcal{E}_k$ the averaging operation w.r.t. k: $\mathcal{E}_k f = \frac{1}{K}\int_K^{2K} f(k)\,\mathrm{d}k$. Following the notation conventions in the proof of Lemma 3.4.1, we have

$$\begin{aligned}16\pi^2\mathcal{E}_k\big(\overline{u_1^\infty(\hat{x},k,\omega)}u_1^\infty(\hat{x},k+\tau,\omega)\big) &= \sum_{j,\ell=0,1}\mathcal{E}_k\big(\overline{F_j(k,\hat{x},\omega)}F_\ell(k+\tau,\hat{x},\omega)\big)\\ &=: X_{0,0}+X_{0,1}+X_{1,0}+X_{1,1}.\end{aligned}\tag{3.63}$$

Recall that $\{K_j\}\in P(2+\gamma)$. Then, for $\forall\tau\geq 0$ and $\forall\hat{x}\in\mathbb{S}^2$, Lemma 3.3.2 implies that $\exists\,\Omega_{\tau,\hat{x}}^{0,0}\subset\Omega\colon \mathbb{P}(\Omega_{\tau,\hat{x}}^{0,0})=0$, $\Omega_{\tau,\hat{x}}^{0,0}$ depending on τ and $\hat{x}$, such that

$$\lim_{j\to+\infty}X_{0,0}(K_j,\tau,\hat{x},\omega) = (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x}), \quad \forall\omega\in\Omega\backslash\Omega_{\tau,\hat{x}}^{0,0}. \tag{3.64}$$

$\{K_j\} \in P(2+\gamma)$ implies $\{K_j\} \in P(5/4+\gamma)$, so Lemma 3.3.5 implies the existence of the sets $\Omega^{p,q}_{\tau,\hat{x}}$ $\big((p,q) \in \{(0,1),\,(1,0),\,(1,1)\}\big)$ with zero probability measures such that $\forall \tau \geq 0$ and $\forall \hat{x} \in \mathbb{S}^2$,

$$\lim_{j \to +\infty} X_{p,q}(K_j, \tau, \hat{x}, \omega) = 0, \quad \forall \omega \in \Omega \backslash \Omega^{p,q}_{\tau,\hat{x}}. \tag{3.65}$$

for all $(p,q) \in \{(0,1),\,(1,0),\,(1,1)\}$. Write $\Omega_{\tau,\hat{x}} = \bigcup_{p,q=0,1} \Omega^{p,q}_{\tau,\hat{x}}$, then $\mathbb{P}(\Omega_{\tau,\hat{x}}) = 0$. From Lemmas 3.3.2 and 3.3.5 we note that $\Omega^{p,q}_{\tau,\hat{x}}$ also depends on K_j, so does $\Omega_{\tau,\hat{x}}$, but we omit this dependence in the notation. Write

$$Z(\tau\hat{x},\omega) := \lim_{j \to +\infty} \frac{16\pi^2}{K_j} \int_{K_j}^{2K_j} \overline{u_1^\infty(\hat{x},k,\omega)} u_1^\infty(\hat{x},k+\tau,\omega)\,\mathrm{d}k - (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x})$$

for short. By (3.63)-(3.65), we conclude that,

$$\forall\, y \in \mathbb{R}^3,\, \exists\, \Omega_y \subset \Omega \colon \mathbb{P}(\Omega_y) = 0,\ \text{s.t.}\ \forall\, \omega \in \Omega \backslash \Omega_y,\, Z(y,\omega) = 0. \tag{3.66}$$

Step 2: exchange the logical order.

To conclude (3.59) from (3.66), we should exchange the logical order between y and ω. To achieve this, we utilize Fubini's Theorem. Denote the usual Lebesgue measure on $\mathbb{R}^3$ as $\mathbb{L}$ and the product measure $\mathbb{L} \times \mathbb{P}$ as μ, and construct the product measure space $\mathbb{M} := (\mathbb{R}^3 \times \Omega, \mathcal{G}, \mu)$ in the canonical way, where $\mathcal{G}$ is the corresponding complete σ-algebra. Write

$$\mathcal{A} := \{(y,\omega) \in \mathbb{R}^3 \times \Omega\,;\, Z(y,\omega) \neq 0\},$$

then $\mathcal{A}$ is a subset of $\mathbb{M}$. Set $\chi_{\mathcal{A}}$ as the characteristic function of $\mathcal{A}$ in $\mathbb{M}$. By (3.66) we obtain

$$\int_{R^3} \big(\int_\Omega \chi_{\mathcal{A}}(y,\omega)\,\mathrm{d}\mathbb{P}(\omega) \big)\,\mathrm{d}\mathbb{L}(y) = 0. \tag{3.67}$$

By (3.67) and [Corollary 7 in Section 20.1, 45] we obtain

$$\int_{\mathbb{M}} \chi_{\mathcal{A}}(y,\omega)\,\mathrm{d}\mu = \int_\Omega \big(\int_{R^3} \chi_{\mathcal{A}}(y,\omega)\,\mathrm{d}\mathbb{L}(y) \big)\,\mathrm{d}\mathbb{P}(\omega) = 0. \tag{3.68}$$

Because $\chi_{\mathcal{A}}(y,\omega)$ is non-negative, (3.68) implies

$$\exists\,\Omega_0\colon \mathbb{P}(\Omega_0)=0,\ \text{s.t.}\ \forall\,\omega\in\Omega\backslash\Omega_0,\ \int_{R^3}\chi_{\mathcal{A}}(y,\omega)\,\mathrm{d}\mathbb{L}(y)=0. \tag{3.69}$$

Formula (3.69) further implies for every $\omega\in\Omega\backslash\Omega_0$,

$$\exists\,S_\omega\subset\mathbb{R}^3\colon \mathbb{L}(S_\omega)=0,\ \text{s.t.}\ \forall\,y\in\mathbb{R}^3\backslash S_\omega,\ Z(y,\omega)=0. \tag{3.70}$$

From (3.70) we arrive at (3.59). □

Proof of Lemma 3.4.3. The symbol $\mathcal{E}_k$ is defined the same as in the proof of Lemma 3.4.2. We have

$$\begin{aligned}
&16\pi^2\mathcal{E}_k\big(\overline{u^\infty(\hat{x},k,\omega)}u^\infty(\hat{x},k+\tau,\omega)\big)\\
=&\ 16\pi^2\mathcal{E}_k\big(\overline{u_1^\infty(\hat{x},k,\omega)}\cdot u_1^\infty(\hat{x},k,\omega)\big)+16\pi^2\mathcal{E}_k\big(\overline{u_1^\infty(\hat{x},k,\omega)}\cdot\mathbb{E}u^\infty(\hat{x},k+\tau)\big)\\
&+16\pi^2\mathcal{E}_k\big(\overline{\mathbb{E}u^\infty(\hat{x},k)}\cdot u_1^\infty(\hat{x},k+\tau,\omega)\big)+16\pi^2\mathcal{E}_k\big(\overline{\mathbb{E}u^\infty(\hat{x},k)}\cdot\mathbb{E}u^\infty(\hat{x},k+\tau)\big)\\
=\!:&\,J_0+J_1+J_2+J_3.
\end{aligned} \tag{3.71}$$

From Lemma 3.4.2 we obtain

$$\begin{aligned}
&\lim_{j\to+\infty}J_0=\lim_{j\to+\infty}\frac{16\pi^2}{K_j}\int_{K_j}^{2K_j}\overline{u_1^\infty(\hat{x},k,\omega)}\cdot u_1^\infty(\hat{x},k+\tau,\omega)\,\mathrm{d}k=(2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x}),\\
&\tau\hat{x}\ \text{a.e.}\in\mathbb{R}^3,\quad \omega\ \text{a.s.}\in\Omega.
\end{aligned} \tag{3.72}$$

We now estimate J_1,

$$\begin{aligned}
|J_1|^2&\simeq\big|\mathcal{E}_k\big(\overline{u_1^\infty(\hat{x},k,\omega)}\cdot\mathbb{E}u^\infty(\hat{x},k+\tau)\big)\big|^2=\big|\frac{1}{K_j}\int_{K_j}^{2K_j}\overline{u_1^\infty(\hat{x},k,\omega)}\cdot\mathbb{E}u^\infty(\hat{x},k+\tau)\,\mathrm{d}k\big|^2\\
&\le\frac{1}{K_j}\int_{K_j}^{2K_j}|u^\infty(\hat{x},k,\omega)-\mathbb{E}u^\infty(\hat{x},k)|^2\,\mathrm{d}k\cdot\frac{1}{K_j}\int_{K_j}^{2K_j}|\mathbb{E}u^\infty(\hat{x},k+\tau)|^2\,\mathrm{d}k.
\end{aligned} \tag{3.73}$$

Combining (3.73) with Lemmas 3.3.1 and 3.4.2, we have

$$|J_1|^2 \lesssim (\widehat{\sigma^2}(0) + o(1)) \cdot o(1) = o(1) \to 0, \quad j \to +\infty. \tag{3.74}$$

The analysis of J_2 is similar to that of J_1 so we skip the details.

Finally, by Lemma 3.3.1, the J_3 can be estimated as

$$\begin{aligned} |J_3|^2 &\simeq \left|\mathcal{E}_k\big(\overline{\mathbb{E}u^\infty(\hat{x}, k)} \cdot \mathbb{E}u^\infty(\hat{x}, k+\tau)\big)\right|^2 \\ &\leq \frac{1}{K_j}\int_{K_j}^{2K_j} \sup_{\kappa \geq K_j} \left|\mathbb{E}u^\infty(\hat{x}, \kappa)\right|^2 \mathrm{d}k \cdot \frac{1}{K_j}\int_{K_j}^{2K_j} \sup_{\kappa \geq K_j+\tau} \left|\mathbb{E}u^\infty(\hat{x}, \kappa)\right|^2 \mathrm{d}k \\ &= \sup_{\kappa \geq K_j} |\mathbb{E}u^\infty(\hat{x}, \kappa)|^2 \cdot \sup_{\kappa \geq K_j+\tau} |\mathbb{E}u^\infty(\hat{x}, \kappa)|^2 \to 0, \quad j \to +\infty. \end{aligned} \tag{3.75}$$

Combining (3.71), (3.72), (3.74) and (3.75), we arrive at (3.60). Our proof is done. □

3.5 Uniqueness of the potential and the random source

In this section, we focus on the recovery of the potential term and the expectation of the random source. Due to the highly nonlinear relation between the total wave and the potential, the active scattering measurements are thus utilized to recover the potential. In the recovery of the potential, the random sample ω is set to be fixed so that a single realization of the random term $\dot{B}_x$ is enough to obtain the unique recovery. Different from the recovery of the potential, the uniqueness of the expectation requires all realizations of the random sample ω. Because the deterministic and random parts of the source are entangled together so that only one realization of the random source cannot reveal exact values of the expectation at each spatial point x.

3.5.1 Recovery of the potential

Now we are in the position to prove Theorem 3.1.2. We are to use the incident plane wave, so α is set to be 1 throughout this section.

Proof of Theorem 3.1.2. The random sample ω is assumed to be fixed. Given two direction d_1 and d_2 of the incident plane waves, we denote the corresponding total wave as u_{d_1} and u_{d_2}, respectively. Then, from (3.1), we have

$$\begin{cases} (-\Delta - k^2)(u_{d_1} - u_{d_2}) = V(u_{d_1} - u_{d_2}) \\ u_{d_1} - u_{d_2} = e^{ikd_1\cdot x} - e^{ikd_2\cdot x} + u^{sc}_{d_1}(x) - u^{sc}_{d_2}(x) \\ u^{sc}_{d_1}(x) - u^{sc}_{d_2}(x): \text{ SRC} \end{cases} \tag{3.76}$$

From (3.76) we have the Lippmann-Schwinger equation,

$$\big(I - \mathcal{R}_k V\big)(u_{d_1} - u_{d_2}) = e^{ikd_1\cdot x} - e^{ikd_2\cdot x}. \tag{3.77}$$

When $k > k^*$, equality (3.77) gives

$$u^{sc}_{d_1} - u^{sc}_{d_2} = \mathcal{R}_k V(e^{ikd_1\cdot x} - e^{ikd_2\cdot x}) + \sum_{j=2}^{\infty}(\mathcal{R}_k V)^j(e^{ikd_1\cdot x} - e^{ikd_2\cdot x}).$$

Therefore the difference between the far-field patterns is

$$\begin{aligned} & u^{\infty}(\hat{x}, k, d_1) - u^{\infty}(\hat{x}, k, d_2) \\ = & \int_D \frac{e^{-ik\hat{x}\cdot y}}{4\pi} V(y)(e^{ikd_1\cdot y} - e^{ikd_2\cdot y})\,\mathrm{d}y + \sum_{j=1}^{\infty}\int_D \frac{e^{-ik\hat{x}\cdot y}}{4\pi} V(y)(\mathcal{R}_k V)^j(e^{ikd_1\cdot(\cdot)} - e^{ikd_2\cdot(\cdot)})\,\mathrm{d}y \\ =: & \sqrt{\frac{\pi}{2}}\widehat{V}\big(k(\hat{x} - d_1)\big) - \sqrt{\frac{\pi}{2}}\widehat{V}\big(k(\hat{x} - d_2)\big) + \sum_{j=1}^{\infty} H_j(k), \end{aligned} \tag{3.78}$$

where

$$H_j(k) := \int_D \frac{e^{-ik\hat{x}\cdot y}}{4\pi} V(y)(\mathcal{R}_k V)^j(e^{ikd_1\cdot(\cdot)} - e^{ikd_2\cdot(\cdot)})\,\mathrm{d}y, \quad j = 1, 2, \cdots. \tag{3.79}$$

For any $p \in \mathbb{R}^3$, when $p = 0$, we let $\hat{x} = (1, 0, 0)$, $d_1 = (1, 0, 0)$, $d_2 = (0, 1, 0)$; when $p \neq 0$, we can always find a $p^{\perp} \in \mathbb{R}^3$ which is perpendicular to p. Let

$$e = p^{\perp}/\|p^{\perp}\| \quad \text{and} \quad \begin{cases} \hat{x} = \sqrt{1 - \|p\|^2/(4k^2)}\cdot e + p/(2k), \\ d_1 = \sqrt{1 - \|p\|^2/(4k^2)}\cdot e - p/(2k), \\ d_2 = p/\|p\|, \end{cases}$$

when $k > \|p\|/2$, we have

$$\begin{cases} \hat{x}, d_1, d_2 \in \mathbb{S}^2, \\ k(\hat{x} - d_1) = p, \\ |k(\hat{x} - d_2)| \to \infty \ (k \to \infty). \end{cases} \tag{3.80}$$

Note that the choices of these two unit vectors $\hat{x}$, d_1 depend on k. For different values of k, we pick up different directions $\hat{x}$, d_1 to guarantee (3.80). Then,

$$\sqrt{\frac{\pi}{2}}\widehat{V}(p) = \lim_{k\to+\infty} \big(\sqrt{\frac{\pi}{2}}\widehat{V}(k(\hat{x} - d_1)) - \sqrt{\frac{\pi}{2}}\widehat{V}(k(\hat{x} - d_2))\big). \tag{3.81}$$

Combining (3.78), (3.81) and Lemma 3.5.1, we conclude

$$\widehat{V}(p) = \sqrt{\frac{2}{\pi}} \lim_{k\to+\infty} \big(u^\infty(\hat{x}, k, d_1) - u^\infty(\hat{x}, k, d_2)\big). \tag{3.82}$$

Formula (3.82) completes the proof. □

It remains to give the estimates of these high-order terms $H_j(k)$, and this is done by Lemma 3.5.1.

Lemma 3.5.1. *The sum of high-order terms $H_j(k)$ defined in* (3.79) *satisfies the following estimate,*

$$\big|\sum_{j\geq 1} H_j(k)\big| \leq Ck^{-1},$$

for some constant C independent of k.

Proof of Lemma 3.5.1. According to Lemma 2.2.4, we have

$$\begin{aligned} |H_j(k)| &\lesssim \int_D |V(y)| \cdot \big|[(\mathcal{R}_k V)^j e^{ikd_1\cdot(\cdot)}](y)\big| \,\mathrm{d}y + \int_D |V(y)| \cdot \big|[(\mathcal{R}_k V)^j e^{ikd_2\cdot(\cdot)}](y)\big| \,\mathrm{d}y \\ &\lesssim \|V\|_{L^\infty} \cdot |D|^{1/2} \cdot \big(k^{-j}\|e^{ikd_1\cdot(\cdot)}\|_{L^2(D)} + k^{-j}\|e^{ikd_2\cdot(\cdot)}\|_{L^2(D)}\big) \\ &= 2\|V\|_{L^\infty} \cdot |D| \cdot k^{-j}. \end{aligned}$$

Therefore,

$$|\sum_{j=1}^{\infty} H_j(k)| \leq \sum_{j=1}^{\infty} |H_j(k)| \leq 2C\|V\|_{L^\infty} \cdot |D| \cdot \sum_{j=1}^{\infty} k^{-j} \leq Ck^{-1}, \quad k \to +\infty.$$

The proof is done. □

3.5.2 Recovery of the random source

The variance function of the random source is recovered in Section 3.4, and now we recover its expectation.

Proof to Theorem 3.1.3. According to Theorem 3.1.2, we have the uniqueness of the potential. Assume that two source f, f' generate same far-field patterns for all $k > 0$. We denote the restriction on D of the corresponding total waves as u and u'. Then,

$$\begin{cases} (\Delta + k^2 + V)(\mathbb{E}u - \mathbb{E}u') = f - f' & \text{in } D \\ \mathbb{E}u - \mathbb{E}u' = \partial_\nu(\mathbb{E}u) - \partial_\nu(\mathbb{E}u') = 0 & \text{on } \partial D \end{cases} \tag{3.83}$$

where ν is the outer normal to ∂D. Let test functions $v_k \in H_0^1(D)$ be the weak solutions of the boundary value problem

$$\begin{cases} (-\Delta - V)v_k = k^2 v_k & \text{in } D \\ v_k = 0 & \text{on } \partial D \end{cases} \tag{3.84}$$

for delicately picked k. The solutions v_k are eigenvectors of the system (3.84). From (3.83) we have

$$\int_D (\Delta + V + k^2)(\mathbb{E}u - \mathbb{E}u') \cdot v_k \, \mathrm{d}x = \int_D (f - f')v_k \, \mathrm{d}x. \tag{3.85}$$

Using integral by parts and noting that the v_k's in (3.85) satisfy (3.84), we have

$$\int_D (f - f')v_k \, \mathrm{d}x = 0. \tag{3.86}$$

When $\|V\|_{L^\infty(D)}$ is less than some constant depending on D, the set of eigenvectors $\{v_k\}$ corresponding to different eigenvalues k^2 forms an orthonormal basis of $L^2(D)$

[Theorem 2.37, 41]. Therefore, from (3.86) we conclude that

$$f = f' \text{ in } L^2(D).$$

The proof is done. $\square$

Chapter 4

Schrödinger operator with rough source

4.1 Introduction

In this chapter, we are mainly concerned with the following random Schrödinger system

$$\begin{cases} (-\Delta - E + V(x))u(x, \sqrt{E}, \omega) = f(x, \omega), \quad x \in \mathbb{R}^3, & (4.1a) \\ \lim_{r \to \infty} r \left(\dfrac{\partial u}{\partial r} - \mathrm{i}\sqrt{E}u \right) = 0, \quad r := |x|, & (4.1b) \end{cases}$$

where ω in (4.1a) is the random sample which is already mentioned in Chapter 3. The source $f(x, \omega)$ in (4.1a) is an unknown generalized Gaussian random field with zero-mean while potential $V(x)$ is an unknown deterministic function. the f and V are supported in bounded convex domains D_f and D_V, respectively. $E \in \mathbb{R}_+$ is the energy level. In the sequel, we follow the convention to replace E with k^2, namely $k := \sqrt{E} \in \mathbb{R}_+$, which can be understood as the wave number. u in (4.1a) is the scattered wave field, which is also random due to the randomness of the source and potential. The limit (4.1b) is the Sommerfeld Radiation Condition (SRC) [13] that characterizes the outgoing nature of the u. The random system (4.1) describes the quantum scattering [16, 19] associated with a source f and potential V at the energy level k^2.

The source f in equation (4.1a) is assumed to be generalized Gaussian random field. It means that f is a random distribution and the mapping

$$\omega \in \Omega \ \mapsto \ \langle f(\cdot, \omega), \varphi \rangle \in \mathbb{C}$$

is Gaussian random variable whose probabilistic measure depends on the test function φ. There are different types of generalized Gaussian random fields [46]. In our setting, we assume that the f is *microlocally isotropic* generalized Gaussian random functions (m.i.g.r. for short); see Definition 2.1 in [11]. Recently, the m.i.g.r. model has been under an intensive study; see, e.g., [11, 31–34]. Two important parameters of the m.i.g.r. are its rough order and *rough strength*. Roughly speaking, the rough order determines the degree of spatial roughness of the m.i.g.r., and the rough strength indicates its spatial correlation length and intensity. The rough strength also captures the micro-structure of the object in interest [32]. We shall give a more detailed introduction to this random model in Section 4.2.1.

The direct problem regarding the system (4.1) is the study of the well-posedness of the mapping

$$(f, V) \ \to \ \{u^{\infty}(\hat{x}, k, \omega)\,;\, \hat{x} \in \mathbb{S}^2, k > 0,\, \omega \in \Omega\}. \tag{4.2}$$

With a good understanding of the direct problem (4.3), we can consider the corresponding inverse problem, which is the main purpose of this chapter. The recovery of the potential has been studied in Section 3.5, our inverse problem in this chapter is to recover the rough strength of f by knowledge of the associated far-field pattern. Therefore, our inverse problem can be formulated as

$$\{u^{\infty}(\hat{x}, k, \omega)\,;\, \hat{x} \in \mathbb{S}^2, k > 0,\, \omega \in \Omega\} \ \to \ \mu. \tag{4.3}$$

where μ is the rough strength of f. We establish unique recovery result for the aforementioned inverse scattering problem. More precisely, we establish sufficient conditions under which the correspondence between μ and u^{∞} is one to one. Our mathematical arguments are constructive and recovery formulas can also be obtained for the inverse problem. The major novelty of our unique recovery results compared to those existing ones in the literature is that both the source and the potential are unknown. This makes the corresponding study radically more challenging. The main result is presented in Theorem 4.1.1.

Theorem 4.1.1. *In system* (4.1)*, the f is assumed to be with rough strength μ and rough order $-m$ such that $1 < m < 3$, and V in system* (4.1) *is in $C_c^\infty(\mathbb{R}^3)$ with* $\operatorname{supp} V \subset D_V$, $\operatorname{dist}(D_V, D_f) > 0$. *Then, the rough strength μ can be uniquely recovered by the data set $\{u^\infty(\hat{x}, k, \omega)\,;\, \hat{x} \in \mathbb{S}^2, k > 0\}$ almost surely.*

Remark 4.1.1. In Theorem 4.1.1, the rough strength μ can be recovered by data of a single realization of the random sample. The data is passive in the sense that no incident wave is needed.

The rest of the chapter is organized as follows. In Section 4.2, we present the well-posedness of the direct scattering problem. Section 4.3 study the asymptotics of leader order term and higher order terms. In Section 4.4 we establish the recovery of the rough strength.

4.2 Well-posedness of the direct problem

In this section, the unique existence of a *mild solution* is established to the random Schrödinger system (4.1). Before that, we present some basic knowledge about the random model and some other preliminaries for the subsequent use.

4.2.1 The random model

As already mentioned in Section 4.1, a generalized Gaussian random field maps test functions to random variables. Assume h is a generalized Gaussian random field. Then both $\langle h(\cdot,\omega), \varphi\rangle$ and $\langle h(\cdot,\omega), \psi\rangle$ are random variables for φ, $\psi \in \mathscr{S}(\mathbb{R}^n)$. From a statistical point of view, the covariance between these two random variables,

$$\mathbb{E}_\omega(\langle \overline{h(\cdot,\omega)}, \varphi\rangle \langle h(\cdot,\omega), \psi\rangle), \tag{4.4}$$

can be understood as the covariance of h, where the $\mathbb{E}_\omega$ means to take expectation on the argument ω. Formula (4.4) induces an operator C_h,

$$C_h\colon \varphi \in \mathscr{S}(\mathbb{R}^n) \;\mapsto\; C_h\varphi \in \mathscr{S}'(\mathbb{R}^n),$$

in a way that

$$C_h\varphi\colon \psi \in \mathscr{S}(\mathbb{R}^n) \;\mapsto\; (C_h\varphi)(\psi) = \mathbb{E}_\omega(\langle \overline{h(\cdot,\omega)}, \varphi\rangle \langle h(\cdot,\omega), \psi\rangle) \in \mathbb{C}.$$

The operator C_h is called the covariance operator of h. See also [11,32] for reference.

We adopt the definition of the m.i.g.r. from [11] with minor modifications to fit our mathematical setting.

Definition 4.2.1. A generalized Gaussian random function h on $\mathbb{R}^n$ is called microlocally isotropic (m.i.g.r.) with rough order $-m$ and rough strength $\mu(x)$ in D, if the following conditions hold:

1. the expectation $\mathbb{E}h$ is in $C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\mathbb{E}h \subset D$;
2. h is supported in D a.s.;
3. the covariance operator C_h is a classical pseudodifferential operator of order m;
4. C_h has a principal symbol of the form $\mu(x)|\xi|^{-m}$ with $\mu \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$, $\operatorname{supp}\mu \subset D$ and $\mu(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Here, the $\mu(x)|\xi|^m$ is a representative of the principal symbol of C_h. Throughout this chapter, the principal symbol of the covariance operator of the $f(\cdot,\omega)$ in (4.1) is assumed to be $\mu(x)|\xi|^{-m}$.

Lemma 4.2.1. *Let h be a m.i.g.r. of rough order $-m$ in D. Then, $h \in H^{s,p}(\mathbb{R}^n)$ almost surely for any $1 < h < +\infty$ and $s < (m-n)/2$.*

Proof of Lemma 4.2.1. See Proposition 2.4 in [11]. □

Lemma 4.2.1 shows the regularity of h according to its rough order.

By the Schwartz kernel theorem (see Theorem 5.2.1 in [22]), there exists a kernel $K_h(x,y)$ with $\operatorname{supp} K_h \subset D \times D$ such that

$$(C_h\varphi)(\psi) = \mathbb{E}_\omega(\langle \overline{h(\cdot,\omega)}, \varphi\rangle \langle h(\cdot,\omega), \psi\rangle) = \iint K_h(x,y)\varphi(x)\psi(y)\,\mathrm{d}x\,\mathrm{d}y, \tag{4.5}$$

for all φ, $\psi \in \mathscr{S}(\mathbb{R}^n)$. It is easy to check that $K_h(x,y) = \overline{K_h(y,x)}$. Denote the symbol of the C_h as c_h, it can be verified [11] that the equalities

$$\begin{cases} K_h(x,y) = (2\pi)^{-n} \displaystyle\int e^{i(x-y)\cdot\xi} c_h(x,\xi)\,\mathrm{d}\xi, & \text{(4.6a)} \\ c_h(x,\xi) = \displaystyle\int e^{-i\xi\cdot(x-y)} K_h(x,y)\,\mathrm{d}x, & \text{(4.6b)} \end{cases}$$

hold in the distributional sense, and the integrals in (4.6) shall be understood as oscillatory integral. Despite the fact that h usually is not a function, intuitively speaking, however, it is helpful to keep in mind the following correspondence,

$$K_h(x,y) \sim \mathbb{E}_\omega\big(\overline{h(x,\omega)}h(y,\omega)\big).$$

4.2.2 Preliminaries

For a generalized Gaussian random field f, we define $\mathcal{R}_k f(x)$ as

$$\mathcal{R}_k f(x) := \langle f, \Phi(x,\cdot)\rangle. \tag{4.7}$$

We may also write $\mathcal{R}_k f(x)$ as $\int_{\mathbb{R}^3} \Phi_k(x,y) f(y)\,\mathrm{d}y$. Note that the resolvent operators defined in (2.1) and in (4.7) are linear. The following lemma shows some basic properties of the $\mathcal{R}_k f$. Note that the μ is the rough strength of f.

Lemma 4.2.2. *We have $\mathcal{R}_k f \in L^2_{-1/2-\epsilon}$ almost surely, and $\mathbb{E}\|\mathcal{R}_k f\|_{L^2(D_f)} < C < +\infty$, for some constant C independent of k.*

Proof. From (4.7), one can compute

$$\begin{aligned} &\mathbb{E}(\|\mathcal{R}_k f(\cdot,\omega)\|^2_{L^2_{-1/2-\epsilon}}) \\ &= \int_{\mathbb{R}^3} \langle x\rangle^{-1-2\epsilon} \mathbb{E}(\langle \overline{f}, \Phi_{-k,x}\rangle\langle f, \Phi_{k,x}\rangle)\,\mathrm{d}x = \int_{\mathbb{R}^3} \langle x\rangle^{-1-2\epsilon} \langle C_f \Phi_{-k,x}, \Phi_{k,x}\rangle\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \int \langle x\rangle^{-1-2\epsilon} \int \big((2\pi)^{-3} \int\int e^{i(y-z)\cdot\xi} c_f(y,\xi)\cdot \Phi_{-k,x}(z)\,\mathrm{d}z\,\mathrm{d}\xi\big)\Phi_{k,x}(y)\,\mathrm{d}y\,\mathrm{d}x \\
&\simeq \int \langle x\rangle^{-1-2\epsilon} \int_{D_f} \big(\int_{D_f} \frac{\mathcal{I}(y,z)e^{-ik|x-z|}}{|x-z|\cdot|y-z|^2}\,\mathrm{d}z\big)\cdot \frac{e^{ik|x-y|}}{|x-y|}\,\mathrm{d}y\,\mathrm{d}x, \qquad (4.8)
\end{aligned}
$$

where the $c_f(y,\xi)$ is the symbol of the covariance operator C_f and

$$
\mathcal{I}(y,z) := \int_{\mathbb{R}^3} |y-z|^2 e^{i(y-z)\cdot\xi} c_f(y,\xi)\,\mathrm{d}\xi.
$$

When $y = z$, we know $\mathcal{I}(y,z) = 0$ because the integrand is zero. Thanks to the condition that $m > 1$, when $y \neq z$ we have

$$
\begin{aligned}
|\mathcal{I}(y,z)| &= \big|\sum_{j=1}^{3} \int_{\mathbb{R}^3} (y_j - z_j)^2 e^{i(y-z)\cdot\xi} c_f(y,\xi)\,\mathrm{d}\xi\big| \\
&= \big|\sum_{j=1}^{3} \int_{\mathbb{R}^3} e^{i(y-z)\cdot\xi} (\partial_{\xi_j}^2 c_f)(y,\xi)\,\mathrm{d}\xi\big| \\
&\leq \sum_{j=1}^{3} \int_{\mathbb{R}^3} C_j \langle\xi\rangle^{-m-2}\,\mathrm{d}\xi \\
&\leq C_0 < +\infty, \qquad (4.9)
\end{aligned}
$$

for some constant C_0 independent of y and z. Note that if D_f is bounded, then for $j = 1, 2$ we can have

$$
\int_{D_f} |x-y|^{-j}\,\mathrm{d}y \leq C_{D,j}\langle x\rangle^{-j}, \quad \forall x \in \mathbb{R}^3. \qquad (4.10)
$$

for some constant $C_{D,j}$ depending only on D, j and the dimension. With the help of (4.9)-(4.10) and Hölder's inequality, we can continue (4.8) as

$$
\begin{aligned}
&\mathbb{E}(\|\mathcal{R}_k f(\cdot,\omega)\|_{L^2_{-1/2-\epsilon}}^2) \\
&\lesssim \int \langle x\rangle^{-1-2\epsilon}\big(\iint_{D_f\times D_f} (|x-z|\cdot|y-z|^2\cdot|x-y|)^{-1}\,\mathrm{d}z\,\mathrm{d}y\big)\,\mathrm{d}x \\
&\leq \int \langle x\rangle^{-1-2\epsilon}\Big(C\iint_{D_f\times D_f} |x-z|^{-2}\cdot|y-z|^{-2}\,\mathrm{d}z\,\mathrm{d}y \\
&\qquad \cdot \iint_{D_f\times D_f} |x-y|^{-2}\cdot|y-z|^{-2}\,\mathrm{d}z\,\mathrm{d}y\Big)^{1/2}\,\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \int \langle x\rangle^{-1-2\epsilon}\Big(C\int_{D_f}\big(\int_{D_f}|y-z|^{-2}\,\mathrm{d}y\big)|x-z|^{-2}\,\mathrm{d}z \\
&\qquad\qquad \cdot\int_{D_f}\big(\int_{D_f}|y-z|^{-2}\,\mathrm{d}z\big)|x-y|^{-2}\,\mathrm{d}y\Big)^{1/2}\mathrm{d}x \\
&= \int \langle x\rangle^{-1-2\epsilon}\Big(C_D\int_{D_f}|x-z|^{-2}\,\mathrm{d}z\cdot\int_{D_f}|x-y|^{-2}\,\mathrm{d}y\Big)^{1/2}\mathrm{d}x \\
&= \int \langle x\rangle^{-1-2\epsilon}C_D\langle x\rangle^{-2}\,\mathrm{d}x \le C_D < +\infty,
\end{aligned}
$$

which is

$$
\mathbb{E}(\|\mathcal{R}_k f(\cdot,\omega)\|^2_{L^2_{-1/2-\epsilon}}) \le C_D < +\infty. \tag{4.11}
$$

By the Hölder inequality applied to the probability measure, (4.11) gives

$$
\mathbb{E}\|\mathcal{R}_k f\|_{L^2_{-1/2-\epsilon}} \le [\mathbb{E}(\|\mathcal{R}_k f\|^2_{L^2_{-1/2-\epsilon}})]^{1/2} \le C_D^{1/2} < +\infty, \tag{4.12}
$$

for some constant C_D independent of k. The formula (4.12) gives that

$$
\mathcal{R}_k f \in L^2_{-1/2-\epsilon} \quad \text{a.s. .}
$$

By replacing $\mathbb{R}^3$ with D_f and deleting the term $\langle x\rangle^{-1-2\epsilon}$ in the derivation above, one easily arrives at $\mathbb{E}\|\mathcal{R}_k f\|_{L^2(D)} < +\infty$. The proof is complete. □

4.2.3 The well-posedness of the direct problem

For a particular realization of the random sample $\omega \in \Omega$, the regularity of m.i.g.r. f could be very rough; see Lemma 4.2.1. Due to this reason, the classical second-order elliptic PDE theories may not have license to be applied to (4.1). To that end, the notion of the *mild solution* is introduced for random PDEs (cf. [3, 35]) . In what follows, we introduce the mild solution for our problem setting (4.1), and we show that this mild solution and the corresponding far-field pattern are well-posed in the proper sense.

Reformulating (4.1) into the Lippmann-Schwinger equation formally (cf. [13]), we have

$$
(I - \mathcal{R}_k V)u = -\mathcal{R}_k f, \tag{4.13}
$$

where the term $\mathcal{R}_k f$ is defined by (4.7). Suppose k is large enough. From Lemma 2.2.3, we know the operator $I-\mathcal{R}_k V$ is an invertible mapping that maps from $L^2_{-1/2-\epsilon}$ to $L^2_{-1/2-\epsilon}$. Moreover, by Lemmas 2.2.3, 2.2.1 and 4.2.2 we know the right-hand side of (4.13) belongs to $L^2_{-1/2-\epsilon}$. We are now in a position to present one of the results concerning the direct scattering problem.

Theorem 4.2.1. *When k is large enough such that $\|\mathcal{R}_k V\|_{\mathcal{L}(L^2_{-1/2-\epsilon}, L^2_{-1/2-\epsilon})} < 1$, there exists a unique stochastic process $u(\cdot,\omega)\colon \mathbb{R}^3 \to \mathbb{C}$ such that $u^{sc}(x)$ satisfies* (4.13) *a.s.. Moreover, $u(\cdot,\omega) \in L^2_{-1/2-\epsilon}$ a.s. for any $\epsilon \in \mathbb{R}_+$. Then $u(x)$ is called the* mild solution *to the random scattering problem* (4.1).

Proof. By Lemmas 2.2.1, 2.2.3 and 4.2.2, we obtain

$$F := -\mathcal{R}_k f \in L^2_{-1-\epsilon}.$$

According to Lemma 2.2.3, there exists a constant $k_0 > 0$ depending on D and V such that for all $k > k_0$, $\|\mathcal{R}_k V\|_{\mathcal{L}(L^2_{-1/2-\epsilon}, L^2_{-1/2-\epsilon})} < 1$. Hence, $\sum_{j=0}^{\infty}(\mathcal{R}_k V)^j$ is well-defined. Therefore, $\sum_{j=0}^{\infty}(\mathcal{R}_k V)^j F \in L^2_{-1/2-\epsilon}$. Because $\sum_{j=0}^{\infty}(\mathcal{R}_k V)^j = (I-\mathcal{R}_k V)^{-1}$, we see $(I-\mathcal{R}_k V)^{-1} F \in L^2_{-1/2-\epsilon}$. Let $u := (I-\mathcal{R}_k V)^{-1} F \in L^2_{-1/2-\epsilon}$, then u^{sc} fulfills requirements. That is, the existence of the mild solution is proven. The uniqueness and stability of the mild solution follows easily from the inequality

$$\|u\|_{L^2_{-1/2-\epsilon}} \leq \sum_{j\geq 0} \|\mathcal{R}_k V\|^j_{\mathcal{L}(L^2_{-1/2-\epsilon}, L^2_{-1/2-\epsilon})} \|\mathcal{R}_k f\|_{L^2_{-1/2-\epsilon}} \leq C \|\mathcal{R}_k f\|_{L^2_{-1/2-\epsilon}}.$$

The proof is complete. □

Next we show that the far-field pattern is well-defined in the L^2 sense. Assume that k is large enough. From (4.13) we deduce that

$$u = -(I-\mathcal{R}_k V)^{-1}(\mathcal{R}_k f) = -\mathcal{R}_k (I - V\mathcal{R}_k)^{-1}(f).$$

Therefore, we define the far-field pattern of the scattered wave $u(x,k,\omega)$ formally in the following manner,

$$u^\infty(\hat{x},k,\omega) := \frac{-1}{4\pi}\int_D e^{-ik\hat{x}\cdot y}(I-V\mathcal{R}_k)^{-1}(f)(y)\,\mathrm{d}y, \quad \hat{x}\in\mathbb{S}^2. \tag{4.14}$$

Theorem 4.2.2. *Define the far-field pattern of the mild solution as in* (4.14). *When k is large enough, there is a subset $\Omega_0\subset\Omega$, with zero measure $\mathbb{P}(\Omega_0)=0$, such that there holds*

$$u^\infty(\hat{x},k,\omega)\in L^2(\mathbb{S}^2), \quad \forall\,\omega\in\Omega\backslash\Omega_0.$$

Proof of Theorem 4.2.2. By Lemma 2.2.4, we have

$$\|V\mathcal{R}_k\|_{\mathcal{L}(L^2(D_f),L^2(D_f))}\le Ck^{-1}<1,$$

when k is sufficiently large. Therefore, we have

$$\begin{aligned}
&\int_{\mathbb{S}^2}|u^\infty(\hat{x},k,\omega)|^2\,\mathrm{d}S(\hat{x})\\
\lesssim\;&\int_{\mathbb{S}^2}\Big|\int_{D_f}e^{-ik\hat{x}\cdot y}(I-V\mathcal{R}_k)^{-1}(f)\,\mathrm{d}y\Big|^2\,\mathrm{d}S(\hat{x})\\
\lesssim\;&\int\Big|\int_{D_f}e^{-ik\hat{x}\cdot y}\sum_{j\ge1}(V\mathcal{R}_k)^j(f)\,\mathrm{d}y\Big|^2\,\mathrm{d}S(\hat{x})+\int|\langle f,e^{-ik\hat{x}\cdot(\cdot)}\rangle|^2\,\mathrm{d}S(\hat{x})\\
=:\;&f_1(\hat{x},k,\omega)+f_2(\hat{x},k,\omega).
\end{aligned}\tag{4.15}$$

Next, we derive estimates on these terms f_j $(j=1,2)$ in (4.15).

$$\begin{aligned}
f_1(\hat{x},k,\omega)&=\int\Big|\int_{D_f}e^{-ik\hat{x}\cdot y}\sum_{j\ge1}(V\mathcal{R}_k)^j(f)\,\mathrm{d}y\Big|^2\,\mathrm{d}S(\hat{x})\\
&\le C\int\int_{D_f}\Big|\sum_{j\ge0}(V\mathcal{R}_k)^jV\mathcal{R}_kf\Big|^2\,\mathrm{d}y\,\mathrm{d}S(\hat{x})\\
&=C\Big\|\sum_{j\ge0}(V\mathcal{R}_k)^jV\mathcal{R}_kf\Big\|^2_{L^2(D_f)}\\
&\le C\Big(\sum_{j\ge0}\|(V\mathcal{R}_k)^jV\mathcal{R}_kf\|_{L^2(D_f)}\Big)^2\\
&\le C\Big(\sum_{j\ge0}k^{-j}\|V\mathcal{R}_kf\|_{L^2(D_f)}\Big)^2\\
&\le C\|V\|^2_{L^\infty(D_f)}\|\mathcal{R}_kf\|^2_{L^2(D_f)}<C_0<+\infty,
\end{aligned}$$

for some constant C_0 independent of k. The independence of C_0 about k can be seen from Lemma 4.2.2.

By (4.5) and Fubini's theorem, the expectation of $f_2(\hat{x}, k, \omega)$ can be computed as

$$\begin{aligned}\mathbb{E}f_2(\hat{x}, k, \omega) &= \mathbb{E}\int |\langle f, e^{-ik\hat{x}\cdot(\cdot)}\rangle|^2 \,\mathrm{d}S(\hat{x}) = \int \mathbb{E}|\langle f, e^{-ik\hat{x}\cdot(\cdot)}\rangle|^2 \,\mathrm{d}S(\hat{x}) \\ &= \int |\langle C(\chi_{D_f} e^{-ik\hat{x}\cdot(\cdot)}), (\chi_{D_f} e^{ik\hat{x}\cdot(\cdot)})\rangle| \,\mathrm{d}S(\hat{x}) \\ &\leq \int \|C(\chi_{D_f} e^{-ik\hat{x}\cdot(\cdot)})\|_{L^2(\mathbb{R}^3)} \cdot \|\chi_{D_f} e^{ik\hat{x}\cdot(\cdot)}\|_{L^2(\mathbb{R}^3)} \,\mathrm{d}S(\hat{x}).\end{aligned}$$

The symbol of the pseudo-differential operator is of order $-m < 0$, thus C is bounded operator from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$; see [Theorem 11.7, 49]. Hence

$$\begin{aligned}\mathbb{E}f_2(\hat{x}, k, \omega) &\leq C \int \|\chi_{D_f} e^{-ik\hat{x}\cdot(\cdot)}\|_{L^2(\mathbb{R}^3)} \cdot \|\chi_{D_f} e^{ik\hat{x}\cdot(\cdot)}\|_{L^2(D_f)} \,\mathrm{d}S(\hat{x}) \\ &\leq C \int \|\chi_{D_f}\|_{L^2(\mathbb{R}^3)} \cdot \|\chi_{D_f}\|_{L^2(D_f)} \,\mathrm{d}S(\hat{x}) \\ &\leq C_D < +\infty,\end{aligned}$$

for some constant independent of $\hat{x}$ and k. Thus, $f_2(\hat{x}, k, \omega) < +\infty$ almost surely.

Combining the estimates on $f_j(\hat{x}, \omega)$ $(j = 1, 2)$, we conclude that

$$\int_{\mathbb{S}^2} |u^\infty(\hat{x}, k, \omega)|^2 \,\mathrm{d}S(\hat{x}) < \infty$$

almost surely. The proof is done. □

4.3 Some asymptotic estimates

Similar to the beginning of Section 3.3, we need to analyze the asymptotics of leader order and higher order terms. To recovery the rough strength, only the passive far-field patterns are utilized. Motivated by [11, 35], our recovery formula of the rough strength is of the form

$$\frac{1}{K}\int_K^{2K} \overline{u^\infty(\hat{x}, k, \omega)} \cdot u^\infty(\hat{x}, k + \tau, \omega) \,\mathrm{d}k. \tag{4.16}$$

We Set

$$u_1^\infty(\hat{x}, k, \omega) := u^\infty(\hat{x}, k, \omega) - \mathbb{E}u^\infty(\hat{x}, k). \tag{4.17}$$

Note that u_1^∞ is independent of the incident direction d. Assume that $k > k^*$, and expanding $\sum_{j=0}^{+\infty}(\mathcal{R}_k V)^j$ in the form of Neumann series,

$$\begin{aligned} u_1^\infty(\hat{x}, k, \omega) &= \frac{-1}{4\pi} \sum_{j=0}^{+\infty} \int_D e^{-ik\hat{x}\cdot y} (\mathcal{R}_k V)^j (\sigma \dot{B}_y) \, \mathrm{d}y, \quad \hat{x} \in \mathbb{S}^2 \\ &:= \frac{-1}{4\pi} \big[F_0(k, \hat{x}) + F_1(k, \hat{x}) \big], \end{aligned} \tag{4.18}$$

where

$$\begin{cases} F_0(k, \hat{x}, \omega) := \langle f(\cdot), e^{-ik\hat{x}\cdot(\cdot)} \rangle, \\ F_1(k, \hat{x}, \omega) := \displaystyle\sum_{j\geq 1} \int_{D_f} e^{-ik\hat{x}\cdot y} (V\mathcal{R}_k)^j (f) \, \mathrm{d}y. \end{cases} \tag{4.19}$$

Meanwhile, the expectation of the far-field pattern $\mathbb{E}u^\infty$ is

$$\mathbb{E}u^\infty(\hat{x}, k) = \frac{-1}{4\pi} \int_{D_f} e^{-ik\hat{x}\cdot y} \big((I - V\mathcal{R}_k)^{-1} \mathbb{E}f \big)(y) \, \mathrm{d}y, \quad \hat{x} \in \mathbb{S}^2. \tag{4.20}$$

Lemma 4.3.1. *We have*

$$\lim_{k\to+\infty} |\mathbb{E}u^\infty(\hat{x}, k)| = 0 \quad \textit{uniformly in } \hat{x} \in \mathbb{S}^2.$$

Proof of Lemma 4.3.1. Due to the fact that $f \in L^\infty(D) \subset L^2(D)$, we know

$$\forall\, \epsilon > 0, \exists\, \varphi_\epsilon \in \mathscr{D}(D), \text{ s.t. } \|f - \varphi_\epsilon\|_{L^2(D)} < \epsilon/(2|D|^{\frac{1}{2}}). \tag{4.21}$$

When $k > k^*$, we know $(I - V\mathcal{R}_k)^{-1}$ equals to $I + \sum_{j=1}^{+\infty}(V\mathcal{R}_k)^j$. By (4.21), Lemma 2.2.4 and utilizing the stationary phase lemma, one can compute

$$\begin{aligned} |\mathbb{E}u^\infty(\hat{x}, k)| &\lesssim \Big| \int_{D_f} e^{-ik\hat{x}\cdot y} \varphi_\epsilon(y) \, \mathrm{d}y \Big| + \Big| \int_{D_f} e^{-ik\hat{x}\cdot y} \big[f(y) - \varphi_\epsilon(y) + \big(\sum_{j\geq 1} (V\mathcal{R}_k)^j \mathbb{E}f \big)(y) \big] \, \mathrm{d}y \Big| \\ &\lesssim \Big| k^{-2} \int_{D_f} e^{-ik\hat{x}\cdot y} \cdot \Delta\varphi_\epsilon(y) \, \mathrm{d}y \Big| + |D_f|^{\frac{1}{2}} \cdot \|f - \varphi_\epsilon + \sum_{j\geq 1} (V\mathcal{R}_k)^j \mathbb{E}f\|_{L^2(D_f)} \end{aligned}$$

$$
\le k^{-2} \cdot |D_f|^{\frac{1}{2}} \cdot \|\Delta\varphi_\epsilon\|_{L^2(D_f)} + |D_f|^{\frac{1}{2}} \cdot \big(\epsilon/(2|D_f|^{\frac{1}{2}}) + C \sum_{j\ge 1} k^{-j} \|\mathbb{E}f\|_{L^2(D_f)}\big)
$$

$$
= k^{-2} \cdot |D_f|^{\frac{1}{2}} \|\Delta\varphi_\epsilon\|_{L^2(D_f)} + \epsilon/2 + C(k-1)^{-1} \cdot \|\mathbb{E}f\|_{L^2(D_f)}. \tag{4.22}
$$

Write $\mathcal{K} := \max\{K_0, \frac{2}{\sqrt{\epsilon}}|D_f|^{\frac{1}{4}}\|\Delta\varphi_\epsilon\|_{L^2(D_f)}^{\frac{1}{2}}, 1 + \frac{4C}{\epsilon}\|\mathbb{E}f\|_{L^2(D_f)}\}$. From (4.22), we have

$$
\forall\, k > \mathcal{K}, \quad |\mathbb{E}u^\infty(\hat{x}, k)| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon, \quad \text{uniformly for} \quad \forall \hat{x} \in \mathbb{S}^2.
$$

Since the ϵ is taken arbitrarily, the conclusion follows. □

By substituting (4.18)-(4.19) into (4.16), we obtain several crossover terms between F_0 and F_1. The asymptotic estimates of these crossover terms are the main purpose of Sections 4.3.1 and 4.3.2. Section 4.3.1 focuses on the estimate of the leading order term while the estimates of the higher order terms are presented in Section 4.3.2.

4.3.1 Asymptotics of the leading order term

Lemma 4.3.2 below is the asymptotic estimate of the crossover leading order term. By utilizing the ergodicity, the result of Lemma 4.3.2 is also statistically stable. In order to keep our arguments flowing, we postpone Lemma 4.3.3 until we finish Lemma 4.3.2.

Lemma 4.3.2. *Let $F_j(k, \hat{x})$ $(j = 0, 1)$ be defined as in (4.19). Write*

$$
X_{0,0}(K, \tau, \hat{x}) = \frac{1}{K} \int_K^{2K} k^m \overline{F_0(k, \hat{x})} \cdot F_0(k + \tau, \hat{x})\, \mathrm{d}k.
$$

Assume $\{K_j\} \in P(1 + \gamma)$, then for any $\tau > 0$, we have

$$
\lim_{j\to+\infty} X_{0,0}(K_j, \tau, \hat{x}) = (2\pi)^{3/2}\widehat{\mu}(\tau\hat{x}) \qquad a.s. \; . \tag{4.23}
$$

In what follows, we may denote $X_{0,0}(K, \tau, \hat{x})$ as $X_{0,0}$ for short if it is clear in the context.

Proof of Lemma 4.3.2. By (4.83), we have

$$\mathbb{E}(X_{0,0}) = \frac{1}{K}\int_K^{2K} k^m \mathbb{E}\big(\overline{F_0(k,\hat{x})}F_0(k+\tau,\hat{x})\big)\,\mathrm{d}k = \frac{1}{K}\int_K^{2K}[(2\pi)^{3/2}\,\widehat{\mu}(\tau\hat{x}) + \mathcal{O}(k^{-1})]\,\mathrm{d}k$$
$$= (2\pi)^{3/2}\,\widehat{\mu}(\tau\hat{x}) + \mathcal{O}(K^{-1}), \quad K\to+\infty.$$

By Isserlis' Theorem and (4.83), and note that $\overline{F_j(k,\hat{x})} = F_j(-k,\hat{x}), F_0(-k,-\hat{x}) = F_0(k,\hat{x})$, one can compute

$$\begin{aligned}
&\mathbb{E}\big(|X_{0,0} - (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x})|^2\big)\\
=&\frac{1}{K^2}\int_K^{2K}\int_K^{2K}\mathbb{E}\Big(\big[k_1^m F_0(k_1+\tau,\hat{x})\overline{F_0(k_1,\hat{x})} - (2\pi)^{3/2}\widehat{\mu}(\tau\hat{x})\big]\\
&\times\big[k_2^m\overline{F_0(k_2+\tau,\hat{x})}F_0(k_2,\hat{x}) - (2\pi)^{3/2}\overline{\widehat{\mu}(\tau\hat{x})}\big]\Big)\,\mathrm{d}k_1\,\mathrm{d}k_2\\
=&\frac{1}{K^2}\int_K^{2K}\int_K^{2K}\mathbb{E}\big(k_1^m F_0(k_1+\tau,\hat{x})\overline{F_0(k_1,\hat{x})}\,\big)\cdot\overline{\mathbb{E}\big(k_2^m F_0(k_2+\tau,\hat{x})\overline{F_0(k_2,\hat{x})}\big)}\\
&+(1+\tau/k_2)^{-m}\mathbb{E}\big((k_2+\tau)^m F_0(k_1+\tau,\hat{x})\overline{F_0(k_2+\tau,\hat{x})}\,\big)\cdot\mathbb{E}\big(k_1^m F_0(k_2,\hat{x})\overline{F_0(k_1,\hat{x})}\big)\\
&+\overline{\mathbb{E}\big(k_2^m F_0(-k_1-\tau,\hat{x})\overline{F_0(k_2,\hat{x})}\,\big)}\cdot\mathbb{E}\big(k_1^m F_0(-k_2-\tau,\hat{x})\,\overline{F_0(k_1,\hat{x})}\,\big)\,\mathrm{d}k_1\,\mathrm{d}k_2\\
&-(2\pi)^3|\widehat{\mu}(\tau\hat{x})|^2 + \mathcal{O}(|\hat{\mu}(\tau\hat{x})|K^{-1})\\
=&\frac{1}{K^2}\int_K^{2K}\int_K^{2K}(1+\frac{\tau}{k_2})^{-m}[(2\pi)^{3/2}\,\widehat{\mu}((k_1-k_2)\hat{x}) + \mathcal{O}(k_2^{-1})]\\
&\qquad\cdot[(2\pi)^{3/2}\overline{\widehat{\mu}((k_1-k_2)\hat{x})} + \mathcal{O}(k_1^{-1})]\\
&\quad+[(2\pi)^{3/2}\,\widehat{\mu}((k_1+k_2+\tau)\hat{x}) + \mathcal{O}(k_2^{-1})]\\
&\qquad\cdot[(2\pi)^{3/2}\,\overline{\widehat{\mu}((k_1+k_2+\tau)\hat{x})} + \mathcal{O}(k_1^{-1})]\,\mathrm{d}k_1\,\mathrm{d}k_2 + \mathcal{O}(|\hat{\mu}(\tau\hat{x})|K^{-1})\\
=&\frac{(2\pi)^3}{K^2}\int_K^{2K}\int_K^{2K}(1+\tau/k_2)^{-m}|\widehat{\mu}((k_1-k_2)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2\\
&+\frac{(2\pi)^3}{K^2}\int_K^{2K}\int_K^{2K}|\widehat{\mu}((k_1+k_2+\tau)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2\\
&+\frac{(2\pi)^{3/2}}{K^2}\int_K^{2K}\int_K^{2K}\widehat{\mu}((k_1-k_2)\hat{x})\cdot(1+\tau/k_2)^{-m}\mathcal{O}(k_1^{-1})\,\mathrm{d}k_1\,\mathrm{d}k_2\\
&+\frac{(2\pi)^{3/2}}{K^2}\int_K^{2K}\int_K^{2K}\overline{\widehat{\mu}((k_1-k_2)\hat{x})}\cdot\mathcal{O}(k_2^{-1})\,\mathrm{d}k_1\,\mathrm{d}k_2 + \mathcal{O}(K^{-1})\\
\le&\frac{(2\pi)^3}{K^2}\int_K^{2K}\int_K^{2K}|\widehat{\mu}((k_1-k_2)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2 + \frac{(2\pi)^3}{K^2}\int_K^{2K}\int_K^{2K}|\widehat{\mu}((k_1+k_2+\tau)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2
\end{aligned}$$

$$
+\Big(\frac{1}{K^2}\int_K^{2K}\int_K^{2K}|\widehat{\mu}((k_1-k_2)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2\Big)^{1/2}\cdot\mathcal{O}(K^{-1})
$$
$$
+\Big(\frac{1}{K^2}\int_K^{2K}\int_K^{2K}|\widehat{\mu}((k_1-k_2)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2\Big)^{1/2}\cdot\mathcal{O}(K^{-1})+\mathcal{O}(K^{-1}). \tag{4.24}
$$

Note that these missing terms involving $\widehat{\mu}((k_1+k_2+\tau)\hat{x})$ in (4.24) is counted into $\mathcal{O}(K^{-1})$ because $\widehat{\mu}((k_1+k_2+\tau)\hat{x})\to 0$ $(k_1,k_2\to+\infty)$. By (4.24) and Lemma 4.3.3, we have

$$
\mathbb{E}\big(|X_{0,0}-(2\pi)^{3/2}\widehat{\mu}(\tau\hat{x})|^2\big)=\mathcal{O}(K^{-1}),\quad K\to+\infty. \tag{4.25}
$$

Fixing an integer $K_0>0$, and by Chebyshev's inequality and (4.25) we have

$$
P\Big(\bigcup_{j\geq K_0}\{|X_{0,0}(K_j)-(2\pi)^{3/2}\widehat{\mu}(\tau\hat{x})|\geq\epsilon\}\Big)\leq\frac{1}{\epsilon^2}\sum_{j\geq K_0}\mathbb{E}\big(|X_{0,0}(K_j)-(2\pi)^{3/2}\widehat{\mu}(\tau\hat{x})|^2\big)
$$
$$
\lesssim\frac{1}{\epsilon^2}\sum_{j\geq K_0}K_j^{-1}=\frac{1}{\epsilon^2}\sum_{j\geq K_0}j^{-1-\gamma}\leq\frac{1}{\epsilon^2}\int_{K_0}^{+\infty}(t-1)^{-1-\gamma}\,\mathrm{d}t=\frac{1}{\epsilon^2\gamma}(K_0-1)^{-\gamma}. \tag{4.26}
$$

Here $X_{0,0}(K_j)$ stands for $X_{0,0}(K_j,\tau,\hat{x})$. By Lemma 2.2.5, (4.26) implies that for any fixed $\tau\geq 0$ and $\hat{x}\in\mathbb{S}^2$, we have

$$
X_{0,0}(K_j,\tau,\hat{x})\to(2\pi)^{3/2}\widehat{\mu}(\tau\hat{x})\quad\text{ a.s. .}
$$

The proof is complete. □

Lemma 4.3.3. *When $\mu\in C_c^\infty(D)$, $\tau\in\mathbb{R}$ and $\hat{x}\in\mathbb{S}^2$, we have*

$$
\frac{(2\pi)^3}{K^2}\int_K^{2K}\int_K^{2K}|\widehat{\mu}((k_1-k_2)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2\leq CK^{-1}, \tag{4.27}
$$
$$
\frac{(2\pi)^3}{K^2}\int_K^{2K}\int_K^{2K}|\widehat{\mu}((k_1+k_2+\tau)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2\leq CK^{-1}, \tag{4.28}
$$

for some constant C independent of τ and $\hat{x}$.

Lemma 4.3.3 gives estimates on terms that arise in the proof of Lemma 4.3.2. The reason that the decaying order in Lemma 4.3.3 is higher than that in Lemma 3.3.3 is due to the smoothness of $\widehat{\mu}$.

Proof of Lemma 4.3.3. To conclude (4.27), we make a change of variable,

$$\begin{cases} s = k_1 - k_2, \\ t = k_2. \end{cases}$$

Write $Q = \{(s,t) \in \mathbb{R}^2 \,\big|\, K \le s+t \le 2K,\ K \le t \le 2K\}$. Q is illustrated as in Figure 4.1.

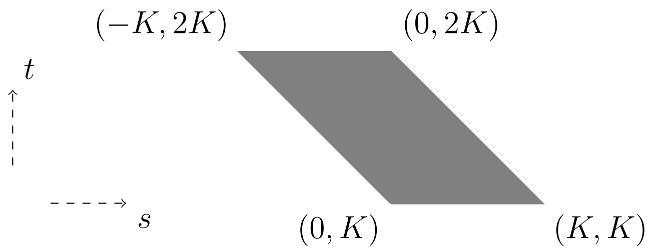

Figure 4.1: Illustration of Q

Recall that $\operatorname{supp}\sigma \subseteq D$, then we have

$$\begin{aligned} &\frac{1}{K^2}\int_K^{2K}\int_K^{2K} |\widehat{\mu}((k_1-k_2)\hat{x})|^2 \,\mathrm{d}k_1\,\mathrm{d}k_2 = \frac{1}{K^2}\iint_Q |\widehat{\mu}(s\hat{x})|^2 \,\mathrm{d}s\,\mathrm{d}t \\ =&\frac{1}{K^2}\int_{-K}^{0}(K+s)|\widehat{\mu}(s\hat{x})|^2\,\mathrm{d}s + \frac{1}{K^2}\int_0^K (K-s)|\widehat{\mu}(s\hat{x})|^2\,\mathrm{d}s \\ \le&\frac{2}{K}\int_{\mathbb{R}} |\widehat{\mu}(s\hat{x})|^2\,\mathrm{d}s. \end{aligned} \tag{4.29}$$

Recall that $\mu \in C_c^\infty(\mathbb{R}^3) \subset \mathscr{S}(\mathbb{R}^3)$, thus $\widehat{\mu}(x)$ decays faster than the reciprocal of any polynomials, especially,

$$|\widehat{\mu}(s\hat{x})| \le C\langle x\rangle^{-1}, \quad \forall\, x \in \mathbb{R}^3. \tag{4.30}$$

Thus (4.29)-(4.30) gives

$$\frac{1}{K^2}\int_K^{2K}\int_K^{2K} |\widehat{\mu}((k_1-k_2)\hat{x})|^2\,\mathrm{d}k_1\,\mathrm{d}k_2 \le \frac{2}{K}\int_{\mathbb{R}} C\langle s\rangle^{-2}\,\mathrm{d}s \le CK^{-1},$$

which is (4.27).

To prove (4.28), again we make a change of variable:

$$\begin{cases} s = k_1 + k_2 + \tau, \\ t = k_2. \end{cases} \tag{4.31}$$

Write $Q' = \{(s,t) \in \mathbb{R}^2 \,\big|\, K \leq s - t - \tau \leq 2K,\ K \leq t \leq 2K\}$. Combining with (4.30), one can compute

$$\begin{aligned}
&\frac{1}{K^2}\int_K^{2K}\int_K^{2K} |\widehat{\mu}((k_1+k_2+\tau)\hat{x})|^2 \,\mathrm{d}k_1\,\mathrm{d}k_2 = \frac{1}{K^2}\iint_{Q'} |\widehat{\mu}(s\hat{x})|^2 \,\mathrm{d}s\,\mathrm{d}s \\
=&\frac{1}{K^2}\int_{2K+\tau}^{3K+\tau} (s-2K-\tau)|\widehat{\mu}(s\hat{x})|^2 \,\mathrm{d}s + \frac{1}{K^2}\int_{3K+\tau}^{4K+\tau} (4K+\tau-s)|\widehat{\mu}(s\hat{x})|^2 \,\mathrm{d}s \\
\leq&\frac{2}{K}\int_{2K-\tau}^{2K+\tau} |\widehat{\mu}(s\hat{x})|^2 \,\mathrm{d}s = \frac{2}{K}\int_{\mathbb{R}} |\widehat{\mu}(s\hat{x})|^2 \,\mathrm{d}s \leq \frac{C}{K}\int_{\mathbb{R}} \langle s\rangle^{-2} \,\mathrm{d}s \leq \frac{C}{K},
\end{aligned}$$

which gives (4.28).

The proof is complete. □

4.3.2 Asymptotics of higher order terms

In Section 4.3.2, Lemmas 4.3.4, 4.3.5 and 4.3.6 are presented and proved. These lemmas play key roles in the proofs to Lemmas 4.4.1, 4.4.2 and 4.4.3.

Lemma 4.3.4. *Define $F_j(k,\hat{x})$ $(j=0,1)$ as in (4.19). For every $\hat{x} \in \mathbb{S}^2$ and every $k_1, k_2 \geq k$, when $k \to +\infty$, we have the following estimates:*

$$\big|\mathbb{E}\big(\overline{F_1(k_2,\hat{x})}F_0(k_1,\hat{x})\big)\big| = \mathcal{O}(k^{-m-1}), \quad \big|\mathbb{E}\big(F_0(k_1,\hat{x}) \cdot F_1(k_2,\hat{x})\big)\big| = \mathcal{O}(k^{-m-1}), \tag{4.32}$$

uniformly for all $\hat{x}$.

Proof of Lemma 4.3.4. We only prove the first equations in (4.32), because for the second equation, the arguments of proving it are similar to the that of the first equation. For simplicity, we may use D to stand for D_V, and D_z to stand for the integral domain D_V of argument z.

In what follows we let $\hat{x}_1, \hat{x}_2 \in \mathbb{S}^2$. In this proof we may drop the arguments $k, \hat{x}$

if it is clear in the context. Write

$$G_0(k,\hat{x}) := \langle f, e^{-ik\hat{x}\cdot(\cdot)}\rangle, \quad G_j(k,\hat{x}) := \int_D e^{-ik\hat{x}\cdot y}\big((V\mathcal{R}_k)^j(f)\big)(y)\,\mathrm{d}y, \tag{4.33}$$

$$r_j(k,\hat{x}) := \sum_{s\geq j} G_s(k,\hat{x}), \tag{4.34}$$

for $j = 1, 2, \cdots$. Thus

$$F_0 = G_0, \quad F_1 = r_1 = G_1 + r_2,$$

so we have

$$\mathbb{E}\big(F_0(k_1,\hat{x}_1)\cdot\overline{F_1(k_2,\hat{x}_2)}\big) = \mathbb{E}\big(G_0(k_1,\hat{x}_1)\cdot\overline{G_1(k_2,\hat{x}_2)}\,\big) + \mathbb{E}\big(G_0(k_1,\hat{x}_1)\cdot\overline{r_2(k_2,\hat{x}_2)}\big). \tag{4.35}$$

To prove (4.32), we need to estimate $\mathbb{E}(G_0\overline{G_1})$ and $\mathbb{E}(G_0\overline{r_2})$. One can compute

$$\begin{aligned}
&|\mathbb{E}\big(G_0(k_1,\hat{x}_1)\cdot\overline{G_1(k_2,\hat{x}_2)}\,\big)| \\
=&\big|\mathbb{E}\big(\int_{D_y} e^{-ik_1\hat{x}_1\cdot y} f(y)\,\mathrm{d}y \times \overline{\int_{D_z} e^{-ik_2\hat{x}_2\cdot z}V(z)\int_{D_t}\Phi(z,t)f(t)\,\mathrm{d}t\,\mathrm{d}z}\big)\big| \\
=&\big|\int_{D_z} e^{ik_2\hat{x}_2\cdot z}\overline{V}(z)\cdot\mathbb{E}\Big(\int_{D_y} e^{-ik_1\hat{x}_1\cdot y}f(y)\,\mathrm{d}y\cdot\int_{D_t}\overline{\Phi}(z,t)f(t)\,\mathrm{d}t\Big)\,\mathrm{d}z\big| \\
=&\big|\int_{D_z} e^{ik_2\hat{x}_2\cdot z}\overline{V}(z)\cdot\big(\iint_{D_f\times D_f} K_f(t,y)e^{-ik_1\hat{x}_1\cdot y}\overline{\Phi}(z,t)\,\mathrm{d}y\,\mathrm{d}t\big)\,\mathrm{d}z\big| \\
=&\big|\int_{D_z} e^{ik_2\hat{x}_2\cdot z}\overline{V}(z)\cdot\big(\int_{D_f}(\mu(t)k_1^{-m} + a(t,-k_1\hat{x}_1))e^{-ik_1\hat{x}_1\cdot t}\overline{\Phi}_{k_2}(z,t)\,\mathrm{d}t\big)\,\mathrm{d}z\big| \\
\leq& k_1^{-m}\|V\|_{L^2(D)}\cdot\|\mathcal{R}_{k_2}(\mu e^{ik_1\hat{x}_1\cdot(\cdot)})\|_{L^2(D)} \\
&\quad + k_1^{-m-1}\|V\|_{L^2(D)}\cdot\|\mathcal{R}_{k_2}(k_1^{m+1}\overline{a(\cdot,-k_1\hat{x}_1)}e^{ik_1\hat{x}_1\cdot(\cdot)}\chi_D)\|_{L^2(D)} \\
\lesssim& k_1^{-m}\|V\|_{L^2(D)}\cdot\|\mathcal{R}_{k_2}(\mu e^{ik_1\hat{x}_1\cdot(\cdot)})\|_{L^2_{-1}(\mathbb{R}^3)} \\
&\quad + k_1^{-m-1}\|V\|_{L^2(D)}\cdot\|\mathcal{R}_{k_2}(k_1^{m+1}\overline{a(\cdot,-k_1\hat{x}_1)}e^{ik_1\hat{x}_1\cdot(\cdot)}\chi_D)\|_{L^2_{-1}(\mathbb{R}^3)} \\
\lesssim& k_1^{-m}\|V\|_{L^2(D)}\cdot k_2^{-1}\|\mu e^{ik_1\hat{x}_1\cdot(\cdot)}\|_{L^2_1(\mathbb{R}^3)} \\
&\quad + k_1^{-m-1}\|V\|_{L^2(D)}\cdot k_2^{-1}\|k_1^{m+1}a(\cdot,-k_1\hat{x}_1)e^{-ik_1\hat{x}_1\cdot(\cdot)}\chi_D\|_{L^2_1(\mathbb{R}^3)} \\
\lesssim& k_1^{-m}\|V\|_{L^2(D)}\cdot k_2^{-1}\|\mu e^{ik_1\hat{x}_1\cdot(\cdot)}\|_{L^2(D)} \\
&\quad + k_1^{-m-1}\|V\|_{L^2(D)}\cdot k_2^{-1}\|k_1^{m+1}a(\cdot,-k_1\hat{x}_1)e^{-ik_1\hat{x}_1\cdot(\cdot)}\|_{L^2(D)} \\
=&\mathcal{O}(k_1^{-m}k_2^{-1}), \quad k\to+\infty.
\end{aligned} \tag{4.36}$$

To estimate $\mathbb{E}\big(G_0(k_1,\hat{x}_1)\cdot\overline{r_2(k_2,\hat{x}_2)}\big)$ we first prove for $j>1$,

$$\mathbb{E}\big(G_0(k_1,\hat{x}_1)\cdot\overline{G_j(k_2,\hat{x}_2)}\,\big)=\overline{\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^j\big(c_f(\cdot,k_1\hat{x}_1)\,e^{ik_1\hat{x}_1\cdot(\cdot)}\,\chi_D\big)(z)\,\mathrm{d}z}, \tag{4.37}$$

$$\mathbb{E}\big(G_1(k_1,\hat{x}_1)\cdot\overline{G_j(k_2,\hat{x}_2)}\big)=\overline{\int_D e^{-ik_2\hat{x}_2\cdot z}\big((V\mathcal{R}_{k_2})^j(\chi_D C_f\overline{\mathcal{R}_{k_1}(Ve^{-ik_1\hat{x}_1\cdot(\cdot)})})\big)(z)\,\mathrm{d}z}. \tag{4.38}$$

We have

$$\begin{aligned}
&\mathbb{E}\big(\overline{G_0(k_1,\hat{x}_1)}\cdot G_j(k_2,\hat{x}_2)\big)\\
=&\mathbb{E}\big(\langle f,e^{ik_1\hat{x}_1\cdot(\cdot)}\rangle\cdot\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^j(f)(z)\,\mathrm{d}z\big)\\
=&\mathbb{E}\big(\langle f,e^{ik_1\hat{x}_1\cdot(\cdot)}\rangle\cdot\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\big(V(\cdot)\langle f(y),\Phi_{k_2}(y,\cdot)\rangle\big)(z)\,\mathrm{d}z\big)\\
=&\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\Big(V(\cdot)\mathbb{E}\big(\langle f(t),e^{ik_1\hat{x}_1\cdot t}\rangle\langle f(y),\Phi_{k_2}(y,\cdot)\rangle\big)\Big)(z)\,\mathrm{d}z\\
=&\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\Big(V(\cdot)\iint_{D_f\times D_f}K_f(y,t)e^{ik_1\hat{x}_1\cdot t}\Phi_{k_2}(y,\cdot)\,\mathrm{d}y\,\mathrm{d}t\Big)(z)\,\mathrm{d}z\\
=&\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\Big(V(\cdot)\int_{D_f}\big(\int_{D_f}K_f(y,t)e^{-i(k_1\hat{x}_1)\cdot(y-t)}\,\mathrm{d}t\big)e^{ik_1\hat{x}_1\cdot y}\Phi_{k_2}(y,\cdot)\chi_D(y)\,\mathrm{d}y\Big)(z)\,\mathrm{d}z\\
=&\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\Big(V(\cdot)\int_{D_f}c_f(y,k_1\hat{x}_1)e^{ik_1\hat{x}_1\cdot y}\Phi_{k_2}(y,\cdot)\chi_D(y)\,\mathrm{d}y\Big)(z)\,\mathrm{d}z\\
=&\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\Big(V(\cdot)\mathcal{R}_{k_2}\big(c_f(\cdot,k_1\hat{x}_1)e^{ik_1\hat{x}_1\cdot(\cdot)}\chi_D\big)(\cdot)\Big)(z)\,\mathrm{d}z\\
=&\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j}\big(c_f(\cdot,k_1\hat{x}_1)e^{ik_1\hat{x}_1\cdot(\cdot)}\chi_D\big)(z)\,\mathrm{d}z.
\end{aligned} \tag{4.39}$$

By taking the conjugate of (4.39), we arrive at (4.37). Then to prove (4.38) one can compute

$$\begin{aligned}
&\mathbb{E}\big(\overline{G_1(k_1,\hat{x}_1)}\cdot G_j(k_2,\hat{x}_2)\big)\\
=&\mathbb{E}\big(\int_D e^{ik_1\hat{x}_1\cdot x}\overline{(V\mathcal{R}_{k_1}f)(x)}\,\mathrm{d}x\cdot\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^j(f)(z)\,\mathrm{d}z\big)\\
=&\mathbb{E}\big(\int_D e^{ik_1\hat{x}_1\cdot x}\overline{V}(x)\langle f(t),\Phi_{-k_1}(t,x)\rangle\,\mathrm{d}x\cdot\int_D e^{-ik_1\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\big(V(\cdot)\langle f(y),\Phi_{k_2}(y,\cdot)\rangle\big)(z)\,\mathrm{d}z\big)\\
=&\mathbb{E}\big(\langle f,\overline{\mathcal{R}_{k_1}(Ve^{-ik_1\hat{x}_1\cdot(\cdot)})}\rangle\cdot\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\big(V(\cdot)\langle f(y),\Phi_{k_2}(y,\cdot)\rangle\big)(z)\,\mathrm{d}z\big)\\
=&\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\Big(V(\cdot)\mathbb{E}\big(\langle f,\overline{\mathcal{R}_{k_1}(Ve^{-ik_1\hat{x}_1\cdot(\cdot)})}\rangle\langle f(y),\chi_D(y)\Phi_{k_2}(y,\cdot)\rangle\big)\Big)(z)\,\mathrm{d}z\\
=&\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\Big(V(\cdot)\langle(C_f\overline{\mathcal{R}_{k_1}(Ve^{-ik_1\hat{x}_1\cdot(\cdot)})})(y),\chi_D(y)\Phi_{k_2}(y,\cdot)\rangle\Big)(z)\,\mathrm{d}z\\
=&\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^{j-1}\big(V(\cdot)(\mathcal{R}_{k_2}\chi_D C_f\overline{\mathcal{R}_{k_1}(Ve^{-ik_1\hat{x}_1\cdot(\cdot)})})(\cdot)\big)(z)\,\mathrm{d}z
\end{aligned}$$

$$= \int_D e^{-ik_2\hat{x}_2\cdot z}\big((V\mathcal{R}_{k_2})^j(\chi_D C_f \overline{\mathcal{R}_{k_1}(Ve^{-ik_1\hat{x}_1\cdot(\cdot)})})\big)(z)\,\mathrm{d}z \tag{4.40}$$

We arrive at (4.38) by taking the conjugate of (4.40). By applying (4.37) we have

$$\begin{aligned}
&\big|\mathbb{E}\big(G_0(k_1,\hat{x}_1)\cdot\overline{r_2(k_2,\hat{x}_2)}\big)\big| \le \sum_{j\ge 2}\big|\mathbb{E}\big(G_0(k_1,\hat{x}_1)\cdot\overline{G_j(k_2,\hat{x}_2)}\big)\big| \\
=&\sum_{j\ge 2}\Big|\int_D e^{-ik_2\hat{x}_2\cdot z}(V\mathcal{R}_{k_2})^j\big(c_f(\cdot,k_1\hat{x}_1)e^{ik_1\hat{x}_1\cdot(\cdot)}\chi_D\big)\,\mathrm{d}z\Big| \\
\le&|D|^{1/2}\cdot\sum_{j\ge 2}\|(V\mathcal{R}_{k_2})^j\big(c_f(\cdot,k_1\hat{x}_1)e^{ik_1\hat{x}_1\cdot(\cdot)}\chi_D\big)\|_{L^2(D)} \\
\le&C|D|^{1/2}\cdot\sum_{j\ge 2}k_2^{-j}\|c_f(\cdot,k_1\hat{x}_1)\chi_D\|_{L^2(D)} \\
\le&C|D|^{1/2}k_1^{-m}\cdot\sum_{j\ge 2}k_2^{-j}\|k_1^m c_f(\cdot,k_1\hat{x}_1)\chi_D\|_{L^2(D)} \\
=&\mathcal{O}(k_1^{-m}k_2^{-2}),\quad k\to+\infty.
\end{aligned} \tag{4.41}$$

By (4.35), (4.36) and (4.41), the formula (4.32) is proved. □

Lemma 4.3.5. *Define $F_j(k,\hat{x})$ $(j=0,1)$ as in (4.19). For every $\hat{x}\in\mathbb{S}^2$ and every $k_1,k_2\ge k$, when $k\to+\infty$, we have the following estimates:*

$$\big|\mathbb{E}\big(\overline{F_1(k_2,\hat{x})}F_1(k_1,\hat{x})\big)\big| = \mathcal{O}(k^{-3}),\quad \big|\mathbb{E}\big(F_1(k_1,\hat{x})\cdot F_1(k_2,\hat{x})\big)\big| = \mathcal{O}(k^{-3}), \tag{4.42}$$

uniformly for all $\hat{x}$.

Proof of Lemma 4.3.5. We only prove the first equations in (4.42), because for the second equation, the arguments of proving it are similar to the that of the first equation.

To prove (4.42), the following equations (4.43)-(4.44) are useful.

$$G_j(k,\hat{x}) = \langle f(s), \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y}\big[(V\mathcal{R}_k)^{j-1}(V(\cdot)\Phi(s,\cdot))\big](y)\,\mathrm{d}y\rangle \quad (j\ge 1). \tag{4.43}$$

$$\begin{aligned}
&\mathbb{E}\big(G_j(k_1,\hat{x}_1)\cdot\overline{G_\ell(k_2,\hat{x}_2)}\big) \\
=&\int_D e^{ik_2\hat{x}_2\cdot z}\Big\{(V\mathcal{R}_{k_2})^{\ell-1}\Big(\int_D e^{-ik_1\hat{x}_1\cdot y}\big[(V\mathcal{R}_{k_1})^{j-1}(V(1)\overline{V}(2)I(2,1))\big](y)\,\mathrm{d}y\Big)\Big\}(z)\,\mathrm{d}z \quad (j,\ell\ge 1),
\end{aligned} \tag{4.44}$$

where

$$I(x,y) := \iint_{U\times U} K_f(s,t)\Phi(s-y)\overline{\Phi}(t-x)\,\mathrm{d}s\,\mathrm{d}t. \tag{4.45}$$

In (4.44), with some abuse of notation, we use "1" (resp. "2") to represent the variable that $V\mathcal{R}_{k_1}$ (resp. $V\mathcal{R}_{k_2}$) acts on.

To prove (4.43), one can compute

$$\begin{aligned}[(V\mathcal{R}_k)^j f](x) &= [(V\mathcal{R}_k)^{j-1}((V\mathcal{R}_k)f)](x) = \big[(V\mathcal{R}_k)^{j-1}(V(\cdot)\langle f(s),\Phi_k(s,\cdot)\rangle)\big](x)\\ &= \langle f(s), \big[(V\mathcal{R}_k)^{j-1}(V(\cdot)\Phi(s,\cdot))\big](x)\rangle. \end{aligned} \tag{4.46}$$

By (4.33) and (4.46), we arrive at (4.43).

To prove (4.44), one can compute

$$\begin{aligned}
&\mathbb{E}\big(G_j(k_1,\hat{x}_1)\cdot\overline{G_\ell(k_2,\hat{x}_2)}\big)\\
&= \mathbb{E}\Big(\big\langle f(s), \int_D e^{-ik_1\hat{x}_1\cdot y}\big[(V\mathcal{R}_{k_1})^{j-1}(V(\cdot)\Phi(s,\cdot))\big](y)\,\mathrm{d}y\big\rangle\\
&\qquad\cdot\big\langle f(t), \int_D e^{ik_2\hat{x}_2\cdot z}\big[(V\mathcal{R}_{k_2})^{\ell-1}(\overline{V}(\cdot)\overline{\Phi}(t,\cdot))\big](z)\,\mathrm{d}z\big\rangle\Big)\\
&= \iint_{D_f\times D_f} K_f(s,t)\int_D e^{-ik_1\hat{x}_1\cdot y}\big[(V\mathcal{R}_{k_1})^{j-1}(V(\cdot)\Phi(s,\cdot))\big](y)\,\mathrm{d}y\\
&\qquad\cdot\int_D e^{ik_2\hat{x}_2\cdot z}\big[(V\mathcal{R}_{k_2})^{\ell-1}(\overline{V}(\cdot)\overline{\Phi}(t,\cdot))\big](z)\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}t\\
&= \iint_{D_f\times D_f}\int_D e^{-ik_1\hat{x}_1\cdot y}\big[(V\mathcal{R}_{k_1})^{j-1}(K(s,t)V(\cdot)\Phi(s,\cdot))\big](y)\,\mathrm{d}y\\
&\qquad\cdot\int_D e^{ik_2\hat{x}_2\cdot z}\big[(V\mathcal{R}_{k_2})^{\ell-1}(\overline{V}(\cdot)\overline{\Phi}(t,\cdot))\big](z)\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}t\\
&= \int_D e^{ik_2\hat{x}_2\cdot z}\Big\{(V\mathcal{R}_{k_2})^{\ell-1}\Big(\int_D e^{-ik_1\hat{x}_1\cdot y}\big[(V\mathcal{R}_{k_1})^{j-1}(V(1)\overline{V}(2)I(2,1))\big](y)\,\mathrm{d}y\Big)\Big\}(z)\,\mathrm{d}z.
\end{aligned}$$

Thus, (4.44) is proved.

Note that

$$\mathbb{E}\big(F_1(k_1,\hat{x}_1)\cdot\overline{F_1(k_2,\hat{x}_2)}\big) = \mathbb{E}\big(G_1(k_1,\hat{x}_1)\cdot\overline{G_1(k_2,\hat{x}_2)}\,\big) + \sum_{\substack{j+\ell\geq 3\\ j,\ell\geq 1}}\mathbb{E}\big(G_j(k_1,\hat{x}_1)\cdot\overline{G_\ell(k_2,\hat{x}_2)}\,\big). \tag{4.47}$$

We are to estimate $\mathbb{E}(G_1\overline{G_1})$, $\mathbb{E}(G_j\overline{G_\ell})$ $(j+\ell\geq 3,\ j,\ell\geq 1)$ in different manners.

To estimate $\mathbb{E}(G_1\overline{G_1})$, fix $\eta_i \in C_c^\infty(\mathbb{R}^3)\,(i = 1, 2, 3)$ such that $\eta_1 \equiv 1$ in D, $\eta_2 \equiv 1$ in U and $\eta_3 \equiv 1$ in $\{s + t \in \mathbb{R}^3; s, t \in U\}$, one can compute

$$
\begin{aligned}
&\mathbb{E}\big(G_1(k_1,\hat{x}_1)\cdot\overline{G_1(k_2,\hat{x}_2)}\big)\\
=&\int_D e^{ik_2\hat{x}_2\cdot z}\int_D e^{-ik_1\hat{x}_1\cdot y}V(y)\overline{V}(z)I(z,y)\,\mathrm{d}y\,\mathrm{d}z\\
\simeq&\iint \eta_3(s+t)\eta_2(s)\eta_2(t)\Big(\int e^{i(s-t)\cdot\xi}c_f(s,\xi)\,\mathrm{d}\xi\Big)\cdot\Big(\int e^{-ik_1(\hat{x}_1\cdot y-|y-s|)}\frac{V(y)}{|y-s|}\eta_1(y)\,\mathrm{d}y\Big)\\
&\quad\cdot\Big(\int e^{ik_2(\hat{x}_2\cdot z-|z-t|)}\frac{\overline{V}(z)}{|z-t|}\eta_1(z)\,\mathrm{d}z\Big)\,\mathrm{d}s\,\mathrm{d}t\\
\simeq&\iint \eta_3(s+t)\Big(\int e^{i(s-t)\cdot\xi}\tilde{c}(s,t,\xi)\,\mathrm{d}\xi\Big)e^{-ik_1\hat{x}_1\cdot s}e^{ik_2\hat{x}_2\cdot t}\cdot\Big(\int e^{-ik_1(\hat{x}_1\cdot y-|y|)}\frac{V(y+s)}{|y|}\eta_1(y+s)\,\mathrm{d}y\Big)\\
&\quad\cdot\Big(\int e^{ik_2(\hat{x}_2\cdot z-|z|)}\frac{\overline{V}(z+t)}{|z|}\eta_1(z+t)\,\mathrm{d}z\Big)\,\mathrm{d}s\,\mathrm{d}t\\
=&\iint \eta_3(s+t)\Big(\int e^{i(s-t)\cdot\xi}\tilde{c}(s,t,\xi)\,\mathrm{d}\xi\Big)e^{-ik_1\hat{x}_1\cdot s}e^{ik_2\hat{x}_2\cdot t}f(s,k_1,\hat{x}_1)\overline{f}(t,k_2,\hat{x}_2)\,\mathrm{d}s\,\mathrm{d}t\\
=&\iint \eta_3(T)e^{i\theta_2\cdot T}e^{-i\theta_1\cdot S}\Big(\int e^{iS\cdot\xi}\tilde{c}(\frac{T+S}{2},\frac{T-S}{2},\xi)\,\mathrm{d}\xi\Big)\cdot f(\frac{T+S}{2},k_1,\hat{x}_1)\cdot\overline{f}(\frac{T-S}{2},k_2,\hat{x}_2)\frac{1}{2}\,\mathrm{d}S\,\mathrm{d}T\\
=&\frac{1}{2}\iint \eta_3(T)e^{i\theta_2\cdot T}e^{-i\theta_1\cdot S}\Big(\int e^{iS\cdot\xi}c_2(T,\xi)\,\mathrm{d}\xi\Big)\cdot f(\frac{T+S}{2},k_1,\hat{x}_1)\cdot\overline{f}(\frac{T-S}{2},k_2,\hat{x}_2)\,\mathrm{d}S\,\mathrm{d}T\\
=&\frac{1}{2}\iint \eta_3(T)e^{i\theta_2\cdot T}e^{-i\theta_1\cdot S}\Big(\int e^{iS\cdot\xi}f(\frac{T+S}{2},k_1,\hat{x}_1)\overline{f}(\frac{T-S}{2},k_2,\hat{x}_2)c_2(T,\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}S\,\mathrm{d}T\\
=&\frac{1}{2}\iint \eta_3(T)e^{i\theta_2\cdot T}e^{-i\theta_1\cdot S}\Big(\int e^{iS\cdot\xi}\tilde{c}_3(S,T,\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}S\,\mathrm{d}T\\
=&\frac{1}{2}\int \eta_3(T)e^{i\theta_2\cdot T}\Big(\int e^{-i\theta_1\cdot S}\Big(\int e^{iS\cdot\xi}c_3(T,\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}S\Big)\,\mathrm{d}T\\
\simeq&\int_{\mathbb{R}^3}\eta_3(T)e^{i\theta_2\cdot T}c_3(T,\theta_1)\,\mathrm{d}T,
\end{aligned}
\tag{4.48}
$$

where

$$
f(s,k,\hat{x}) := \int_{\mathbb{R}^3} e^{-ik(\hat{x}\cdot y-|y|)}\frac{V(y+s)}{|y|}\eta_1(y+s)\,\mathrm{d}y,
$$

$$
\begin{cases}\theta_1 := (k_1\hat{x}_1 + k_2\hat{x}_2)/2\\ \theta_2 := (k_1\hat{x}_1 - k_2\hat{x}_2)/2\end{cases}\quad\text{and}\quad\begin{cases}S := s-t\\ T := s+t\end{cases}
$$

and

$$\begin{cases} \tilde{c}(s,t,\xi) := \eta_2(s)\eta_2(t)c_f(s,\xi), \\ c_2(T,\xi) = \tilde{c}(T/2,T/2,\xi) + S^{-m-1} = (\eta_2(T/2))^2 c(T/2,\xi) + S^{-m-1}, \\ \tilde{c}_3(S,T,\xi) = \tilde{c}_3(S,T,\xi;k_1,\hat{x}_1,k_2,\hat{x}_2) := f(\frac{T+S}{2},k_1,\hat{x}_1)\overline{f}(\frac{T-S}{2},k_2,\hat{x}_2)c_2(T,\xi) \\ c_3(T,\xi) = \tilde{c}_3(0,T,\xi) + S^{-m-1} \end{cases}$$

and thus

$$\begin{aligned} c_3(T,\xi) &= f(T/2,k_1,\hat{x}_1)\overline{f}(T/2,k_2,\hat{x}_2)c_2(T,\xi) + S^{-m-1} \\ &= (\eta_2(T/2))^2 f(T/2,k_1,\hat{x}_1)\overline{f}(T/2,k_2,\hat{x}_2)c(T/2,\xi) \\ &\quad + \big(1 + f(T/2,k_1,\hat{x}_1)\overline{f}(T/2,k_2,\hat{x}_2)\big)\cdot S^{-m-1} \\ &= (\eta_2(T/2))^2 f(T/2,k_1,\hat{x}_1)\overline{f}(T/2,k_2,\hat{x}_2)c(T/2,\xi) + S^{-m-1}, \end{aligned} \tag{4.49}$$

Set $\hat{x} = \hat{x}_1 = \hat{x}_2$, from (4.48) and (4.49) we obtain

$$\begin{aligned} &|\mathbb{E}\big(G_1(k_1,\hat{x})\cdot\overline{G_1(k_2,\hat{x})}\big)| \leq C|\operatorname{supp}\eta_3|\cdot\sup_{T\in\operatorname{supp}\eta_3}|c_3(T,\theta_1)| \\ &\leq C\cdot|\operatorname{supp}\eta_3|\cdot\sup_{T\in\operatorname{supp}\eta_3}|f(T/2,k_1,\hat{x})|\cdot|\overline{f}(T/2,k_2,\hat{x})|\cdot\langle\theta_1\rangle^{-m} + C|\operatorname{supp}\eta_3|\cdot\langle\theta_1\rangle^{-m-1} \\ &\leq C_U\sup_{T\in\operatorname{supp}\eta_3}|f(T/2,k_1,\hat{x})|\cdot|\overline{f}(T/2,k_2,\hat{x})|\cdot k^{-m} + C_U k^{-m-1} \end{aligned} \tag{4.50}$$

Now we are to show that $f(T/2,k,\hat{x}) = \mathcal{O}(k^{-1})$. For any $\hat{x}\in\mathbb{S}^2$, we can always find two unit vectors $\hat{x}^{\perp,1},\hat{x}^{\perp,2}\in\mathbb{S}^2$ such that these three unit vector are perpendicular to each other. Write the 3×3 matrix $\Phi = (\hat{x},\hat{x}^{\perp,1},\hat{x}^{\perp,2})$, then $\Phi^T\hat{x} = (1,0,0)^T =: e_1$. We have

$$\begin{aligned} &f(s,k,\hat{x}) \\ &= \int_{\mathbb{R}^3} e^{-ik(\hat{x}\cdot y-|y|)}\frac{V(y+s)}{|y|}\eta_1(y+s)\,\mathrm{d}y \\ &= \int_{|y|\leq k^{-1/2}} e^{-ik(\hat{x}\cdot y-|y|)}\frac{V(y+s)}{|y|}\eta_1(y+s)\,\mathrm{d}y \\ &\quad + \int_{|y|>k^{-1/2}} e^{-ik(\hat{x}\cdot y-|y|)}\frac{V(y+s)}{|y|}\eta_1(y+s)\,\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}\Big(\int_{|y|\le k^{-1/2}} \frac{1}{|y|}\,\mathrm{d}y\Big) + \int_{k^{-1/2}}^{+\infty} \frac{e^{ikr}}{r}\,\mathrm{d}r \cdot \int_{\mathbb{S}^2} e^{ikr\hat{x}\cdot w}\eta_1(rw+s)V(rw+s)r^2\,\mathrm{d}S(w) \\
&= \mathcal{O}(k^{-1}) + \int_{k^{-1/2}}^{+\infty} re^{ikr}\,\mathrm{d}r \cdot \int_{\mathbb{S}^2} e^{ikr\hat{x}\cdot w}\eta_1(rw+s)V(rw+s)\,\mathrm{d}S(w) \\
&= \mathcal{O}(k^{-1}) + \int_{k^{-1/2}}^{+\infty} re^{ikr}\,\mathrm{d}r \cdot \int_{\mathbb{S}^2} e^{ikr\hat{x}\cdot \Phi w}\eta_1(r\Phi w+s)V(r\Phi w+s)\,\mathrm{d}S(w) \\
&= \mathcal{O}(k^{-1}) + \int_{k^{-1/2}}^{|s|+\mathrm{diam}D} re^{ikr}\,\mathrm{d}r \cdot \int_{\mathbb{S}^2} e^{ikre_1\cdot w}\eta_1(r\Phi w+s)V(r\Phi w+s)\,\mathrm{d}S(w).
\end{aligned}
$$

We cover the 2-dimensional unit sphere $\mathbb{S}^2$ by six (relative) open parts:

$$
\Gamma_{p,q} := \{(w_1,w_2,w_3)\in\mathbb{R}^3;\sum_{j=1}^3 w_j^2 = 1, (-1)^q w_p > \sqrt{3}/6\},\ p=1,2,3,\ q=0,1.
$$

It is easy to check that $\{\Gamma_{p,q}\}$ is an open cover of $\mathbb{S}^2$: $\mathbb{S}^2 \subset \cup_{p,q}\Gamma_{p,q}$. There exists an partition of unity $\{\rho_{p,q}\}$ subject to the open cover $\{\Gamma_{p,q}\}$, and write

$$
g_{p,q}(r,k,\hat{x},s) := \int_{\Gamma_{p,q}} e^{ikre_1\cdot w}\rho_{p,q}(w)\eta_1(r\Phi w+s)V(r\Phi w+s)\,\mathrm{d}S(w),
$$

thus

$$
f(s,k,\hat{x}) = \mathcal{O}(k^{-1}) + \sum_{p,q}\int_{k^{-1/2}}^{|s|+\mathrm{diam}D} re^{ikr}g_{p,q}(r,k,\hat{x},s)\,\mathrm{d}r. \tag{4.51}
$$

Now we analyze $g_{1,0}$ and $g_{3,0}$. The analysis of $g_{1,1}$ is similar to that of $g_{1,0}$, and $g_{p,q}\,(p=2,3,\ q=0,1)$ is similar to $g_{3,0}$, so we skip analyses of these terms.

In what follows, we write $w=(w_1,w_2,w_3)^T\in\mathbb{S}^2$ as a vertical vector. With a slight abuse of notation, we may write $w=w(w_1,w_2)=(\phi(w_2,w_3),w_2,w_3)^T$. In $\Gamma_{1,0}$, there exists an unique function $\phi\in C^\infty$ such that $w_1=\phi(w_2,w_3)$. Denote the projection of $\Gamma_{1,0}$ onto the (w_2,w_3)-coordinate as $\Pi_{1,0}$. We know $\Pi_{1,0}\subset(-1,1)^2$. We have

$$
\phi(w_2,w_3)\in(\sqrt{30}/6,1],\ \forall\,(w_2,w_3)\in\Pi_{1,0}.
$$

We can fix some $\rho_{1,0}\in C_c^\infty((-1,1)^2)$ s.t. $\rho_{1,0}\equiv 1$ in $\Pi_{1,0}$. Then

$$
\begin{aligned}
g_{1,0} = \int_{\mathbb{R}^2} & e^{ikr\phi(w_2,w_3)}\rho_{1,0}(w_2,w_3)\eta_1(r\Phi w+s)V(r\Phi w+s) \\
&\cdot\sqrt{\det[(\partial_{w_2}w,\partial_{w_3}w)^T(\partial_{w_2}w,\partial_{w_3}w)]}\,\mathrm{d}w_2\,\mathrm{d}w_3.
\end{aligned}
$$

According to $\phi^2 + w_2^2 + w_3^2 = 1$ we have

$$\begin{cases} \phi_{w_2} = -w_2/\phi \\ \phi_{w_3} = -w_3/\phi \end{cases} \quad \text{and} \quad \begin{cases} \phi_{w_2 w_2} = -(1+\phi_{w_2}^2)/\phi \\ \phi_{w_2 w_3} = -\phi_{w_2}\phi_{w_3}/\phi \\ \phi_{w_3 w_3} = -(1+\phi_{w_3}^2)/\phi \end{cases}$$

Note that $\phi > \sqrt{30}/6$. Thus we have that $|\nabla\phi| = 0$ only when $w_2 = w_3 = 0$ and that $\det[\frac{\partial_2 \phi}{\partial w_2 \partial w_3}] = (1 + \phi_{w_2}^2 + \phi_{w_3}^2)/\phi^2 \neq 0$. This means that $(0,0)$ is the only critical point of the phase function $kre_1 \cdot (\phi(w_2, w_3), w_2, w_3)^T$ when $w \in \Gamma_{1,0}$. According to the stationary phase lemma [16, Lemma 19.4], we have

$$\begin{aligned} g_{1,0}(r,k,\hat{x},s) =& \Big(\frac{2\pi}{kr}\Big) C_1 \left(C_2 + C_3(kr)^{-1}\right) + \mathcal{O}((kr)^{-3}) \\ =& C_1(kr)^{-1} + C_2(kr)^{-2} + \mathcal{O}((kr)^{-3}). \end{aligned} \tag{4.52}$$

Now we analyze $g_{3,0}$. We may write $w = w(w_1, w_2) = (w_1, w_2, \phi(w_1, w_2))^T$. Recall that $V \in C_c^\infty(\mathbb{R}3)$, then,

$$\begin{aligned} & g_{3,0}(r,k,\hat{x},s) \\ =& \int_{\mathbb{R}^2} e^{ikrw_1} \rho_{3,0}^2(w_1,w_2)\eta_1(r\Phi w + s)V(r\Phi w + s) \\ & \cdot \sqrt{\det[(\partial_{w_1} w, \partial_{w_2} w)^T(\partial_{w_1} w, \partial_{w_2} w)]}\, \mathrm{d}w_1\, \mathrm{d}w_2 \\ =& \frac{1}{ikr}\int_{\mathbb{R}^2} \partial_{w_1}(e^{ikrw_1})\rho_{3,0}(w)\eta_1(r\Phi w + s)V(r\Phi w+s)\mathcal{C}_1(w_1,w_2)\, \mathrm{d}w_1\, \mathrm{d}w_2 \\ =& \frac{i}{kr}\int_{\mathbb{R}^2} e^{ikrw_1}\partial_{w_1}\big(\rho_{3,0}(w)\eta_1(r\Phi w + s)V(r\Phi w+s)\mathcal{C}_1(w_1,w_2)\big)\, \mathrm{d}w_1\, \mathrm{d}w_2 \\ =& \frac{i}{kr}\int_{\mathbb{R}^2} e^{ikrw_1}\partial_{w_1}\big(\mathcal{C}_2(w_1,w_2;|\hat{x}|,V)\big)\, \mathrm{d}w_1\, \mathrm{d}w_2 \quad , \end{aligned}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are two functions such that $\mathcal{C}_1 \in C^\infty$ and $\mathcal{C}_2 \in C_c^1((-1,1)^2)$, and $\rho_{3,0}$ is chosen in the same manner as $\rho_{1,0}$. Therefore the partial derivative of the function $\mathcal{C}_2$ is bounded above, thus

$$|g_{3,0}(r,k,\hat{x},s)| \le \frac{C}{kr}. \tag{4.53}$$

Combining (4.51) with (4.52) and (4.53), one can compute

$$\begin{aligned}
|f(s,k,\hat{x})| &= \mathcal{O}(k^{-1}) + |\sum_{p,q}\int_{k^{-1/2}}^{|s|+\mathrm{diam}D} re^{ikr}g_{p,q}(r,k,\hat{x},s)\,\mathrm{d}r| \\
&\le \mathcal{O}(k^{-1}) + \sum_{p,q}\int_{k^{-1/2}}^{|s|+\mathrm{diam}D} r[C_1(kr)^{-1} + C_2(kr)^{-2} + \mathcal{O}((kr)^{-3})]\,\mathrm{d}r \\
&= \mathcal{O}(k^{-1}) + \frac{C_1(|s|+\mathrm{diam}D - k^{-1/2})}{k} + \frac{C_2(\ln(|s|+\mathrm{diam}D) + \frac{1}{2}\ln k)}{k^2} \\
&\quad + \frac{C_3(k^{1/2} - (|s|+\mathrm{diam}D)^{-1})}{k^3} \\
&\le \max\{|s|, \mathrm{diam}D\}\cdot\mathcal{O}(k^{-1}), \quad k\to\infty.
\end{aligned}$$

Thus

$$f(s,k,\hat{x}) = \max\{|s|, \mathrm{diam}D\}\cdot\mathcal{O}(k^{-1}), \quad k\to+\infty. \tag{4.54}$$

By (4.50) and (4.54), we arrive at

$$|\mathbb{E}\big(G_1(k_1,\hat{x}_1)\cdot\overline{G_1(k_2,\hat{x}_2)}\big)| \le Ck^{-2}\cdot k^{-m} + C|\operatorname{supp}\eta_3|\cdot k^{-m-1} = \mathcal{O}(k^{-m-1}),\ k\to\infty. \tag{4.55}$$

To estimate $\mathbb{E}(G_j\overline{G_\ell})$ for $j+\ell\ge 3$, $j,\ell\ge 1$, we estimate $I(z,y)$ first, which is defined in (4.45). Recall that $\mathrm{dist}(D_f, D_V) > 0$, and choose $\eta_1,\eta_2\in C_c^\infty(\mathbb{R}^3)$ as before. Then for $z,y\in D$, we have

$$\begin{aligned}
&I(z,y) = I(z,y)\eta_1(z)\eta_1(y) \\
=& \iint_{D_f\times D_f} K(s,t)\eta_1(s)\eta_2(t)\eta_1(z)\eta_1(y)\Phi(s-y)\overline{\Phi}(t-z)\,\mathrm{d}s\,\mathrm{d}t \\
=& \iint_{D_f\times D_f} (2\pi)^{-3/2}\mathcal{F}^{-1}\big\{c(s,\cdot)\big\}(s-t)\cdot\eta_2(s)\eta_2(t)\eta_1(z)\eta_1(y)\Phi(s-y)\overline{\Phi}(t-z)\,\mathrm{d}s\,\mathrm{d}t \\
\simeq& \iint_{D_f\times D_f} \big(\int e^{i(s-t)\cdot\xi}c(s,\xi)\eta_2(s)\eta_2(t)\eta_1(z)\eta_1(y)\,\mathrm{d}\xi\big)\frac{e^{ik_1|s-y|}}{|s-y|}\frac{e^{-ik_2|t-z|}}{|t-z|}\,\mathrm{d}s\,\mathrm{d}t \\
=:& \iint_{D_f\times D_f} e^{ik_1|s-y|-ik_2|t-z|}\big(\int e^{i(s-t)\cdot\xi}c_1(s,t,z,y,\xi)\,\mathrm{d}\xi\big)\mathcal{C}\,\mathrm{d}s\,\mathrm{d}t,
\end{aligned} \tag{4.56}$$

where $c_1(s,t,z,y,\xi) := c(s,\xi)\eta_2(s)\eta_2(t)\eta_1(z)\eta_1(y)$ and $\mathcal{C} = \mathcal{C}(s,t,y,z)$ is within $C_c^\infty(\mathbb{R}^{3\times 4})$ due to the fact that $\mathrm{dist}(D,U) > 0$. Define a differential operator $L :=$

$\frac{(t-z)\cdot\nabla_t}{-ik_2|t-z|}$, it is easy to check that

$$L(e^{ik_1|s-y|-ik_2|t-z|}) = e^{ik_1|s-y|-ik_2|t-z|}.$$

Thus, we can continue (4.56) as

$$\begin{aligned} &|I(z,y)| \\ \simeq &\,\Big|\iint_{D_f\times D_f} L^2(e^{ik_1|s-y|-ik_2|t-z|})\Big(\int e^{i(s-t)\cdot\xi} c_1(s,t,z,y,\xi)\,\mathrm{d}\xi\Big)\mathcal{C}\,\mathrm{d}s\,\mathrm{d}t\Big| \\ \simeq &\, k^{-2}\iint_{D_f\times D_f} e^{ik_1|s-y|-ik_2|t-z|}\big(\mathcal{J}_1\,\mathcal{C} + \vec{\mathcal{J}}_2\cdot\vec{\mathcal{C}}_c + \sum_{a,b=1,2,3}\mathcal{J}_{3;a,b}\cdot\mathcal{C}_{a,b}\big)\,\mathrm{d}s\,\mathrm{d}t, \end{aligned} \tag{4.57}$$

where the integral domain $\mathcal{D}\subset\mathbb{R}^{3\times 4}$ is bounded and

$$\begin{aligned} \mathcal{J}_1 &:= \int e^{i(s-t)\cdot\xi}\,c_1(s,t,z,y,\xi)\,\mathrm{d}\eta, \\ \vec{\mathcal{J}}_2 &:= \nabla_t\int e^{i(s-t)\cdot\xi}\,c_1(s,t,z,y,\xi)\,\mathrm{d}\eta, \\ \mathcal{J}_{3;a,b} &:= \partial^2_{t_a,t_b}\int e^{i(s-t)\cdot\xi}\,c_1(s,t,z,y,\xi)\,\mathrm{d}\eta. \end{aligned}$$

For the case where $s\neq t$, these three quantities, $\mathcal{J}_1$, $\vec{\mathcal{J}}_2$ and $\mathcal{J}_{3;a,b}$, can be estimated as follows,

$$\begin{aligned} |\mathcal{J}_1| &= |s-t|^{-2}\cdot\Big|\int e^{i(s-t)\cdot\xi}(\Delta_\xi c_1)(s,t,z,y,\xi)\,\mathrm{d}\xi\Big| \\ &\le |s-t|^{-2}\int |(\Delta_\xi c_1)(s,t,z,y,\xi)|\,\mathrm{d}\xi \\ &\lesssim |s-t|^{-2}\int\langle\xi\rangle^{-m-2}\,\mathrm{d}\xi \lesssim |s-t|^{-2}, \end{aligned} \tag{4.58}$$

and

$$\begin{aligned} |\vec{\mathcal{J}}_2| &= |s-t|^{-2}\cdot\Big|\int \Delta_\xi(e^{i(s-t)\cdot\xi})\,\xi c_1(s,t,z,y,\xi)\,\mathrm{d}\xi\Big| \\ &\le |s-t|^{-2}\cdot\int|\Delta_\xi(\xi c_1(s,t,z,y,\xi))|\,\mathrm{d}\xi \\ &\lesssim |s-t|^{-2}\int\langle\xi\rangle^{-m+1-2}\,\mathrm{d}\xi \lesssim |s-t|^{-2}, \end{aligned} \tag{4.59}$$

and similarly

$$\begin{aligned}
\mathcal{J}_{3;a,b} &\simeq \int e^{i(s-t)\cdot\xi}\cdot c_1(s,t,z,y,\xi)\xi_a\xi_b\,\mathrm{d}\xi \\
&\simeq |s-t|^{-2}\int \Delta_\xi(e^{i(t-s)\cdot\xi})\cdot c_1(s,t,z,y,\xi)\xi_a\xi_b\,\mathrm{d}\xi \\
&= |s-t|^{-2}\int e^{i(s-t)\cdot\xi}\cdot \Delta_\xi(c_1(s,t,z,y,\xi)\xi_a\xi_b)\,\mathrm{d}\xi. \qquad (4.60)
\end{aligned}$$

By using Lemmas 5.3.1 and 5.3.2, we can continue (4.60) as

$$\begin{aligned}
|\mathcal{J}_{3;a,b}| &\simeq |s-t|^{-2}\cdot\big||s-t|^{-s}\int(-\Delta_\xi)^{s/2}(e^{i(s-t)\cdot\xi})\cdot\Delta_\xi(c_1(s,t,z,y,\xi)\xi_a\xi_b)\,\mathrm{d}\xi\big| \\
&= |s-t|^{-2-s}\cdot|\int e^{i(s-t)\cdot\xi}\cdot(-\Delta_\eta)^{s/2}\big(\Delta_\xi(c_1(s,t,z,y,\xi)\xi_a\xi_b)\big)\,\mathrm{d}\xi| \\
&\lesssim |s-t|^{-2-s}\int\langle\xi\rangle^{-m+2-2-s}\,\mathrm{d}\xi = |s-t|^{-2-s}\int\langle\xi\rangle^{-m-s}\,\mathrm{d}\xi, \qquad (4.61)
\end{aligned}$$

where the number s is chosen to satisfy $0<3-m<s<1$. Therefore, we have

$$\begin{cases} -m-s<-3, & (4.62\text{a}) \\ -2-s>-3. & (4.62\text{b}) \end{cases}$$

Thanks to the condition (4.62a), we can continue (4.61) as

$$|\mathcal{J}_{3;a,b}| \lesssim |s-t|^{-2-s}. \qquad (4.63)$$

Now, combining (4.57), (4.58), (4.59) and (4.63), we arrive at

$$\begin{aligned}
|I(z,y)| &\lesssim\ k^{-2}\iint_{D_f\times D_f}(|\mathcal{J}_1|+|\vec{\mathcal{J}}_2|+\sum_{a,b=1,2,3}|\mathcal{J}_{3;a,b}|)\,\mathrm{d}s\,\mathrm{d}t, \\
&\lesssim k^{-2}\iint_{D_f\times D_f}|s-t|^{-2-s}\,\mathrm{d}s\,\mathrm{d}t \\
&\lesssim k^{-2}. \qquad (4.64)
\end{aligned}$$

The last inequality is due to the condition (4.62b) and that $U\in\mathbb{R}^3$ is bounded. Note that the integral (4.64) should be understood as singular integral because of the presence of the singularities happening when $s=t$.

Combine (4.44) and (4.64), one can compute

$$
\begin{aligned}
&|\mathbb{E}\big(G_j(k_1,\hat{x}_1)\cdot\overline{G_\ell(k_2,\hat{x}_2)}\,\big)| \\
=&\Big|\int_D e^{ik_2\hat{x}_2\cdot z}\Big\{(V\mathcal{R}_{k_2})^{\ell-1}\Big(\int_D e^{-ik_1\hat{x}_1\cdot y}\big[(V\mathcal{R}_{k_1})^{j-1}(V(1)\overline{V}(2)I(2,1))\big](y)\,\mathrm{d}y\Big)\Big\}(z)\,\mathrm{d}z\Big| \\
\le&|D|^{1/2}\big\|(V\mathcal{R}_{k_2})^{\ell-1}\Big(\int_D e^{-ik_1\hat{x}_1\cdot y}\big[(V\mathcal{R}_{k_1})^{j-1}(V(1)\overline{V}(2)I(2,1))\big](y)\,\mathrm{d}y\Big)\big\|_{L^2(D)} \\
=&|D|^{1/2}(C_{D,V}k_2)^{-\ell+1}\Big(\int_D\big|\int_D e^{-ik_1\hat{x}_1\cdot y}\big[(V\mathcal{R}_{k_1})^{j-1}(V(1)\overline{V}(z)I(z,1))\big](y)\,\mathrm{d}y\big|^2\,\mathrm{d}z\Big)^{1/2} \\
\le&|D|^{1/2}(C_{D,V}k_2)^{-\ell+1}\Big(\int_D|D|\cdot\int_D\big|\big[(V\mathcal{R}_{k_1})^{j-1}(V(1)\overline{V}(z)I(z,1))\big](y)\big|^2\,\mathrm{d}y\,\mathrm{d}z\Big)^{1/2} \\
=&|D|(C_{D,V}k_2)^{-\ell+1}\Big(\int_D\big\|(V\mathcal{R}_{k_1})^{j-1}(V(1)\overline{V}(z)I(z,1))\big\|^2_{L^2(D)}\,\mathrm{d}z\Big)^{1/2} \\
\le&|D|(C_{D,V}k_2)^{-\ell+1}\Big(\int_D(C_{D,V}k_1)^{-2j+2}\|V(1)\overline{V}(z)I(z,1)\|^2_{L^2(D;1)}\,\mathrm{d}z\Big)^{1/2} \\
\le&|D|(C_{D,V}k_2)^{-\ell+1}(C_{D,V}k_1)^{-j+1}\Big(\|V\|^4_{L^\infty(D)}\int_D\int_D|I(z,y)|^2\,\mathrm{d}y\,\mathrm{d}z\Big)^{1/2} \\
\le&\|V\|^2_{L^\infty(D)}|D|(C_{D,V}k_2)^{-\ell+1}(C_{D,V}k_1)^{-j+1}\sup_{y,z\in D}|I(z,y)| \\
\le&(C_{D,V}k_2)^{-\ell+1}\cdot(C_{D,V}k_1)^{-j+1}\cdot\mathcal{O}(k^{-2}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|\sum_{j+\ell\ge3,\,j,\ell\ge1}\mathbb{E}\big(G_j(k_1,\hat{x}_1)\cdot\overline{G_\ell(k_2,\hat{x}_2)}\big)| \\
=&\sum_{j=1,\,\ell\ge2}(C_{D,V}k_2)^{-\ell+1}\cdot(C_{D,V}k_1)^{-j+1}\cdot\mathcal{O}(k^{-2}) \\
&+\sum_{j\ge2}\sum_{\ell\ge1}(C_{D,V}k_2)^{-\ell+1}\cdot(C_{D,V}k_1)^{-j+1}\cdot\mathcal{O}(k^{-2}) \\
=&\sum_{\ell\ge1}(C_{D,V}k_2)^{-\ell}\cdot\mathcal{O}(k^{-m})+\sum_{j\ge1}\big[(\sum_{\ell\ge0}(C_{D,V}k_2)^{-\ell})\cdot(C_{D,V}k_1)^{-j}\big]\cdot\mathcal{O}(k^{-2}) \\
=&C_{D,V}k_2^{-1}\cdot\mathcal{O}(k^{-2})+C_{D,V}k_1^{-1}\cdot\mathcal{O}(k^{-2})=\mathcal{O}(k^{-3}),\quad k\to+\infty.
\end{aligned}
\tag{4.65}
$$

Combining (4.47), (4.55) and (4.65), we conclude (4.42). □

Lemma 4.3.6 is the ergodic version of Lemmas 4.3.4 and 4.3.5.

Lemma 4.3.6. *Define $F_j(k,\hat{x})$ $(j=0,1)$ as in (4.19). Write*

$$X_{p,q}(K,\tau,\hat{x}) = \frac{1}{K}\int_K^{2K} k^m \overline{F_q(k,\hat{x})}\cdot F_p(k+\tau,\hat{x})\,\mathrm{d}k, \quad \text{for} \quad (p,q)\in\{(0,1),(1,0),(1,1)\}.$$

Then for any $\hat{x}\in\mathbb{S}^2$ and any $\tau\geq 0$, when $K\to+\infty$, we have the following estimates:

$$\big|\mathbb{E}(X_{p,q}(K,\tau,\hat{x}))\big| = \mathcal{O}(K^{-1}), \quad \big|\mathbb{E}(|X_{p,q}(K,\tau,\hat{x})|^2)\big| = \mathcal{O}(K^{-3/2}), \quad (p,q)\in\{(0,1),(1,0)\} \tag{4.66}$$

$$\big|\mathbb{E}(X_{1,1}(K,\tau,\hat{x}))\big| = \mathcal{O}(K^{m-3}), \quad \big|\mathbb{E}(|X_{1,1}(K,\tau,\hat{x})|^2)\big| = \mathcal{O}(K^{2(m-3)}). \tag{4.67}$$

Let $\{K_j\}\in P\big(\max\{2/3,\,(3-m)^{-1}/2\}+\gamma\big)$, then for any $\tau\geq 0$, we have

$$\lim_{j\to+\infty} X_{p,q}(K_j,\tau,\hat{x}) = 0 \quad a.s.\ , \tag{4.68}$$

for every $(p,q)\in\{(0,1),(1,0),(1,1)\}$.

We may denote $X_{p,q}(K,\tau,\hat{x})$ as $X_{p,q}$ for short if it is clear in the context.

Proof of Lemma 4.3.6. According to Lemmas 4.3.4 and 4.3.5, we have

$$\begin{aligned}\mathbb{E}(X_{0,1}) &= \frac{1}{K}\int_K^{2K} k^m \mathbb{E}\big(\overline{F_1(k,\hat{x})}\cdot F_0(k+\tau,\hat{x})\big)\,\mathrm{d}k = \frac{1}{K}\int_K^{2K}\mathcal{O}(k^{-1})\,\mathrm{d}k \\ &= \mathcal{O}(K^{-1}), \quad K\to+\infty.\end{aligned} \tag{4.69}$$

By formula (4.83), Isserlis' Theorem and Lemma 4.3.3, we compute the secondary moment of $X_{0,1}$,

$$\begin{aligned}&\mathbb{E}\big(|X_{0,1}|^2\big) = \mathbb{E}\Big(\frac{1}{K}\int_K^{2K} k_1^m F_0(k_1+\tau,\hat{x})\cdot\overline{F_1(k_1,\hat{x})}\,\mathrm{d}k_1\cdot\frac{1}{K}\int_K^{2K} k_2^m\overline{F_0(k_2+\tau,\hat{x})}\cdot F_1(k_2,\hat{x})\,\mathrm{d}k_2\Big)\\
=&\frac{1}{K^2}\int_K^{2K}\int_K^{2K}\mathbb{E}\big(k_1^m F_0(k_1+\tau,\hat{x})\overline{F_1(k_1,\hat{x})}\big)\cdot\mathbb{E}\big(k_2^m\overline{F_0(k_2+\tau,\hat{x})}F_1(k_2,\hat{x})\big)\\
&+\mathbb{E}\big(k_2^m F_0(k_1+\tau,\hat{x})\overline{F_0(k_2+\tau,\hat{x})}\big)\cdot\mathbb{E}\big(k_1^m F_1(k_2,\hat{x})\overline{F_1(k_1,\hat{x})}\big)\\
&+\mathbb{E}\big(k_2^m F_0(k_1+\tau,\hat{x})F_1(k_2,\hat{x})\big)\cdot\mathbb{E}\big(k_1^m\overline{F_0(k_2+\tau,\hat{x})}\,\overline{F_1(k_1,\hat{x})}\big)\,\mathrm{d}k_1\,\mathrm{d}k_2 \quad \text{(Isserlis' Theorem)}\\
=&\frac{1}{K^2}\int_K^{2K}\int_K^{2K}\mathcal{O}(K^{-2})+(2\pi)^{3/2}\widehat{\mu}((k_1-k_2)\hat{x})\cdot\mathcal{O}(K^{-1})+\mathcal{O}(K^{-2})\,\mathrm{d}k_1\,\mathrm{d}k_2\\
=&\frac{1}{K^2}\int_K^{2K}\int_K^{2K}(2\pi)^{3/2}\widehat{\mu}((k_1-k_2)\hat{x})\,\mathrm{d}k_1\,\mathrm{d}k_2\cdot\mathcal{O}(K^{-1})+\mathcal{O}(K^{-2})\\
=&\mathcal{O}(K^{-1/2})\cdot\mathcal{O}(K^{-1})+\mathcal{O}(K^{-2}) \quad \text{(Hölder ineq. and (4.27))}\end{aligned}$$

$$=\mathcal{O}(K^{-3/2}), \quad K \to +\infty. \tag{4.70}$$

From (4.69)-(4.70) we obtain (4.66) for $(p, q) = (0, 1)$. Similarly, formula (4.66) for $(p, q) = (1, 0)$ can be proved and we skip the details.

By Chebyshev's inequality and (4.70), for any $\epsilon > 0$, we have

$$\begin{aligned} &P\big(\bigcup_{j \geq K_0} \{|X_{0,1}(K_j, \tau, \hat{x}) - 0| \geq \epsilon\}\big) \leq \frac{C}{\epsilon^2} \sum_{j \geq K_0} K_j^{-3/2} \leq \frac{C}{\epsilon^2} \sum_{j \geq K_0} j^{-1-3\gamma/2} \\ \leq & \frac{C}{\epsilon^2} \int_{K_0}^{+\infty} (t-1)^{-1-3\gamma/2} \, \mathrm{d}t \to 0, \quad K_0 \to +\infty. \end{aligned} \tag{4.71}$$

According to Lemma 2.2.5, (4.71) implies (4.68) for $(p, q) = (0, 1)$. Similarly, formula (4.68) for $(p, q) = (1, 0)$ can be proved.

We now prove (4.67). We have:

$$\mathbb{E}\big(X_{1,1}\big) = \frac{1}{K} \int_K^{2K} k^m \mathbb{E}\big(\overline{F_1(k, \hat{x})} \cdot F_1(k + \tau, \hat{x})\big) \, \mathrm{d}k = \frac{1}{K} \int_K^{2K} \mathcal{O}(K^{m-3}) \, \mathrm{d}k = \mathcal{O}(K^{m-3}). \tag{4.72}$$

Compute the secondary moment:

$$\begin{aligned} &\mathbb{E}\big(|X_{1,1}|^2\big) = \mathbb{E}\big(\frac{1}{K} \int_K^{2K} k_1^m F_1(k_1 + \tau, \hat{x}) \cdot \overline{F_1(k_1, \hat{x})} \, \mathrm{d}k_1 \cdot \frac{1}{K} \int_K^{2K} k_2^m \overline{F_1(k_2 + \tau, \hat{x})} \cdot F_1(k_2, \hat{x}) \, \mathrm{d}k_2\big) \\ =&\frac{1}{K^2} \int_K^{2K} \int_K^{2K} k_1^m \mathbb{E}\big(F_1(k_1 + \tau, \hat{x}) \overline{F_1(k_1, \hat{x})}\big) \cdot k_2^m \mathbb{E}\big(\overline{F_1(k_2 + \tau, \hat{x})} F_1(k_2, \hat{x})\big) \\ &+ \mathbb{E}\big(k_1^m F_1(k_1, \hat{x}) \overline{F_1(k_2, \hat{x})}\big) \cdot \mathbb{E}\big(k_2^m \overline{F_1(k_1 + \tau, \hat{x})} F_1(k_2 + \tau, \hat{x})\big) \\ &+ \mathbb{E}\big(k_1^m F_1(k_1, \hat{x}) F_1(k_2 + \tau, \hat{x})\big) \cdot \mathbb{E}\big(k_2^m \overline{F_1(k_1 + \tau, \hat{x})} \, \overline{F_1(k_2, \hat{x})}\big) \, \mathrm{d}k_1 \, \mathrm{d}k_2 \\ =&\frac{1}{K^2} \int_K^{2K} \int_K^{2K} \mathcal{O}(K^{m-3}) \cdot \mathcal{O}(K^{m-3}) \, \mathrm{d}k_1 \, \mathrm{d}k_2 \quad \text{(Lemmas 4.3.4, 4.3.5)} \\ =&\mathcal{O}(K^{2(m-3)}), \quad K \to +\infty. \end{aligned} \tag{4.73}$$

Formulae (4.72)-(4.73) gives (4.67).

By Chebyshev's inequality and (4.73), for any $\epsilon > 0$, we have

$$\begin{aligned} &P\big(\bigcup_{j \geq K_0} \{|X_{1,1} - 0| \geq \epsilon\}\big) \leq \frac{C}{\epsilon^2} \sum_{j \geq K_0} K_j^{2(m-3)} \leq \frac{C}{\epsilon^2} \sum_{j \geq K_0} j^{-1-\gamma'} \\ \leq&\frac{C}{\epsilon^2} \int_{K_0}^{+\infty} (t-1)^{-1-\gamma'} \, \mathrm{d}t \to 0, \quad K_0 \to +\infty., \end{aligned} \tag{4.74}$$

where γ' is some positive constant depending on m. According to Lemma 2.2.5, (4.74) implies (4.68) for $(p,q) = (1,1)$. The proof is complete. $\square$

4.4 The recovery of the rough strength

In this section we focus on the recovery of the rough strength $\mu(x)$ of the random source. We employ only a single passive scattering measurement. Namely, there is no incident plane wave sent and the random sample ω is fixed. The data set $\{u^\infty(\hat{x}, k, \omega) \,\big|\, \hat{x} \in \mathbb{S}^2, k \in \mathbb{R}_+\}$ is utilized to achieve the unique recovery result. In what follows, we present the main results of recovering $\mu(x)$ in Section 4.4.1, and put the corresponding proofs in Section 4.4.2. Several lemmas about the asymptotics of those high-order terms are put separately in Section 4.3.2 to emphasize the key role to the proofs in Section 4.4.2.

4.4.1 Main unique recovery results

The first main result is in the following.

Lemma 4.4.1. *We have the following asymptotic identity,*

$$4\sqrt{2\pi} \lim_{k\to+\infty} \mathbb{E}\big(k^m \big[\, \overline{u^\infty(\hat{x}, k)} - \overline{\mathbb{E}u^\infty(\hat{x}, k)}\,\big] \cdot \big[u^\infty(\hat{x}, k+\tau) - \mathbb{E}u^\infty(\hat{x}, k+\tau)\big]\big) = \widehat{\mu}(\tau\hat{x}), \tag{4.75}$$

where $\tau \geq 0$, $\hat{x} \in \mathbb{S}^2$.

Lemma 4.4.1 clearly yields a recovery formula for the rough strength μ. However, it requires many realizations and it is lack of practical usefulness. The result in Lemma 4.4.1 can be improved by using the ergodicity. See, i.e. [11, 32].

Recall the notation $\{K_j\} \in P(t)$ already mentioned in Section 2.1.

Lemma 4.4.2. *Let* $m^* = \max\{2/3,\, (3-m)^{-1}/2\}$. *Assume* $\{K_j\} \in P(m^*+\gamma)$. *Then* $\exists\, \Omega_0 \subset \Omega$: $\mathbb{P}(\Omega_0) = 0$, Ω_0 *depends only on* $\{K_j\}_{j\in\mathbb{N}^+}$, *such that for any* $\omega \in \Omega\backslash\Omega_0$, *there exists* $S_\omega \subset \mathbb{R}^3$: $m(S_\omega) = 0$, *such that for* $\forall x \in \mathbb{R}^3\backslash S_\omega$,
when $x \neq 0$,

$$4\sqrt{2\pi} \lim_{j\to+\infty} \frac{1}{K_j} \int_{K_j}^{2K_j} k^m \big[\, \overline{u^\infty(\hat{x}, k, \omega)} - \overline{\mathbb{E}u^\infty(\hat{x}, k)}\,\big] \cdot \big[u^\infty(\hat{x}, k+\tau, \omega) - \mathbb{E}u^\infty(\hat{x}, k+\tau)\big]\, \mathrm{d}k$$

$$= \widehat{\mu}(x), \tag{4.76}$$

where $\tau = |x|$ and $\hat{x} := x/|x|$;
when $x = 0$,

$$4\sqrt{2\pi} \lim_{j\to+\infty} \frac{1}{K_j} \int_{K_j}^{2K_j} k^m |u^\infty(\hat{x}, k, \omega) - \mathbb{E}u^\infty(\hat{x}, k, \omega)|^2 \, \mathrm{d}k = \widehat{\mu}(0), \tag{4.77}$$

holds for any $\hat{x} \in \mathbb{S}^2$.

The recovery formula presented in (4.76) still involves every realization of the random sample ω. To recover $\mu(x)$ by only one realization of the passive scattering measurement, the $\mathbb{E}u^\infty(\hat{x}, k)$ term should be further relaxed in (4.76), and this is done by Lemma 4.4.3 in the following.

Lemma 4.4.3. *Under the same condition as in Lemma 4.4.2, we have*

$$4\sqrt{2\pi} \lim_{j\to+\infty} \frac{1}{K_j} \int_{K_j}^{2K_j} k^m \overline{u^\infty(\hat{x}, k, \omega)} \cdot u^\infty(\hat{x}, k+\tau, \omega) \, \mathrm{d}k = \widehat{\mu}(x). \tag{4.78}$$

Now Theorem 4.1.1 becomes a direct consequence of Lemma 4.4.3.

Proof of Theorem 4.1.1. Lemma 4.4.3 provides a recovery formula for the local strength μ by the far-field data $\{u^\infty(\hat{x}, k, \omega);\, \forall\, \hat{x} \in \mathbb{S}^2,\, \forall\, k \in \mathbb{R}_+\}$ with a single fixed $\omega \in \Omega$. □

4.4.2 Proofs of the main theorems

In this subsection, we present the proofs of The proofs of Lemmas 4.4.1, 4.4.2 and 4.4.3.

Proof of Lemma 4.4.1. We assume $\mathbb{E}f = 0$ for the time being. Let k be large enough such that $(I - \mathcal{R}_k V)^{-1} = \sum_{j=0}^{+\infty} (\mathcal{R}_k V)^j$, and let $\tau \in \mathbb{R}_+$. According to the analysis at the beginning of Section 4.3, one can compute

$$16\pi^2 \mathbb{E}\big(\overline{u^\infty(\hat{x}, k)} u^\infty(\hat{x}, k+\tau)\big) = \sum_{j,\ell=0,1} \mathbb{E}\big(\overline{F_\ell(k, \hat{x})} F_j(k+\tau, \hat{x})\big)$$

$$=: I_{0,0} + I_{0,1} + I_{1,0} + I_{1,1}. \tag{4.79}$$

From Lemmas 4.3.4 and 4.3.5, we have that $I_{0,1}$, $I_{1,0}$, $I_{1,1}$ are all of order no less than k^{-3}, hence

$$16\pi^2 \mathbb{E}\big(k^m \overline{u^\infty(\hat{x},k)} u^\infty(\hat{x},k+\tau)\big) = k^m I_{0,0} + \mathcal{O}(k^{m-3}), \quad k \to +\infty. \tag{4.80}$$

By (4.5)-(4.6), we can compute $I_{0,0}$,

$$\begin{aligned} I_{0,0} &= \mathbb{E}\big(\overline{F_0(k,\hat{x})} F_0(k+\tau,\hat{x})\big) = \mathbb{E}\big(\overline{\langle f, e^{-ik\hat{x}\cdot(\cdot)}\rangle} \cdot \langle f, e^{-i(k+\tau)\hat{x}\cdot(\cdot)}\rangle\big) \\ &= \iint_{D\times D} K_f(y,z) e^{-i(k+\tau)\hat{x}\cdot y} e^{ik\hat{x}\cdot z} \,\mathrm{d}y\,\mathrm{d}z \\ &= \int_D \Big(\int_D K_f(y,z) e^{-ik\hat{x}\cdot(y-z)} \,\mathrm{d}z\Big) e^{-i\tau\hat{x}\cdot y} \,\mathrm{d}y \\ &= \int_D c_f(y,k\hat{x}) e^{-i\tau\hat{x}\cdot y} \,\mathrm{d}y = (2\pi)^{3/2}\, \widehat{\mu}(\tau\hat{x}) k^{-m} + \int_D a(y,k\hat{x}) e^{i\tau\hat{x}\cdot y} \,\mathrm{d}y. \end{aligned} \tag{4.81}$$

The symbol a is of order $-m-1$, thus

$$\Big|\int_D a(y,k\hat{x}) e^{i\tau\hat{x}\cdot y} \,\mathrm{d}y\Big| \leq |D| \cdot |a(y,k\hat{x})| \leq |D| C \langle k\hat{x}\rangle^{-m-1} = |D| C \langle k\rangle^{-m-1}. \tag{4.82}$$

By (4.81)-(4.82) we obtain

$$k^m I_{0,0} = \mathbb{E}\big(k^m \overline{F_0(k,\hat{x})} F_0(k+\tau,\hat{x})\big) = (2\pi)^{3/2}\, \widehat{\mu}(\tau\hat{x}) + \mathcal{O}(k^{-1}), \quad k \to +\infty. \tag{4.83}$$

Formulae (4.80) and (4.83) gives

$$16\pi^2 \mathbb{E}\big(k^m \overline{u^\infty(\hat{x},k)} u^\infty(\hat{x},k+\tau)\big) = (2\pi)^{3/2}\, \widehat{\mu}(\tau\hat{x}) + \mathcal{O}(k^{m-3}) + \mathcal{O}(k^{-1}), \quad k \to +\infty. \tag{4.84}$$

Noting that $m \in (1,3)$, thus (4.84) implies

$$16\pi^2 \lim_{k\to+\infty} \mathbb{E}\big(k^m \overline{u^\infty(\hat{x},k)} u^\infty(\hat{x},k+\tau)\big) = (2\pi)^{3/2}\, \widehat{\mu}(\tau\hat{x}). \tag{4.85}$$

immediately.

Recall that at the beginning of this proof we assume $\mathbb{E}f = 0$. Now we cancel this assumption and note that the far-field pattern corresponding to $\mathbb{E}f$ is $\mathbb{E}u^\infty$, hence (4.85) implies (4.75) immediately. □

The proof of Lemma 4.4.2 involves Lemmas 4.3.2 and 4.3.3 and Lemma 2.2.6 in the following.

Proof of Lemma 4.4.2. We denote as the averaging operation w.r.t. k: $\mathcal{E}_k f = \frac{1}{K}\int_K^{2K} f(k)\,\mathrm{d}k$ by $\mathcal{E}_k$. Following the arguments in the proof of Lemma 4.4.1, we assume $\mathbb{E}f = 0$ and still denote as u the the scattered wave by the f with zero-mean, thus we have

$$\begin{aligned}&16\pi^2\mathcal{E}_k\big(k^m\overline{u^\infty(\hat{x},k)}u^\infty(\hat{x},k+\tau)\big) = \sum_{j,\ell=0,1}\mathcal{E}_k\big(k^m\overline{F_\ell(k,\hat{x})}F_j(k+\tau,\hat{x})\big)\\ &=: X_{0,0} + X_{0,1} + X_{1,0} + X_{1,1}. \end{aligned} \tag{4.86}$$

Recall that $\{K_j\} \in P(m^* + \gamma)$. For $\forall\tau \geq 0$ and $\forall\hat{x} \in \mathbb{S}^2$, Lemma 4.3.2 implies that $\exists\Omega^{0,0}_{\tau,\hat{x}} \subset \Omega$: $\mathbb{P}(\Omega^{0,0}_{\tau,\hat{x}}) = 0$, $\Omega^{0,0}_{\tau,\hat{x}}$ depending on τ and $\hat{x}$, such that

$$\lim_{j\to+\infty} X_{0,0}(K_j,\tau,\hat{x}) = (2\pi)^{3/2}\widehat{\mu}(\tau\hat{x}), \quad \forall\omega \in \Omega\backslash\Omega^{0,0}_{\tau,\hat{x}}. \tag{4.87}$$

Lemma 4.3.6 implies the existence of the sets $\Omega^{p,q}_{\tau,\hat{x}}$ $\big((p,q) \in \{(0,1),\,(1,0),\,(1,1)\}\big)$ with zero probability measures such that $\forall\tau \geq 0$ and $\forall\hat{x} \in \mathbb{S}^2$,

$$\lim_{j\to+\infty} X_{p,q}(K_j,\tau,\hat{x}) = 0, \quad \forall\omega \in \Omega\backslash\Omega^{p,q}_{\tau,\hat{x}}. \tag{4.88}$$

for all $(p,q) \in \{(0,1),\,(1,0),\,(1,1)\}$. Write $\Omega_{\tau,\hat{x}} = \bigcup_{p,q=0,1}\Omega^{p,q}_{\tau,\hat{x}}$, then $\mathbb{P}(\Omega_{\tau,\hat{x}}) = 0$. From Lemmas 4.3.2 and 4.3.6 we note that $\Omega^{p,q}_{\tau,\hat{x}}$ also depends on K_j, so does $\Omega_{\tau,\hat{x}}$, but we omit this dependence in the notation. Write

$$Z(\tau\hat{x},\omega) := \lim_{j\to+\infty}\frac{16\pi^2}{K_j}\int_{K_j}^{2K_j} k^m\overline{u^\infty(\hat{x},k)}u^\infty(\hat{x},k+\tau)\,\mathrm{d}k - (2\pi)^{3/2}\widehat{\mu}(\tau\hat{x})$$

for short. By (4.86)-(4.88), we conclude that,

$$\forall\, y \in \mathbb{R}^3,\ \exists\,\Omega_y \subset \Omega\colon \mathbb{P}(\Omega_y) = 0,\ \text{s.t.}\ \forall\,\omega \in \Omega\backslash\Omega_y,\ Z(y,\omega) = 0. \tag{4.89}$$

To conclude (4.76)-(4.77) from (4.89), we should exchange the logical order between y and ω. To achieve this, we utilize the Fubini's Theorem. Denote the usual Lebesgue measure on $\mathbb{R}^3$ as $\mathbb{L}$ and the product measure $\mathbb{L}\times\mathbb{P}$ as μ, and construct the product measure space $\mathbb{M} := (\mathbb{R}^3 \times \Omega, \mathcal{G}, \mu)$ in the canonical way, where $\mathcal{G}$ is the corresponding complete σ-algebra. Write

$$\mathcal{A} := \{(y,\omega) \in \mathbb{R}^3 \times \Omega\,;\ Z(y,\omega) \neq 0\},$$

then $\mathcal{A}$ is a subset of $\mathbb{M}$. Set $\chi_{\mathcal{A}}$ as the characteristic function of $\mathcal{A}$ in $\mathbb{M}$. By (4.89) we obtain

$$\int_{R^3} \Big(\int_{\Omega} \chi_{\mathcal{A}}(y,\omega)\,\mathrm{d}\mathbb{P}(\omega) \Big)\,\mathrm{d}\mathbb{L}(y) = 0. \tag{4.90}$$

By (4.90) and [Corollary 7 in Section 20.1, 45], we obtain

$$\int_{\mathbb{M}} \chi_{\mathcal{A}}(y,\omega)\,\mathrm{d}\mu = \int_{\Omega} \Big(\int_{R^3} \chi_{\mathcal{A}}(y,\omega)\,\mathrm{d}\mathbb{L}(y) \Big)\,\mathrm{d}\mathbb{P}(\omega) = 0. \tag{4.91}$$

Because $\chi_{\mathcal{A}}(y,\omega)$ is nonnegative, (4.91) implies

$$\exists\, \Omega_0 \colon \mathbb{P}(\Omega_0) = 0, \text{ s.t. } \forall\, \omega \in \Omega \backslash \Omega_0,\ \int_{R^3} \chi_{\mathcal{A}}(y,\omega)\,\mathrm{d}\mathbb{L}(y) = 0. \tag{4.92}$$

Formula (4.92) further implies for every $\omega \in \Omega\backslash\Omega_0$,

$$\exists\, S_\omega \subset \mathbb{R}^3 \colon \mathbb{L}(S_\omega) = 0, \text{ s.t. } \forall\, y \in \mathbb{R}^3 \backslash S_\omega,\ Z(y,\omega) = 0. \tag{4.93}$$

This is (4.76)-(4.77) for $\mathbb{E}u^\infty = 0$.

Recall that at the beginning of this proof we assume $\mathbb{E}f = 0$. Now we cancel this assumption and note that the far-field pattern corresponding to $\mathbb{E}f$ is $\mathbb{E}u^\infty$, hence (4.93) implies (4.76)-(4.77) immediately. □

Proof of Lemma 4.4.3. The symbol $\mathcal{E}_k$ is defined same as in the proof of Lemma 4.4.2. In addition to the notation u_1, we write $u_2 = \mathbb{E}u$, so does the far-field patterns. Therefore $u^\infty = u_1^\infty + u_2^\infty$ and we have

$$16\pi^2 \mathcal{E}_k\big(k^m \overline{u^\infty(\hat{x},k)} u^\infty(\hat{x},k+\tau)\big) = \sum_{(j,\ell)\in\{(0,0),(0,1),(1,0),(1,1)\}} \mathbb{E}\big(k^m \overline{F_j(k,\hat{x})} F_\ell(k+\tau,\hat{x})\big)$$

$$=: J_0 + J_1 + J_2 + J_3.$$

From Lemma 4.4.2 we obtain

$$\lim_{j\to+\infty} J_0 = \lim_{j\to+\infty} \int_{K_j}^{2K_j} \overline{u_1^\infty(\hat{x},k)} \cdot u_1^\infty(\hat{x},k+\tau)\,\mathrm{d}k = (2\pi)^{3/2}\widehat{\sigma^2}(\tau\hat{x}),$$
$$\tau\hat{x} \text{ a.e. } \in \mathbb{R}^3, \quad \omega \text{ a.s. } \in \Omega. \tag{4.94}$$

We now study J_1,

$$\begin{aligned}|J_1|^2 &\simeq \big|\mathcal{E}_k\big(\overline{u_1^\infty(\hat{x},k)}u_2^\infty(\hat{x},k+\tau)\big)\big|^2 = \big|\frac{1}{K_j}\int_{K_j}^{2K_j} \overline{u_1^\infty(\hat{x},k)}u_2^\infty(\hat{x},k+\tau)\,\mathrm{d}k\big|^2 \\ &\leq \frac{1}{K_j}\int_{K_j}^{2K_j} |u_1^\infty(\hat{x},k)|^2\,\mathrm{d}k \cdot \frac{1}{K_j}\int_{K_j}^{2K_j} |u_2^\infty(\hat{x},k+\tau)|^2\,\mathrm{d}k. \end{aligned}\tag{4.95}$$

Recall that $u_1^\infty = u^\infty - \mathbb{E}u^\infty$. Combining (4.95) with Lemmas 4.4.2 and 3.3.1, we have

$$|J_1|^2 \lesssim (\widehat{\sigma^2}(0) + o(1)) \cdot o(1) = 0(1) \to 0, \quad j \to +\infty. \tag{4.96}$$

The analysis to J_2 is similar to that of J_1 so we skip the details.

Then we study J_3. By Lemma 3.3.1, we have

$$\begin{aligned}|J_3|^2 &\simeq \big|\mathcal{E}_k\big(\overline{u_2^\infty(\hat{x},k)}u_2^\infty(\hat{x},k+\tau)\big)\big|^2 = \big|\frac{1}{K_j}\int_{K_j}^{2K_j} \overline{u_2^\infty(\hat{x},k)}u_2^\infty(\hat{x},k+\tau)\,\mathrm{d}k\big|^2 \\ &\leq \frac{1}{K_j}\int_{K_j}^{2K_j} |u_2^\infty(\hat{x},k)|^2\,\mathrm{d}k \cdot \frac{1}{K_j}\int_{K_j}^{2K_j} |u_2^\infty(\hat{x},k+\tau)|^2\,\mathrm{d}k \\ &\leq \frac{1}{K_j}\int_{K_j}^{2K_j} \sup_{\kappa\geq K_j} \big|u_2^\infty(\hat{x},\kappa)\big|^2\,\mathrm{d}k \cdot \frac{1}{K_j}\int_{K_j}^{2K_j} \sup_{\kappa\geq K_j+\tau} \big|u_2^\infty(\hat{x},\kappa)\big|^2\,\mathrm{d}k \\ &= \sup_{\kappa\geq K_j} |u_2^\infty(\hat{x},\kappa)|^2 \cdot \sup_{\kappa\geq K_j+\tau} |u_2^\infty(\hat{x},\kappa)|^2 \to 0, \quad j\to+\infty. \end{aligned}\tag{4.97}$$

Combining (4.94), (4.96) and (4.97), we conclude (4.78). The proof is complete. □

Chapter 5

Schrödinger operator with random potential and random source

5.1 Introduction

5.1.1 Mathematical formulations

In this chapter, we are mainly concerned with the following random Schrödinger system

$$
\begin{cases}
(-\Delta - E + q(x,\omega))u(x,\sqrt{E},d,\omega) = f(x,\omega), \quad x \in \mathbb{R}^3, & (5.1a) \\
u(x,\sqrt{E},d,\omega) = \alpha e^{\mathrm{i}\sqrt{E}x\cdot d} + u^{sc}(x,\sqrt{E},d,\omega), & (5.1b) \\
\lim_{r\to\infty} r\left(\dfrac{\partial u^{sc}}{\partial r} - \mathrm{i}\sqrt{E}u^{sc}\right) = 0, \quad r := |x|. & (5.1c)
\end{cases}
$$

The mathematical setting of system (5.1) is almost identical to the setting of system (4.1) in Chapter 4, except for the fact that in (5.1a), the potential $q(\cdot,\omega)$ is also generalized Gaussian random field with zero-mean and is assumed to be unknown, while in (4.1a), the potential is assumed to be deterministic. In (5.1a), we use letter q to denoted the potential instead of using letter V to distinguish with the deterministic case, i.e., (4.1a). $f(x,\omega)$ in (4.1a) is also assumed to be generalized Gaussian random field with zero-mean, and $f(x,\omega)$ and $q(x,\omega)$ are assumed to be m.i.g.r. (see Definition 4.2.1) and are independently distributed.

In this chapter, we denote the rough order of the f (resp. the q) as $-m_f$ (resp. $-m_q$), and the rough strength as μ_f (resp. μ_q). The main purpose of this chapter is to recover the rough strengths of both the source and the potential using either passive or active far-field measurements.

5.1.2 Statement of the main results

In order to study the inverse scattering problem, i.e., the recovery of μ_f and μ_q, we need to have a thorough understanding of the direct scattering problem first. For the case where both the source and potential are deterministic and L^∞ functions with compact supports, the well-posedness of the direct problem of system (5.1) is well known; see, e.g., [13, 16, 41]. Moreover, there holds the following asymptotic expansion of the outgoing radiating field u^{sc} as $|x| \to +\infty$,

$$u^{sc}(x) = \frac{e^{\mathrm{i}k|x|}}{|x|} u^\infty(\hat{x}, k, d) + \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad \text{in } \mathbb{R}^3.$$

$u^\infty(\hat{x}, k, d)$ is referred to as the far-field pattern, which encodes information of the potential and source. The $\hat{x}$ and d in $u^\infty(\hat{x}, k, d)$ stand for the observation direction and the direction of the incident wave. When $d = -\hat{x}$, the $u^\infty(\hat{x}, k, -\hat{x})$ is called the backscattering far-field pattern.

In our settings, however, due to the randomness of the source and potential, their regularities are much worse. This makes standard PDEs theories invalid for the direct problems of system (5.1). To this end, we shall reformulate the direct problem and show that the direct problem is still well-posed in a proper sense. Therefore, our direct problem can be formulated as

$$(f, q) \to \{u^{sc}(\hat{x}, k, d, \omega), u^\infty(\hat{x}, k, d, \omega)\,;\, \omega \in \Omega,\ \hat{x} \in \mathbb{S}^2, k \in \mathbb{R}_+,\ d \in \mathbb{S}^2\}.$$

The well-posedness of the direct scattering problem enables us to explore our main purpose. Due to the fact that the precise values of a random function provide little information about its statistical properties, we are concerned with the recovery of the rough strength of the source and potential by the knowledge of the far-field patterns.

In the recovery procedure, we recover the μ_f and μ_q in a *sequential* way by the knowledge of the associated far-field pattern measurements $u^\infty(\hat{x}, k, d, \omega)$. By sequential, we mean the μ_f and μ_q are recovered by their corresponding data sets one-by-one. In addition to this, in the recovery procedure, both the *passive* and *active* measurements are utilized. When $\alpha = 0$, the incident wave is suppressed and the scattering is solely generated by the unknown source. The corresponding far-field pattern is referred to as the passive measurement. In this case, the far-field pattern is independent of the incident direction d, and we denote it as $u^\infty(\hat{x}, k, \omega)$. When $\alpha = 1$, the scattering is generated by both the active source and the incident wave, and the far-field pattern is referred to as the active measurement, and is denoted as $u^\infty(\hat{x}, k, d, \omega)$.

Under these settings, our inverse problem can be formulated as

$$\begin{cases} \mathcal{M}_f(\omega) := \{\, u^\infty(\hat{x}, k, \omega)\,;\, \forall \hat{x} \in \mathbb{S}^2,\, \forall k \in \mathbb{R}_+ \,\} & \to \quad \mu_f, \\ \mathcal{M}_q(\omega) := \{\, u^\infty(\hat{x}, k, -\hat{x}, \omega)\,;\, \forall \hat{x} \in \mathbb{S}^2,\, \forall k \in \mathbb{R}_+ \,\} & \to \quad \mu_q. \end{cases}$$

The data set $\mathcal{M}_f(\omega)$ (abbr. $\mathcal{M}_f$) corresponds to the passive measurement ($\alpha = 0$), while the data set $\mathcal{M}_q(\omega)$ (abbr. $\mathcal{M}_q$) corresponds to the active measurement ($\alpha = 1$). Different random sample ω generate different data sets. Both the m_f and m_q are assumed to be unknown, and our study shows that the data sets $\mathcal{M}_f$, $\mathcal{M}_q$ can recover μ_f, μ_q, respectively.

With the potential being unknown, the inverse source problem, i.e., the recovery of μ_f, becomes highly nonlinear and thus more challenging. One possibility to facilitate this situation is to put some geometrical assumptions on the locations of the source and potential. If there is a positive distance between the convex hulls of the supports of f and q, i.e.,

$$\operatorname{dist}(D_f, D_q) := \inf\{\, |x - y|\,;\, x \in D_f,\, y \in D_q \,\} > 0, \tag{5.2}$$

we can find a plane separating the D_f and D_q. Denote as $\boldsymbol{n}$ the normal vector of this plane, pointing from the half-space containing D_f into the half-space containing D_q.

In system (5.1), both the source and potential are assume to be unknown. Moreover, the source and potential are generalized random function of the same type. These issues make the decoupling of μ_f and μ_q far more difficult. However, some *a* *priori* information about the rough orders of f and q may help us achieve our recoveries. This is indeed the case, see (5.3) below. Now we are ready to present our main results.

Theorem 5.1.1. *Assume that f and q in system* (5.1) *are m.i.g.r. of order $-m_f$ and $-m_q$, respectively, satisfying*

$$m_q < m_f < 5m_q - 11, \;\; m_f < 3. \tag{5.3}$$

Assume that (5.2) *is satisfied. Then, without knowing the μ_q, the data set $\mathcal{M}_f(\omega)$ can recover μ_f almost surely. Moreover, the recovering formula is*

$$\widehat{\mu}_f(\tau\hat{x}) = \begin{cases} \lim\limits_{K\to+\infty} \dfrac{4\sqrt{2\pi}}{K} \displaystyle\int_K^{2K} k^{m_f} \overline{u^\infty(\hat{x},k,\omega)} u^\infty(\hat{x},k+\tau,\omega)\,\mathrm{d}k, & \hat{x}\cdot\boldsymbol{n} \geq 0, \\ \overline{\widehat{\mu}_f(-\tau\hat{x})}, \quad \hat{x}\cdot\boldsymbol{n} < 0, & \end{cases} \tag{5.4}$$

where $\tau \geq 0$ and $u^\infty(\hat{x},k,\omega) \in \mathcal{M}_f(\omega)$.

Readers may refer to Definition 4.2.1 for the particular definition of m.i.g.r..

Remark 5.1.1. In Theorem 5.1.1, the μ_f can be uniquely recovered without *a priori* knowledge of q. Moreover, since $\alpha = 0$, Theorem 5.1.1 indicates that the μ_f can be uniquely recovered by a single realization of the passive scattering measurement. Due to the requirement $\hat{x}\cdot\boldsymbol{n} \geq 0$, only half of all the observation direction are needed. Besides, for the sake of simplicity, we set the wave number k in the definition of $\mathcal{M}_f$ to be running over all positive real numbers. But, in practice, it is enough to let the k be greater than any fixed positive number. These remarks also applies to Theorem 5.1.2. Also, note that in the definition of m.i.g.r., μ is defined as a real-valued function. Therefore, the $\widehat{\mu}_f$ in Theorem 5.1.1 (and $\widehat{\mu}_q$ in Theorem 5.1.2 below) is a conjugate-symmetric function.

To recover the μ_q, the active scattering measurement is needed in our recovery procedure.

Theorem 5.1.2. *Under the same condition as in Theorem 5.1.1, and without knowing μ_f, the data set $\mathcal{M}_q(\omega)$ can recover μ_q almost surely. Moreover, the recovering* *formula is*

$$\widehat{\mu}_q(\tau\hat{x}) = \begin{cases} \lim\limits_{K\to+\infty} \dfrac{4\sqrt{2\pi}}{K} \displaystyle\int_K^{2K} k^{m_q} \overline{u^\infty(\hat{x},k,-\hat{x},\omega)} u^\infty(\hat{x},k+\tfrac{\tau}{2},-\hat{x},\omega)\,\mathrm{d}k, & \hat{x}\cdot\boldsymbol{n}\geq 0, \\ \overline{\widehat{\mu}_f(-\tau\hat{x})}, \quad \hat{x}\cdot\boldsymbol{n}<0, \end{cases} \tag{5.5}$$

where $\tau \geq 0$ *and* $u^\infty(\hat{x},k,-\hat{x},\omega) \in \mathcal{M}_q(\omega)$*.*

Remark 5.1.2. Theorem 5.1.2 shows that the μ_q can be uniquely recovered without knowing the random source. Moreover, we only make use of a single realization of the active scattering measurement.

5.1.3 Discussion and connection to the existing results

There are abundant literature for the inverse scattering problem associated with either the passive or active measurements. Given an known potential, the recovery of an unknown source term by the corresponding passive measurement is referred to as the inverse source problem. We refer to [4–6,12,18,24–26,28,48,51] and the references therein for both theoretical uniqueness/stability results and computational methods for the inverse source problem in the deterministic setting. The simultaneous recovery of an unknown source and its surrounding potential was also investigated in the literature. In [27, 38], motivated by applications in thermo- and photo-acoustic tomography, the simultaneous recovery of an unknown source and its surrounding medium parameter was considered. The simultaneous recovery study in [27, 38] was confined to the deterministic setting and associated mainly with the passive measurement. For the random/stochastic case, the determination of a random source by the corresponding passive measurement was also recently studied in [3, 33, 34, 39, 50]. In [33], the homogeneous Helmholtz system with a random source is studied. Compared with [33], system (5.1) in this chapter comprises of both unknown source and unknown potential, making the corresponding study radically more challenging. The determination of a random potential, without source term, by the corresponding active measurement was established in [11]. And we also refer to [7–9, 31, 32] and the references therein for other relevant studies on random inverse medium problems.

We are particularly interested in the case with a single realization of the random sample, namely the ω is fixed in the recovery of the potential and source; see the recovery formulae (5.4)-(5.5). In our approach, we assume that the backscattering far-field data $u^\infty(\hat{x}, k, -\hat{x}, \omega)$ for different observation directions are generated by a single realization of the random sample [11]. Intuitively, a particular realization of q and f provides little information about the statistical properties of the random potential and source. However, our study indicates that a *single realization* of the far-field measurement can be used to uniquely recover the rough strengths in certain scenarios. A crucial assumption to make the single realization recovery possible is that the randomness is independent of the wave number k. Indeed, there are variant applications in which the randomness changes slowly or is independent of time [11, 32], and by temporal Fourier transforming into the frequency domain, they actually correspond to the aforementioned situation. The single realization recovery has been studied in the literature; see, e.g., [11, 31, 32, 35]. The idea of this chapter is mainly motivated by [11, 35].

Compared with our previous work [35], the differences are that, first, the random models are different. In [35], the random part of the source is assumed to be a Gaussian white noise, while in system (5.1), the potential and the source are assumed to be microlocally isotropic generalized Gaussian random function (m.i.g.r.). The m.i.g.r. can fit larger range of randomness by tuning its rough order. Second, in system (5.1), both the source and potential are random, while in [35], the potential is assumed to be deterministic. These two facts make the problem in this chapter much more difficult than [35]. The estimates of higher order terms (see Section 5.3) are pseudodifferential operators and microlocal analysis, which are more involved compared to that in [35].

In our setting, both the source and potential are random and unknown. This makes our study more intriguing compared to those existing ones in the literature. We use both the passive and active far-field measurements, and rough strengths of both the source and the potential are recovered with explicit recovering formulae (5.4)-(5.5). The major novelty of our unique recovery results compared to others is that on the one hand, both the source and the potential are random and unknown, and on the other hand, we use only two realizations of the random sample obtaining two unique recoveries. The mathematical arguments of our study are constructive and we derive explicitly recovery formulas, which can be employed for numerical reconstruction in future work.

The rest of the chapter is organized as follows. In Section 5.2, we first give an introduction to the random model and present some preliminaries; then we show the well-posedness of the direct scattering problem. Section 5.3 establishes the asymptotics of different terms appeared in the recovery formula. In Section 5.4, we recover the rough strength of the source. Section 5.5 is devoted to the recovery of the rough strength of the potential.

5.2 Mathematical analysis of the direct problem

In this section, we show that the direct problem is well-posed. Before showing that, we first present some preliminaries for the subsequent use and give the introduction to our random model.

5.2.1 Preliminaries

For convenient reference and self-containedness, we collect some preliminaries knowledge here. First, we introduce the Schwartz space and pseudodifferential operators. Readers may refer to [2, 49] for a more comprehensive study of the pseudodifferential operators.

If $\varphi \in C^\infty(\mathbb{R}^n)$, and for any multi-indices α and β, there holds

$$\sup_{x\in\mathbb{R}^n} |x^\beta \big(D^\alpha \varphi\big)(x)| < +\infty,$$

we call φ a Schwartz function. Denote the space of Schwartz function as $\mathscr{S}(\mathbb{R}^n)$. To construct the Schwartz space, we shall equip $\mathscr{S}(\mathbb{R}^n)$ with a topology $\mathcal{T}$. The $\mathcal{T}$ is induced by the set of the semi-norms $\{|\cdot|_m\,;\, m \in \{0\} \cup \mathbb{N}^+\}$ defined as

$$|\varphi|_m := \sum_{|\alpha|\le m, |\beta|\le m} \sup_{x\in\mathbb{R}^n} |x^\beta \big(\partial^\alpha \varphi\big)(x)|,\ \varphi \in \mathscr{S}(\mathbb{R}^n),$$

where α, β are multi-indices. It's straight forward to see that $|\varphi|_m \le |\varphi|_{m+1}$. Denote

$N(m;\epsilon) := \{\varphi \in \mathscr{S}(\mathbb{R}^n)\,;\, |\varphi|_m < \epsilon\}$ as the open neighborhood of the point $0 \in \mathscr{S}(\mathbb{R}^n)$. Define $\mathcal{N} := \{N(m;\epsilon)\,;\, m \in \{0\} \cup \mathbb{N}^+, \epsilon > 0\}$ as the open neighborhood basis of 0, and $\varphi + \mathcal{N}$ the open neighborhood basis of $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then the topology generated by these open neighborhood basis is the topology $\mathcal{T}$. The function space $\mathscr{S}(\mathbb{R}^n)$ together with the topology $\mathcal{T}$ is called the Schwartz space. We may denote the Schwartz space just as $\mathscr{S}(\mathbb{R}^n)$ if it is clear in the context.

A linear functional T is called a tempered distribution (or generalized function) if for any sequence $\{\varphi_j\}_{j=1}^{\infty} \subset \mathscr{S}(\mathbb{R}^n)$ s.t. $\varphi_j \to 0$ in $\mathcal{T}$, we have

$$T(\varphi_j) \to 0 \ (j \to +\infty).$$

The set of tempered distribution, denoted as $\mathscr{S}'(\mathbb{R}^n)$, is the dual of Schwartz Space $\mathscr{S}(\mathbb{R}^n)$.

Let $m \in (-\infty, +\infty)$. Then we define S^m to be the set of all functions $\sigma(x,\xi) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n; \mathbb{C})$ such that for any two multi-indices α and β, there is a positive constant $C_{\alpha,\beta}$, depending on α and β only, for which

$$\left|(D_x^\alpha D_\xi^\beta \sigma)(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}, \quad \forall x, \xi \in \mathbb{R}^n.$$

We call any function σ in $\bigcup_{m\in\mathbb{R}} S^m$ a *symbol*. A *principal symbol* of σ is an equivalent class $[\sigma] = \{\tilde{\sigma} \in S^m\}$ such that $a - \tilde{a} \in S^{m-1}$. In what follows, we may use one representative $\tilde{\sigma}$ in $[\sigma]$ to represent the equivalent class $[\sigma]$. Let σ be a symbol. Then the *pseudodifferential operator* T, defined on $\mathscr{S}(\mathbb{R}^n)$ and associated with σ, is defined by

$$\begin{aligned}(T_\sigma\varphi)(x) &:= (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\sigma(x,\xi)\hat{\varphi}(\xi)\,\mathrm{d}\xi \\ &= (2\pi)^{-n}\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{i(x-y)\cdot\xi}\sigma(x,\xi)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad \forall\varphi \in \mathscr{S}(\mathbb{R}^n).\end{aligned}$$

5.2.2 Preparation for the direct problem

Theorem 5.2.1. *For any $0 < s < 1/2$ and any $\epsilon > 0$, when $k > 2$,*

$$\|\mathcal{R}_k\varphi\|_{H^s_{-1/2-\epsilon}(\mathbb{R}^3)} \leq C_{\epsilon,s}k^{-(1-2s)}\|\varphi\|_{H^{-s}_{1/2+\epsilon}(\mathbb{R}^3)}, \quad \varphi \in H^{-s}_{1/2+\epsilon}(\mathbb{R}^3).$$

Theorem 5.2.2. *Assume that $q(\cdot,\omega)$ is microlocally isotropic of order $-m$. Then for every $s > (n-m)/2$ and every $\epsilon \in (0, 3/2]$, $q\colon H^s_{-1/2-\epsilon}(\mathbb{R}^n) \to H^{-s}_{1/2+\epsilon}(\mathbb{R}^n)$ is bounded almost surely,*

$$\|q(\cdot,\omega)\varphi(\cdot)\|_{H^{-s}_{1/2+\epsilon}(\mathbb{R}^3)} \leq C_{\epsilon,s}(\omega)\|\varphi\|_{H^s_{-1/2-\epsilon}(\mathbb{R}^3)}, \quad \varphi \in H^{-s}_{1/2+\epsilon}(\mathbb{R}^3), \qquad a.s.\ \omega \in \Omega.$$

The random variable $C_{\epsilon,s}(\omega)$ is finite almost surely.

The proofs of Theorem 5.2.1 and 5.2.2 follows both [11] and [§29, 16].

Proof of Theorem 5.2.1. Define a operator $\mathcal{R}_{k,\epsilon}\varphi := (2\pi)^{-n}\int_{\mathbb{R}^3} e^{ix\cdot\xi}\frac{\hat{\varphi}(\xi)\,\mathrm{d}\xi}{|\xi|^2-k^2-i\epsilon}$. Fix some function χ satisfying

$$\begin{cases} \chi \in C_c^\infty(\mathbb{R}^n),\ 0 \leq \varphi \leq 1, \\ \chi(x) = 1 \text{ when } |x| \leq 1, \\ \chi(x) = 0 \text{ when } |x| \geq 2. \end{cases} \tag{5.6}$$

Write $\mathfrak{R}\psi(x) := \psi(-x)$. Fix some $p \in (1, +\infty)$, we have

$$\begin{aligned}
&(\mathcal{R}_{k,\epsilon}\varphi, \psi)_{L^2(\mathbb{R}^3)} \\
=\ & \int_{\mathbb{R}^3} \mathcal{R}_{k,\epsilon}\varphi(x)\overline{\psi(x)}\,\mathrm{d}x = \int_{\mathbb{R}^3} \mathcal{F}\{\mathcal{R}_{k,\epsilon}\varphi\}(\xi)\cdot\mathcal{F}\{\mathfrak{R}\overline{\psi}\}(\xi)\,\mathrm{d}\xi \\
=\ & \int_{\mathbb{R}^3} \frac{\hat{\varphi}(\xi)\widehat{\mathfrak{R}\overline{\psi}}(\xi)}{|\xi|^2-k^2-i\epsilon}\,\mathrm{d}\xi = \int_{\mathbb{R}^3} \frac{\langle\xi\rangle^{2s}[\langle\xi\rangle^{-s}\hat{\varphi}(\xi)][\langle\xi\rangle^{-s}\widehat{\mathfrak{R}\overline{\psi}}(\xi)]}{|\xi|^2-k^2-i\epsilon}\,\mathrm{d}\xi \\
=\ & \int_0^\infty r^2\,\mathrm{d}r\cdot\int_{\mathbb{S}^2} \frac{\langle r\rangle^{2s}[\langle r\rangle^{-s}\hat{\varphi}(r\omega)][\langle r\rangle^{-s}\widehat{\mathfrak{R}\overline{\psi}}(r\omega)]}{r^2-k^2-i\epsilon}\,\mathrm{d}S(\xi) \\
=\ & \int_0^\infty \frac{(1-\chi^2(r-k))}{r^2-k^2-i\epsilon}\,\mathrm{d}r\cdot\int_{\mathbb{S}^2}\hat{\varphi}(\xi)\cdot\widehat{\mathfrak{R}\overline{\psi}}(\xi)\,\mathrm{d}S(\xi) \\
&+ \int_0^\infty \frac{\langle r\rangle^{1/p}r^2\chi^2(r-k)}{r^2-k^2-i\epsilon}\,\mathrm{d}r\cdot\int_{\mathbb{S}^2}[\langle k\rangle^{\frac{-1}{2p}}\hat{\varphi}(k\omega)][\langle k\rangle^{\frac{-1}{2p}}\widehat{\mathfrak{R}\overline{\psi}}(r\omega)]\,\mathrm{d}S(\omega)
\end{aligned}$$

$$+\int_0^\infty \frac{\langle r\rangle^{1/p} r^2\chi^2(r-k)}{r^2-k^2-i\epsilon}\,\mathrm{d}r\cdot\int_{\mathbb{S}^2}\{[\langle r\rangle^{\frac{-1}{2p}}\hat{\varphi}(r\omega)][\langle r\rangle^{\frac{-1}{2p}}\widehat{\mathfrak{R}\overline{\psi}}(r\omega)]$$
$$-[\langle k\rangle^{\frac{-1}{2p}}\hat{\varphi}(k\omega)][\langle k\rangle^{\frac{-1}{2p}}\widehat{\mathfrak{R}\overline{\psi}}(k\omega)]\}\,\mathrm{d}S(\xi)$$
$$=: I_1(\epsilon)+I_2(\epsilon)+I_3(\epsilon). \tag{5.7}$$

Now we estimate $I_1(\epsilon)$. By Young's inequality we have

$$ab\le a^p/p+b^q/q \quad\Rightarrow\quad (p^{1/p}q^{1/q})a^{1/p}b^{1/q}\le a+b \tag{5.8}$$

for $a,b>0$, $p,q>1$, $1/p+1/q=1$. Note that $|\widehat{\mathfrak{R}\overline{\psi}}(\xi)|=|\hat{\psi}(\xi)|$, one can compute

$$\begin{aligned}
|I_1(\epsilon)| &\le \int_0^\infty \frac{1-\chi^2(r-k)}{|r-k||r+k|}\,\mathrm{d}r\cdot\int_{|\xi|=r}|\hat{\varphi}(\xi)|\cdot|\widehat{\mathfrak{R}\overline{\psi}}(\xi)|\,\mathrm{d}S(\xi)\\
&\le \int_0^\infty \frac{1-\chi^2(r-k)}{1\cdot p^{1/p}q^{1/q}(r+1)^{1/p}(k-1)^{1/q}}\,\mathrm{d}r\cdot\int_{|\xi|=r}|\hat{\varphi}(\xi)|\cdot|\hat{\psi}(\xi)|\,\mathrm{d}S(\xi)\quad\text{(by (5.8))}\\
&\le C_pk^{-1/q}\int_0^\infty\langle r\rangle^{-1/p}\,\mathrm{d}r\cdot\int_{|\xi|=r}|\hat{\varphi}(\xi)|\cdot|\hat{\psi}(\xi)|\,\mathrm{d}S(\xi)\\
&\le C_pk^{1/p-1}\Big(\int_{\mathbb{R}^3}\langle\xi\rangle^{-1/p}|\hat{\varphi}(\xi)|^2\,\mathrm{d}\xi\Big)^{\frac12}\cdot\Big(\int_{\mathbb{R}^3}\langle\xi\rangle^{-1/p}|\hat{\psi}(\xi)|^2\,\mathrm{d}\xi\Big)^{\frac12}\\
&\le C_pk^{1/p-1}\|\varphi\|_{H^{-1/(2p)}(\mathbb{R}^3)}\|\psi\|_{H^{-1/(2p)}(\mathbb{R}^3)}\\
&\le C_pk^{1/p-1}\|\varphi\|_{H_\delta^{-1/(2p)}(\mathbb{R}^3)}\|\psi\|_{H_\delta^{-1/(2p)}(\mathbb{R}^3)}
\end{aligned}\tag{5.9}$$

where $1<p<+\infty$ and $\delta>0$.

Then we estimate $I_2(\epsilon)$. One has

$$I_2(\epsilon)=\int_{\mathbb{S}^2}[\langle k\rangle^{\frac{-1}{2p}}\hat{\varphi}(k\omega)][\langle k\rangle^{\frac{-1}{2p}}\widehat{\mathfrak{R}\overline{\psi}}(r\omega)]\int_0^\infty\frac{\langle r\rangle^{\frac1p}r^2\chi^2(r-k)\,\mathrm{d}r}{r^2-k^2-i\epsilon}\,\mathrm{d}S(\omega). \tag{5.10}$$

Let $\epsilon_0\in(0,1)$ be a fixed number whose shall be specified later. Write $p(r):=r^2-k^2-i\epsilon$. Recall that $\chi(r-k)=0$ when $|r-k|>2$. When $\epsilon_0\le|r-k|\le 2$, we have

$$|p(r)|\ge|\Re p(r)|=|r-k||r+k|\ge\epsilon_0(2k-2)\ge\epsilon_0 k. \tag{5.11}$$

Write $\Gamma_{k,\epsilon_0}:=\{r\in\mathbb{C};|r-k|=\epsilon_0,\Im r\le 0\}$. When $r\in\Gamma_{k,\epsilon_0}$ and $0<\epsilon<\epsilon_0$, we have

$$|p(r)|\ge|r-k|\,|2k+(r-k)|-\epsilon_0=\epsilon_0(2k-\epsilon_0)-\epsilon_0\ge\epsilon_0 k. \tag{5.12}$$

Combining (5.11) and (5.12), we conclude that $\forall \epsilon \in (0, \epsilon_0), \forall k > 2$,

$$|p(r)| \geq \epsilon_0 k, \quad \forall\, r \in \{r \in \mathbb{R}^+; 2 \geq |r-k| \geq \epsilon_0\} \cup \Gamma_{k,\epsilon_0}. \tag{5.13}$$

By using the Cauchy integral theorem, we change the integral domain w.r.t r from $\mathbb{R}_+$ in (5.10) to $\{r \in \mathbb{R}^+; 2 \geq |r-k| \geq \epsilon_0\} \cup \Gamma_{k,\epsilon_0}$. Combining this with the estimate (5.13) and noting that $\chi(r-k) = 1$ when $r \in \{r \in \mathbb{R}; |r-k| \leq 1\}$, we have

$$\begin{aligned}
|I_2(\epsilon)| = \Big| & \int_{\mathbb{S}^2} [\langle k \rangle^{\frac{-1}{2p}} \hat{\varphi}(k\omega)][\langle k \rangle^{\frac{-1}{2p}} \widehat{\mathcal{R}\overline{\psi}}(r\omega)] \int_{\{r \in \mathbb{R}^+; 2 \geq |r-k| \geq \epsilon_0\}} \frac{\langle r \rangle^{\frac{1}{p}} r^2 \chi^2(r-k)\, \mathrm{d}r}{r^2 - k^2 - i\epsilon} \,\mathrm{d}S(\omega) \\
& + \int_{\mathbb{S}^2} [\langle k \rangle^{\frac{-1}{2p}} \hat{\varphi}(k\omega)][\langle k \rangle^{\frac{-1}{2p}} \widehat{\mathcal{R}\overline{\psi}}(r\omega)] \int_{\Gamma_{k,\epsilon_0}} \frac{(1+r^2)^{\frac{1}{2p}} r^2 \,\mathrm{d}r}{r^2 - k^2 - i\epsilon} \,\mathrm{d}S(\omega) \Big| \\
\leq & \int_{\mathbb{S}^2} \langle k \rangle^{\frac{-1}{2p}} |\hat{\varphi}(k\omega)| \cdot \langle k \rangle^{\frac{-1}{2p}} |\widehat{\mathcal{R}\overline{\psi}}(r\omega)| \Big(\int_{\{r \in \mathbb{R}^+; 2 \geq |r-k| \geq \epsilon_0\}} \frac{\langle r \rangle^{\frac{1}{2p}} r^2}{\epsilon_0 k} \,\mathrm{d}r \Big) \,\mathrm{d}S(\omega) \\
& + \int_{\mathbb{S}^2} \langle k \rangle^{\frac{-1}{2p}} |\hat{\varphi}(k\omega)| \cdot \langle k \rangle^{\frac{-1}{2p}} |\widehat{\mathcal{R}\overline{\psi}}(r\omega)| \Big(\int_{\Gamma_{k,\epsilon_0}} \frac{|(1+r^2)^{\frac{1}{2p}} r^2|}{\epsilon_0 k} \,\mathrm{d}r \Big) \,\mathrm{d}S(\omega) \\
\leq & \int_{|\xi|=k} \langle \xi \rangle^{\frac{-1}{2p}} |\hat{\varphi}(\xi)| \cdot \langle \xi \rangle^{\frac{-1}{2p}} |\hat{\psi}(\xi)| \Big(\int_{\{r \in \mathbb{R}^+; 2 \geq |r-k| \geq \epsilon_0\}} \frac{\langle r \rangle^{\frac{1}{p}} (\frac{r}{k})^2}{\epsilon_0 k} \,\mathrm{d}r \Big) \,\mathrm{d}S(\xi) \\
& + \int_{|\xi|=k} \langle \xi \rangle^{\frac{-1}{2p}} |\hat{\varphi}(\xi)| \cdot \langle \xi \rangle^{\frac{-1}{2p}} |\hat{\psi}(\xi)| \Big(\int_{\Gamma_{k,\epsilon_0}} \frac{(1+|r|^2)^{\frac{1}{2p}} (\frac{r}{k})^2}{\epsilon_0 k} \,\mathrm{d}r \Big) \,\mathrm{d}S(\xi)
\end{aligned} \tag{5.14}$$

for all $\epsilon \in (0, \epsilon_0)$ and for all $\forall k > 2$.

Note that in $\{r \in \mathbb{R}^+; 2 \geq |r-k| \geq \epsilon_0\}$ we have

$$\langle r \rangle^{2s} \leq 5^s \langle k \rangle^{2s}, \quad 1 \leq (r/k)^2 \leq 4. \tag{5.15}$$

For $r \in \Gamma_{k,\epsilon_0}$ the complex number $(1+r^2)$ can be expressed as $R(r)e^{i\theta(r)}$ for real valued function $R(r)$ and $\theta(r)$. Now we choose ϵ_0 small enough such that $|\theta(r)| < \frac{\pi}{10}$ in Γ_{2,ϵ_0}, then $|\theta(r)| < \frac{\pi}{10}$ in Γ_{k,ϵ_0} for all $k \geq 2$. This can be easily seen from geometric view. Thus $(1+r^2)^s$ is well-defined for all $|s| \leq 2$, and

$$\forall r \in \Gamma_{k,\epsilon_0}, \quad |(1+r^2)^s| = |1+r^2|^s \leq (1+|r|^2)^s \leq \langle k+\epsilon_0 \rangle^{2s} \leq C \langle k \rangle^{2s} \tag{5.16}$$

for some constant C. Similarly, we have

$$\forall r \in \Gamma_{k,\epsilon_0}, \quad |r/k|^2 \leq (k+\epsilon_0)^2/k^2 \leq C \tag{5.17}$$

for some constant C.

Thus by (5.15)-(5.17) and Remark 13.1 in [16], we can continue (5.14) as

$$\begin{aligned}
|I_2(\epsilon)| &\leq C \int_{|\xi|=k} \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\varphi}(\xi)| \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\psi}(\xi)| \Big(\int_{\{r\in\mathbb{R}^+;2\geq|r-k|\geq\epsilon_0\}} \frac{\langle k\rangle^{1/p}}{\epsilon_0 k}\,\mathrm{d}r\Big)\,\mathrm{d}S(\xi) \\
&\quad + C \int_{|\xi|=k} \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\varphi}(\xi)| \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\psi}(\xi)| \Big(\int_{\Gamma_{k,\epsilon_0}} \frac{\langle k\rangle^{1/p}}{\epsilon_0 k}\,\mathrm{d}r\Big)\,\mathrm{d}S(\xi) \\
&\leq Ck^{1/p-1} \int_{|\xi|=k} \langle\xi\rangle^{-\frac{1}{2p}} |\hat{\varphi}(\xi)| \langle\xi\rangle^{-\frac{1}{2p}} |\hat{\psi}(\xi)|\,\mathrm{d}S(\xi) \\
&\leq Ck^{1/p-1} \Big(\int_{|\xi|=k} |\langle\xi\rangle^{\frac{-1}{2p}} \hat{h}(\xi)|^2\,\mathrm{d}S(\xi)\Big)^{\frac{1}{2}} \Big(\int_{|\xi|=k} |\langle\xi\rangle^{\frac{-1}{2p}} \hat{\psi}(\xi)|^2\,\mathrm{d}S(\xi)\Big)^{\frac{1}{2}} \\
&\leq Ck^{1/p-1} \|\langle\cdot\rangle^{-1/(2p)}\hat{\varphi}(\cdot)\|_{H^{1/2+\epsilon}(\mathbb{R}^3)} \|\langle\cdot\rangle^{-1/(2p)}\hat{\psi}(\cdot)\|_{H^{1/2+\epsilon}(\mathbb{R}^3)} \\
&= Ck^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}.
\end{aligned} \tag{5.18}$$

The last equality in (5.18) used (2.3).

Finally, we estimate $I_3(\epsilon)$. Denote $\mathbb{F}(r) = \mathbb{F}_r := \langle r\rangle^{-1/(2p)}\hat{\varphi}(r\omega)$ and $\mathbb{G}(r) = \mathbb{G}_r := \langle r\rangle^{-1/(2p)}\widehat{\mathfrak{R}\bar{\psi}}(r\omega)$. One can compute

$$\begin{aligned}
|I_3(\epsilon)| &= \Big| \int_0^\infty \frac{\langle r\rangle^{1/p}\, r^2 \chi^2(r-k)}{r^2-k^2-i\epsilon}\,\mathrm{d}r \cdot \int_{\mathbb{S}^2} (\mathbb{F}_r\mathbb{G}_r - \mathbb{F}_k\mathbb{G}_k)\,\mathrm{d}S(\omega)\Big| \\
&\leq \int_0^\infty \frac{\langle r\rangle^{1/p}\, r^2 \chi^2(r-k)}{|r^2-k^2|}\,\mathrm{d}r \cdot \int_{\mathbb{S}^2} |\mathbb{F}_r| \cdot |\mathbb{G}_r - \mathbb{G}_k|\,\mathrm{d}S(\omega) \\
&\quad + \int_0^\infty \frac{\langle r\rangle^{1/p}\, r^2 \chi^2(r-k)}{|r^2-k^2|}\,\mathrm{d}r \cdot \int_{\mathbb{S}^2} |\mathbb{F}_r - \mathbb{F}_k| \cdot |\mathbb{G}_k|\,\mathrm{d}S(\omega) \\
&\leq \int_0^\infty \frac{\langle r\rangle^{1/p} \chi^2(r-k)}{|r^2-k^2|} \cdot \|\mathbb{F}\|_{L^2(\mathbb{S}^2_r)} \cdot \Big(r^2 \int_{\mathbb{S}^2} |\mathbb{G}_r - \mathbb{G}_k|^2\,\mathrm{d}S(\omega)\Big)^{\frac{1}{2}}\,\mathrm{d}r \\
&\quad + \int_0^\infty \frac{\langle r\rangle^{1/p} \chi^2(r-k)}{|r^2-k^2|} \cdot \Big(r^2 \int_{\mathbb{S}^2} |\mathbb{F}_r - \mathbb{F}_k|^2\,\mathrm{d}S(\omega)\Big)^{\frac{1}{2}} \cdot \Big(\frac{r}{k}\Big)^2 \|\mathbb{G}\|_{L^2(\mathbb{S}^2_k)}\,\mathrm{d}r,
\end{aligned} \tag{5.19}$$

where $\mathbb{S}^2_r$ signifies the sphere with radius r. Combining Remark 13.1, (13.28) in [16] and (5.8), we can continue (5.19) as

$$
\begin{aligned}
|I_3(\epsilon)| &\lesssim \int_0^\infty \frac{\langle r\rangle^{1/p}\chi^2(r-k)}{|r-k|(r+k)} \cdot \|\mathbb{F}\|_{H^{1/2+\epsilon}(\mathbb{R}^3)} \cdot |r-k|^\alpha \cdot \|\mathbb{G}\|_{H^{1/2+\epsilon}(\mathbb{R}^3)} \,\mathrm{d}r \\
&\lesssim \int_0^\infty \frac{\langle r\rangle^{1/p}\chi^2(r-k)}{|r-k|^{1-\alpha}(r+1)^{1/p}(k-1)^{1-1/p}} \,\mathrm{d}r \cdot \|\mathbb{F}\|_{H^{1/2+\epsilon}(\mathbb{R}^3)}\|\mathbb{G}\|_{H^{1/2+\epsilon}(\mathbb{R}^3)} \\
&\lesssim k^{1/p-1} \int_0^\infty \frac{\chi^2(r-k)}{|r-k|^{1-\alpha}} \,\mathrm{d}r \cdot \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \cdot \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \\
&\le C_\epsilon k^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \cdot \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}, \qquad (5.20)
\end{aligned}
$$

where $0 < \alpha \le \epsilon$.

Combining (5.7), (5.9), (5.18) and (5.20), we arrive at

$$
|(\mathcal{R}_{k,\epsilon}\varphi, \psi)_{L^2(\mathbb{R}^3)}| \le |I_1(\epsilon)| + |I_2(\epsilon)| + |I_3(\epsilon)| \le C_\epsilon k^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)},
$$

which implies that

$$
\|\mathcal{R}_{k,\epsilon}\varphi\|_{H^{1/(2p)}_{-1/2-\epsilon}(\mathbb{R}^3)} \le C_\epsilon k^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}. \qquad (5.21)
$$

Now we study the limiting case $\lim_{\epsilon\to 0^+} \mathcal{R}_{k,\epsilon}\varphi$. For any two positive real numbers $\epsilon_1, \epsilon_2 < \tilde{\epsilon}$, we study $|I_j(\epsilon_1) - I_j(\epsilon_2)|$ for $j = 1, 2, 3$.

Similar to our previous derivation, we have

$$
\begin{aligned}
|I_1(\epsilon_1) - I_1(\epsilon_2)| &\le \int_0^\infty \frac{|\epsilon_1-\epsilon_2|(1-\chi^2(r-k))}{|r^2-k^2| \cdot p^{\frac{1}{p}} q^{\frac{1}{q}} (r+1)^{\frac{1}{p}} (k-1)^{\frac{1}{q}}} \,\mathrm{d}r \cdot \int_{|\xi|=r} |\hat{\varphi}(\xi)| \cdot |\hat{\psi}(\xi)| \,\mathrm{d}S(\xi) \\
&\le \tilde{\epsilon}\, C_p k^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{\delta}(\mathbb{R}^3)} \|\psi\|_{H^{-1/(2p)}_{\delta}(\mathbb{R}^3)}, \qquad (5.22)
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2(\epsilon_1) - I_2(\epsilon_2)| &\le C \int_{|\xi|=k} \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\varphi}(\xi)| \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\psi}(\xi)| \Big(\int_{\{r\in\mathbb{R}^+; 2\ge|r-k|\ge\epsilon_0\}} \frac{|\epsilon_1-\epsilon_2| \langle k\rangle^{\frac{1}{p}}}{(\epsilon_0 k)^2} \,\mathrm{d}r \Big) \,\mathrm{d}S(\xi) \\
&\quad + C \int_{|\xi|=k} \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\varphi}(\xi)| \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\psi}(\xi)| \Big(\int_{\Gamma_{k,\epsilon_0}} \frac{|\epsilon_1-\epsilon_2| \langle k\rangle^{\frac{1}{p}}}{(\epsilon_0 k)^2} \,\mathrm{d}r \Big) \,\mathrm{d}S(\xi) \\
&\le \tilde{\epsilon}\, C k^{\frac{1}{p}-1} \int_{|\xi|=k} \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\varphi}(\xi)| \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\psi}(\xi)| \Big(\int_{\{r\in\mathbb{R}^+; 2\ge|r-k|\ge\epsilon_0\}} 1 \,\mathrm{d}r \Big) \,\mathrm{d}S(\xi)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tilde{\epsilon}\, C k^{\frac{1}{p}-1} \int_{|\xi|=k} \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\varphi}(\xi)| \langle\xi\rangle^{\frac{-1}{2p}} |\hat{\psi}(\xi)| \Big(\int_{\Gamma_{k,\epsilon_0}} 1 \,\mathrm{d}r \Big) \,\mathrm{d}S(\xi) \\
&= \tilde{\epsilon}\, C k^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{\frac{1}{2}+\epsilon}(\mathbb{R}^3)} \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}.
\end{aligned} \tag{5.23}
$$

To analyze $I_3(\epsilon)$ as ϵ goes to zero, we note that by (5.8) we have

$$
C_\beta (\Re z)^\beta (\Im z)^{1-\beta} \le |z|, \quad \forall z \in \mathbb{C}
$$

holds for all $\beta \in (0,1)$ and some constant C. Without loss of generality, we assume $\epsilon_1 \le \epsilon_2$. Hence we can compute

$$
\begin{aligned}
\Big|\frac{1}{r^2-k^2-i\epsilon_1} - \frac{1}{r^2-k^2-i\epsilon_2}\Big| &= \frac{1}{|r^2-k^2-i\epsilon_1|} \cdot \frac{|\epsilon_1-\epsilon_2|}{|r^2-k^2-i\epsilon_2|} \\
&\le \frac{1}{|r^2-k^2|} \cdot \frac{C\epsilon_2}{|r^2-k^2|^\beta \cdot \epsilon_2^{1-\beta}} \le \frac{C\epsilon_2^\beta}{|r^2-k^2|^{1+\beta}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&|I_3(\epsilon_1) - I_3(\epsilon_2)| \\
&\lesssim \epsilon_2^\beta \int_0^\infty \frac{\langle r\rangle^{\frac{1}{p}} \chi^2(r-k)}{|r^2-k^2|^{1+\beta}} \,\mathrm{d}r \cdot \big(r^2 \int_{\mathbb{S}^2} |\mathbb{F}_r|^2 \,\mathrm{d}S(\omega) \big)^{\frac{1}{2}} \Big(r^2 \int_{\mathbb{S}^2} |\mathbb{G}_r - \mathbb{G}_k|^2 \,\mathrm{d}S(\omega) \Big)^{\frac{1}{2}} \\
&\quad + \epsilon_2^\beta \int_0^\infty \frac{\langle r\rangle^{\frac{1}{p}} \chi^2(r-k)}{|r^2-k^2|^{1+\beta}} \,\mathrm{d}r \cdot \Big(r^2 \int_{\mathbb{S}^2} |\mathbb{F}_r - \mathbb{F}_k|^2 \,\mathrm{d}S(\omega) \Big)^{\frac{1}{2}} \Big(k^2 \int_{\mathbb{S}^2} |\mathbb{G}_k|^2 \,\mathrm{d}S(\omega) \Big)^{\frac{1}{2}} \\
&\lesssim \epsilon_2^\beta \int_0^\infty \frac{\langle r\rangle^{\frac{1}{p}} \chi^2(r-k)}{|r-k|^{1+\beta} (r+1)^{\frac{1}{p}} (k-1)^{1-\frac{1}{p}}} \cdot |r-k|^\alpha \cdot \|\mathbb{F}\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R}^3)} \cdot \|\mathbb{G}\|_{H^{\frac{1}{2}+\epsilon}(\mathbb{R}^3)} \,\mathrm{d}r \\
&\lesssim \epsilon_2^\beta k^{\frac{1}{p}-1} \int_0^\infty \frac{\chi^2(r-k)}{|r-k|^{1+\beta-\alpha}} \,\mathrm{d}r \cdot \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \\
&\lesssim \tilde{\epsilon}^\beta k^{1/p-1} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)} \|\psi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}.
\end{aligned} \tag{5.24}
$$

The last inequality in (5.24) holds when $0 < \beta < \alpha$.

From (5.22), (5.23) and (5.24) we arrive at

$$
\|\mathcal{R}_{k,\epsilon_1}\varphi - \mathcal{R}_{k,\epsilon_2}\varphi\|_{H^{-1/(2p)}_{-1/2-\epsilon}(\mathbb{R}^3)} \lesssim \tilde{\epsilon} \|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}, \quad \forall \epsilon_1, \epsilon_2 \in (0, \tilde{\epsilon}), \tag{5.25}
$$

and thus

$$
\lim_{\tilde{\epsilon} \to 0^+} \mathcal{R}_{k,\tilde{\epsilon}}\varphi = \mathcal{R}_k \varphi \quad \text{in} \quad H^{1/(2p)}_{-1/2-\epsilon}(\mathbb{R}^3). \tag{5.26}
$$

The relationships (5.25) and (5.26) sometimes refer to as the limiting absorption principle. Hence from (5.21) and (5.26) we conclude

$$\|\mathcal{R}_k\varphi\|_{H^{1/(2p)}_{-1/2-\epsilon}(\mathbb{R}^3)} \leq C_{\epsilon,p}k^{-(1-1/p)}\|\varphi\|_{H^{-1/(2p)}_{1/2+\epsilon}(\mathbb{R}^3)}$$

holds for any $1 < p < +\infty$ and any $\epsilon > 0$. The proof is complete. □

Proof of Theorem 5.2.2. Let $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$, define $\langle q\varphi, \psi\rangle := \langle q, \varphi\psi\rangle$. Choose some function χ such that $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi(x) = 1$ when $x \in \operatorname{supp} q$. Choose s' satisfying $-s' < (m-n)/2$ and p, p' satisfying $1 < p < +\infty$, $1/p' + 1/p = 1$, then according to [Proposition 2.4, 11], $\|q\|_{H^{-s',p'}(\mathbb{R}^n)} < +\infty$ almost surely. Denote $\|q\|_{H^{-s',p'}(\mathbb{R}^n)}$ as $C_s(\omega)$, one can compute

$$\begin{aligned} |\langle q\varphi, \psi\rangle| &= |\langle q, (\chi\varphi)(\chi\psi)\rangle| = |\langle (I-\Delta)^{-s'}q, (I-\Delta)^{s'}\big((\chi\varphi)(\chi\psi)\big)\rangle| \\ &\leq \|q\|_{H^{-s',p'}(\mathbb{R}^n)} \cdot \|(I-\Delta)^{s'}\big((\chi\varphi)(\chi\psi)\big)\|_{L^p(\mathbb{R}^n)} \\ &= C_s(\omega)\|(I-\Delta)^{s'}\big((\chi\varphi)(\chi\psi)\big)\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{5.27}$$

According to the fractional Leibniz rule [17], when $1/p = 1/2 + 1/q$, we have

$$\begin{aligned} \|(I-\Delta)^{s'}\big((\chi\varphi)(\chi\psi)\big)\|_{L^p(\mathbb{R}^n)} \leq C_s(\omega)\big(&\|\chi\varphi\|_{L^2(\mathbb{R}^n)}\|\chi\psi\|_{H^{s',q}(\mathbb{R}^n)} \\ &+ \|\chi\psi\|_{L^2(\mathbb{R}^n)}\|\chi\varphi\|_{H^{s',q}(\mathbb{R}^n)}\big). \end{aligned} \tag{5.28}$$

By (5.27)-(5.28) and noting the Sobolev embedding $H^s(\mathbb{R}^n) \hookrightarrow H^{s',q}(\mathbb{R}^n)$ when $s - n/2 \geq s' - n/q$, $s > s'$, we can continue (5.27) as

$$\begin{aligned} |\langle q\varphi, \psi\rangle| &\lesssim C_s(\omega)\big(\|\chi\varphi\|_{L^2(\mathbb{R}^n)} \cdot \|\chi\psi\|_{H^{s',q}(\mathbb{R}^n)} + \|\chi\psi\|_{L^2(\mathbb{R}^n)} \cdot \|\chi\varphi\|_{H^{s',q}(\mathbb{R}^n)}\big) \\ &\lesssim C_s(\omega)\big(\|\chi\varphi\|_{L^2(\mathbb{R}^n)} \cdot \|\chi\psi\|_{H^s(\mathbb{R}^n)} + \|\chi\psi\|_{L^2(\mathbb{R}^n)} \cdot \|\chi\varphi\|_{H^s(\mathbb{R}^n)}\big) \\ &\lesssim C_s(\omega)\|\chi\varphi\|_{H^s(\mathbb{R}^n)} \cdot \|\chi\psi\|_{H^s(\mathbb{R}^n)}. \end{aligned} \tag{5.29}$$

Because $1 < p' < +\infty$ and $s' > -\frac{m-n}{2}$, the real number s should satisfy

$$s \geq s' + \frac{n}{2} - \frac{n}{q} = s' + \frac{n}{2} - n(\frac{1}{p} - \frac{1}{2}) = s' + n - \frac{n}{p} = s' + \frac{n}{p'} \geq s' > \frac{n-m}{2}.$$

Now we mimic the proof of Lemma 3.7 in [11] to show that

$$\|\chi\varphi\|_{H^s(\mathbb{R}^n)} \le C\|\varphi\|_{H^s_{-2}(\mathbb{R}^n)}, \quad \varphi \in \mathscr{S}(\mathbb{R}^n). \tag{5.30}$$

Rewrite the right-hand-sight of (5.30) in terms of the L^2-norm forms, we obtain

$$\|\chi\varphi\|_{H^s(\mathbb{R}^n)} \le C\|\langle\cdot\rangle^{-2}(I-\Delta)^{s/2}\varphi\|_{L^2(\mathbb{R}^n)}.$$

Write $\psi(x) := \langle x\rangle^{-2}(I-\Delta)^{s/2}\varphi(x)$, Obviously, $\varphi \in \mathscr{S}(\mathbb{R}^n)$ is equivalent to $\psi \in \mathscr{S}(\mathbb{R}^n)$. Define $T_a\psi := \chi\cdot(I-\Delta)^{-s/2}(\langle\cdot\rangle^2\psi)$, then $\chi\varphi = T_N\psi$ and (5.30) is equivalent with

$$\|T_a\psi\|_{H^s(\mathbb{R}^n)} \le C\|\psi\|_{L^2(\mathbb{R}^n)}. \tag{5.31}$$

The T_a is a pseudo-differential operator with

$$a(x,\xi) := \chi(x)\big(\langle x\rangle^2\langle\xi\rangle^{-s} - 2ix\cdot\nabla_\xi\langle\xi\rangle^{-s} - \Delta_\xi\langle\xi\rangle^{-s}\big)$$

as its symbol. It is easy to see that $a \in S^{-s}$, thus according to properties of pseudo-differential operators [16], (5.31) holds, and so does (5.30).

Now we can continue (5.29) as

$$\begin{aligned}|\langle q\varphi,\psi\rangle| &\lesssim C_s(\omega)\|\chi\varphi\|_{H^s(\mathbb{R}^n)}\cdot\|\chi\psi\|_{H^s(\mathbb{R}^n)} \lesssim C_s(\omega)\|\varphi\|_{H^s_{-2}(\mathbb{R}^n)}\cdot\|\psi\|_{H^s_{-2}(\mathbb{R}^n)} \\ &\le C_s(\omega)\|\varphi\|_{H^s_{-1/2-\epsilon}(\mathbb{R}^n)}\cdot\|\psi\|_{H^s_{-1/2-\epsilon}(\mathbb{R}^n)}, \quad \forall\varphi,\psi\in\mathscr{S}(\mathbb{R}^n),\end{aligned}$$

which implies that

$$\|q\varphi\|_{H^{-s}_{1/2+\epsilon}(\mathbb{R}^n)} \le C_{\epsilon,s}(\omega)\|\varphi\|_{H^s_{-1/2-\epsilon}(\mathbb{R}^n)}, \quad \forall\varphi\in\mathscr{S}(\mathbb{R}^n). \tag{5.32}$$

Now we show that $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s_{-1/2-\epsilon}(\mathbb{R}^n)$. Fix some function φ satisfying (5.6). For any $\varphi \in H^s_{-1/2-\epsilon}(\mathbb{R}^n)$, we have $\langle\cdot\rangle^{-1/2-\epsilon}(I-\Delta)^{s/2}\varphi \in L^2(\mathbb{R}^n)$. Then for any $\delta > 0$ there exists a constant M, depending on φ, such that

$$\|\langle\cdot\rangle^{-1/2-\epsilon}(I-\Delta)^{s/2}\varphi - \varphi^{(1)}\|_{L^2(\mathbb{R}^n)} < \frac{\delta}{2},$$

where $\varphi^{(1)} = \varphi(\cdot/M)\langle\cdot\rangle^{-1/2-\epsilon}(I-\Delta)^{s/2}\varphi$. Note that $\varphi^{(1)} \in L^2(\mathbb{R}^n)$ with compact support. Further, there exists a constant m small enough such that $\|\varphi^{(1)} - \varphi^{(2)}\|_{L^2(\mathbb{R}^n)} < \frac{\delta}{2}$ where $\varphi^{(2)} = (\frac{1}{m^n}\varphi(\frac{\cdot}{m})) * \varphi^{(1)}$. The function $\varphi^{(2)}$ is in $C^\infty(\mathbb{R}^n)$ with compact support, thus is in $\mathscr{S}(\mathbb{R}^n)$. Write $\varphi^{(3)} = (I-\Delta)^{-s/2}\big(\langle\cdot\rangle^{1/2+\epsilon}\varphi^{(2)}\big)$, thus $\varphi^{(3)} \in \mathscr{S}(\mathbb{R}^n)$ and

$$\begin{aligned}\|\varphi - \varphi^{(3)}\|_{H^s_{-1/2-\epsilon}(\mathbb{R}^n)} &= \|\langle\cdot\rangle^{-1/2-\epsilon}(I-\Delta)^{s/2}\varphi - \langle\cdot\rangle^{-1/2-\epsilon}(I-\Delta)^{s/2}\varphi^{(3)}\|_{L^2(\mathbb{R}^n)} \\ &= \|\langle\cdot\rangle^{-1/2-\epsilon}(I-\Delta)^{s/2}\varphi - \varphi^{(2)}\|_{L^2(\mathbb{R}^n)} \\ &\le \|\langle\cdot\rangle^{-1/2-\epsilon}(I-\Delta)^{s/2}\varphi - \varphi^{(1)}\|_{L^2(\mathbb{R}^n)} + \|\varphi^{(1)} - \varphi^{(2)}\|_{L^2(\mathbb{R}^n)} \\ &< \delta/2 + \delta/2 = \delta.\end{aligned}$$

Therefore $\mathscr{S}(\mathbb{R}^n)$ is dense in $H^s_{-1/2-\epsilon}(\mathbb{R}^n)$. It is well-known that the space $H^s_{-1/2-\epsilon}(\mathbb{R}^n)$ is a Banach space, thus by a density argument, the inequality (5.32) can be extended to all $\varphi \in H^s_{-1/2-\epsilon}(\mathbb{R}^n)$. The proof is complete. $\square$

5.2.3 The well-posedness of the direct problem

To study the well-posedness of the direct scattering problem, we reformulate (5.1) into the Lippmann-Schwinger equation formally (cf. [13]) to obtain

$$(I - \mathcal{R}_k q)u^{sc} = \alpha\mathcal{R}_k q u^i - \mathcal{R}_k f. \tag{5.33}$$

We are now in a position to present one of the results concerning the direct scattering problem.

Theorem 5.2.3. *When k is large enough such that $\|\mathcal{R}_k q\|_{\mathcal{L}(H^{-s}_{1/2+\epsilon}(\mathbb{R}^3), H^{-s}_{1/2+\epsilon}(\mathbb{R}^3))} < 1$, there exists a unique stochastic process $u^{sc}(\cdot,\omega)\colon \mathbb{R}^3 \to \mathbb{C}$ such that $u^{sc}(x)$ satisfies* (5.33) *almost surely. Moreover,*

$$\|u^{sc}(\cdot,\omega)\|_{H^{-s}_{1/2+\epsilon}(\mathbb{R}^3)} \lesssim \|\alpha\mathcal{R}_k q u^i - \mathcal{R}_k f\|_{H^{-s}_{1/2+\epsilon}(\mathbb{R}^3)} \quad a.s., \tag{5.34}$$

for any $\epsilon \in \mathbb{R}_+$.

Proof. By Theorems 5.2.1 and 5.2.2, we know

$$F := \alpha \mathcal{R}_k q u^i - \mathcal{R}_k f \in H^{-s}_{1/2+\epsilon}(\mathbb{R}^3).$$

From Theorems 5.2.1 and 5.2.2, we know the operator $I - \mathcal{R}_k q$ is invertible from $H^{-s}_{1/2+\epsilon}(\mathbb{R}^3)$ to itself, and the right-hand side of (5.33) belongs to $H^{-s}_{1/2+\epsilon}(\mathbb{R}^3)$.

Let $u^{sc} := (I - \mathcal{R}_k q)^{-1} F \in H^{-s}_{1/2+\epsilon}(\mathbb{R}^3)$, then u^{sc} fulfills requirements. The existence of the solution is proved. The (5.34) can be verified easily from Theorems 5.2.1, 5.2.2 and (5.33). The uniqueness follows easily from (5.34). The proof is complete. □

5.3 Asymptotic analysis of high-order terms

Following the ideas of Corollary 4.4 in [11] and of Lemma 4.3 in [35], we wish to recovery μ_f, μ_q from the data of the form

$$\frac{1}{K} \int_K^{2K} k^m \overline{u^\infty(k,\omega)} u^\infty(k+\tau,\omega) \,\mathrm{d}k, \tag{5.35}$$

where the $u^\infty(k,\omega)$ stands for $u^\infty(\hat{x},k,\omega) \in \mathcal{M}_f$ in the case of $\alpha = 0$ and stands for $u^\infty(\hat{x},k,-\hat{x},\omega) \in \mathcal{M}_q$ in the case of $\alpha = 1$. The Lippmann-Schwinger equation corresponding to (5.1) is

$$(I - \mathcal{R}_k q) u^s(k,\omega) = \alpha \mathcal{R}_k q u^i + \mathcal{R}_k f. \tag{5.36}$$

When k is large enough such that $\|\mathcal{R}_k q\|_{\mathcal{L}(H^s_{-1/2-\epsilon}, H^s_{-1/2-\epsilon})} < 1$, from (5.36) we obtain

$$u^s(k,\omega) = \sum_{j \geq 0} \mathcal{R}_k\big(q(\mathcal{R}_k q)^j f\big) + \alpha \sum_{j \geq 0} \mathcal{R}_k\big(q(\mathcal{R}_k q)^j u^i\big), \tag{5.37}$$

$$u^\infty(k,\omega) = (4\pi)^{-1} \sum_{j=0,1,2} F_j(\hat{x},k,\omega) + \alpha (4\pi)^{-1} \sum_{j=0,1,2} G_j(\hat{x},k,\omega), \tag{5.38}$$

where

$$
\begin{cases}
F_j(\hat{x}, k, \omega) := \displaystyle\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot z} \big[(q\mathcal{R}_k)^j f\big](z)\,\mathrm{d}z, \quad j = 0, 1 \\
F_2(\hat{x}, k, \omega) := \displaystyle\sum_{j\geq 2} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot z} \big[(q\mathcal{R}_k)^j f\big](z)\,\mathrm{d}z, \\
G_j(\hat{x}, k, d, \omega) := \displaystyle\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot z} \big[(q\mathcal{R}_k)^j u^i\big](z)\,\mathrm{d}z, \quad j = 0, 1 \\
G_2(\hat{x}, k, d, \omega) := \displaystyle\sum_{j\geq 2} \int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot z} \big[(q\mathcal{R}_k)^j u^i\big](z)\,\mathrm{d}z.
\end{cases}
\tag{5.39}
$$

Substitute (5.38) into (5.35), we obtain several crossover terms comprised by F_j and G_j. To recover μ_f and μ_q, it is necessary to establish the asymptotics of F_j and G_j in terms of k. The asymptotic analysis of G_j ($j = 0, 1, 2$) are already given in [11].

This section is devoted to the asymptotic analysis of F_1 and F_2, which are given in Lemmas 5.3.3 and 5.3.5, respectively. These two lemma play key roles in the proofs to Theorems 5.1.1 and 5.1.2.

5.3.1 Asymptotics of F_1

In order to establish the asymptotics of F_1, we need two auxiliary lemmas, i.e., Lemmas 5.3.1 and 5.3.2. First, let us recall the notion of the fractional Laplacian [43] of order $s \in (0, 1)$,

$$
(-\Delta)^{s/2}\varphi(x) := (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} |\xi|^s \varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi, \tag{5.40}
$$

where the integration is defined as an oscillatory integral. When $\varphi \in \mathscr{S}(\mathbb{R}^n)$, (5.40) can be understood as a usual Lebesgue integral if one integrate w.r.t. y first and then integrate w.r.t. ξ. By duality arguments, the fractional Laplacian can be generalized to act on more wide range of functions and distributions, see [49] for reference. It can be verified that the fractional Laplacian is self-adjoint.

Lemma 5.3.1. *For any $s \in (0, 1)$, we have*

$$
(-\Delta_\xi)^{s/2}(e^{ix\cdot\xi}) = |x|^s e^{ix\cdot\xi}
$$

in the distributional sense.

Proof of Lemma 5.3.1. For any $\varphi \in \mathscr{S}(\mathbb{R}^n)$, because $(-\Delta_\xi)^{s/2}$ is self-adjoint, we have

$$\begin{aligned}
\big((-\Delta_\xi)^{s/2}(e^{ix\cdot\xi}), \varphi(\xi)\big) &= \big(e^{ix\cdot\xi}, (-\Delta_\xi)^{s/2}\varphi(\xi)\big) \\
&= \int e^{ix\cdot\xi} \cdot \big[(2\pi)^{-n} \iint e^{i(\xi-y)\cdot\eta}|\eta|^s \varphi(y)\,\mathrm{d}y\,\mathrm{d}\eta\big]\,\mathrm{d}\xi \\
&= \int e^{ix\cdot\xi} \cdot \big\{(2\pi)^{-n/2} \int \big[(2\pi)^{-n/2} \int e^{i(\xi-y)\cdot\eta}|\eta|^s\,\mathrm{d}\eta\big]\varphi(y)\,\mathrm{d}y\big\}\,\mathrm{d}\xi \\
&= (2\pi)^{-n/2} \int e^{ix\cdot\xi} \cdot \int \mathcal{F}^{-1}\{|\cdot|^s\}(\xi - y)\cdot\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
&= (2\pi)^{-n/2} \iint e^{ix\cdot\xi}\mathcal{F}^{-1}\{|\cdot|^s\}(\xi - y)\cdot\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
&= \int \big[(2\pi)^{-n/2} \int e^{ix\cdot\xi}\mathcal{F}^{-1}\{|\cdot|^s\}(\xi - y)\,\mathrm{d}\xi\big]\cdot\varphi(y)\,\mathrm{d}y \\
&= \int e^{ix\cdot y}\big[(2\pi)^{-n/2} \int e^{-i(-x)\cdot\xi}\mathcal{F}^{-1}\{|\cdot|^s\}(\xi)\,\mathrm{d}\xi\big]\cdot\varphi(y)\,\mathrm{d}y \\
&= \int e^{ix\cdot y}\mathcal{F}\mathcal{F}^{-1}\{|\cdot|^s\}(-x)\cdot\varphi(y)\,\mathrm{d}y \\
&= \int |x|^s e^{ix\cdot y}\cdot\varphi(y)\,\mathrm{d}y \\
&= \big(|x|^s e^{ix\cdot\xi}, \varphi(\xi)\big).
\end{aligned}$$

Readers should note that in the derivation above, some integrals should be understood as oscillatory integral. □

Lemma 5.3.2. *For any $m < 0$, any $s \in (0, 1)$ and any $c(x, \xi) \in S^m$, we have*

$$|\big((-\Delta_\xi)^{s/2}c\big)(x, \xi)| \le C\langle\xi\rangle^{m-s},$$

where the constant C is independent of x, ξ.

Proof of Lemma 5.3.2. Our arguments are divided into two steps.

Step 1: For the case where $|\xi| \ge 1$.

In this step, we set $|\xi|$ to be greater than 1. By the definition (5.40), we have

$$\begin{aligned}
\big((-\Delta_\xi)^{s/2}c\big)(x, \xi) &\simeq \iint e^{i(\xi-\eta)\cdot\gamma}|\gamma|^s\, c(x, \eta)\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&= \iint e^{-i\eta\cdot\gamma}|\gamma|^s\, c(x, \eta + \xi)\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&= \iint e^{-i\eta\cdot\gamma}\Big|\frac{\gamma}{|\xi|}\Big|^s\, c(x, |\xi|\eta + \xi)\,\mathrm{d}(|\xi|\eta)\,\mathrm{d}(\gamma/|\xi|)
\end{aligned}$$

$$\simeq |\xi|^{-s} \iint e^{-i\eta\cdot\gamma} |\gamma|^s \, c(x, |\xi|(\eta + \hat{\xi})) \, \mathrm{d}\eta \, \mathrm{d}\gamma \tag{5.41}$$

where $\hat{\xi} = \xi/|\xi|$. Fix some $\chi_0 \in C_c^\infty(\mathbb{R})$ with $\chi_0(|x|) \equiv 1$ when $1/2 \leq |x| \leq 3/2$ and $\chi_0(|x|) \equiv 1$ when $|x| \leq 0$ or $|x| \geq 2$. We can continue (5.41) as

$$\begin{aligned}
\big((-\Delta_\xi)^{s/2} c\big)(x, \xi) \simeq |\xi|^{m-s} & \iint e^{-i\eta\cdot\gamma} \chi_0(|\eta|) |\gamma|^s \, c(x, |\xi|(\eta + \hat{\xi})) \, |\xi|^{-m} \, \mathrm{d}\eta \, \mathrm{d}\gamma \\
& + |\xi|^{m-s} \iint e^{-i\eta\cdot\gamma} \big(1 - \chi_0(|\eta|)\big) \, |\gamma|^s \, c(x, |\xi|(\eta + \hat{\xi})) \, |\xi|^{-m} \, \mathrm{d}\eta \, \mathrm{d}\gamma \\
& := |\xi|^{m-s} (\mathcal{B}_1 + \mathcal{B}_2).
\end{aligned} \tag{5.42}$$

We estimate $\mathcal{B}_1$, $\mathcal{B}_2$ seperately. For $\mathcal{B}_1$, one can compute

$$\begin{aligned}
\mathcal{B}_1 &= \iint e^{-i(\eta - \hat{\xi})\cdot\gamma} \chi_0(|\eta - \hat{\xi}|) |\gamma|^s \, c(x, |\xi|\eta) \, |\xi|^{-m} \, \mathrm{d}\eta \, \mathrm{d}\gamma \\
&= \int e^{i\hat{\xi}\cdot\gamma} |\gamma|^s \Big(\int e^{-i\eta\cdot\gamma} \chi_0(|\eta - \hat{\xi}|) \, c(x, |\xi|\eta) \, |\xi|^{-m} \, \mathrm{d}\eta \Big) \, \mathrm{d}\gamma \\
&=: \int e^{i\hat{\xi}\cdot\gamma} |\gamma|^s J(\gamma; |\xi|, x) \, \mathrm{d}\gamma,
\end{aligned} \tag{5.43}$$

where $J(\gamma; |\xi|, x) = \int e^{-i\eta\cdot\gamma} \chi_0(|\eta - \hat{\xi}|) \, c(x, |\xi|\eta) \, |\xi|^{-m} \, \mathrm{d}\eta$. We claim that the $J(\gamma; |\xi|, x)$ is rapidly decaying w.r.t. $|\gamma|$, that is

$$\forall N \in \mathbb{N}, \quad |\gamma|^{2N} \, |J(\gamma; |\xi|, x)| \leq C_N < +\infty, \tag{5.44}$$

for some constant C_N independent of γ, ξ and x. This can be seen from

$$\begin{aligned}
|\gamma|^{2N} \, |J(\gamma; |\xi|, x)| &\simeq \Big| \int \Delta_\eta^N (e^{-i\eta\cdot\gamma}) \cdot \chi_0(|\eta - \hat{\xi}|) \, c(x, |\xi|\eta) \, |\xi|^{-m} \, \mathrm{d}\eta \Big| \\
&= \Big| \int e^{-i\eta\cdot\gamma} \cdot \Delta_\eta^N \big(\chi_0(|\eta - \hat{\xi}|) \, c(x, |\xi|\eta) \big) \, |\xi|^{-m} \, \mathrm{d}\eta \Big| \\
&\leq \int_{\frac{1}{2} \leq |\eta - \hat{\xi}| \leq 2} |\Delta_\eta^N \big(\chi_0(|\eta - \hat{\xi}|) \, c(x, |\xi|\eta) | \cdot |\xi|^{-m} \, \mathrm{d}\eta \\
&\lesssim \int_{\frac{1}{2} \leq |\eta - \hat{\xi}| \leq 2} \sum_{|\alpha| \leq 2N} |(\partial_\xi^\alpha c)(x, |\xi|\eta)| \cdot |\xi|^{|\alpha| - m} \, \mathrm{d}\eta \\
&\lesssim \sum_{|\alpha| \leq 2N} \int_{\frac{1}{2} \leq |\eta - \hat{\xi}| \leq 2} (1 + |\xi| \, |\eta|)^{m - |\alpha|} \cdot |\xi|^{|\alpha| - m} \, \mathrm{d}\eta
\end{aligned}$$

$$= \sum_{|\alpha|\le 2N} \int_{\frac{1}{2}\le|\eta-\hat{\xi}|\le 2} (|\xi|^{-1}+|\eta|)^{m-|\alpha|}\,\mathrm{d}\eta, \tag{5.45}$$

where the N is an arbitrary non-negative integer. The condition $|\xi|\ge 1$ gives

$$(|\xi|^{-1}+|\eta|)^{m-|\alpha|} \le \begin{cases} (1+|\eta|)^{m-|\alpha|}, & \text{when } |\alpha|\le m, \\ |\eta|^{m-|\alpha|}, & \text{when } |\alpha|>m. \end{cases} \tag{5.46}$$

By (5.45) and (5.46), we obtain (5.44). Therefore, $J(\gamma;|\xi|,x)$ is indeed rapidly decaying. Now, combining (5.43) and (5.44), we arrive at

$$|\mathcal{B}_1| \lesssim \int_{|\gamma|\ge 1} |\gamma|^s\,\mathrm{d}\gamma + \int_{|\gamma|>1} |\gamma|^s|\gamma|^{-4}\,\mathrm{d}\gamma \le C < +\infty, \tag{5.47}$$

for some constant C independent of x, ξ.

To estimate $\mathcal{B}_2$, we split $\mathcal{B}_2$ into two terms, say, $\mathcal{B}_{21}$ and $\mathcal{B}_{22}$, in the following way,

$$\begin{aligned} \mathcal{B}_2 &= \iint_{\gamma\le 1} e^{-i\eta\cdot\gamma}\big(1-\chi_0(|\eta|)\big)\,|\gamma|^s\,c(x,|\xi|(\eta+\hat{\xi}))\,|\xi|^{-m}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\ &\quad+ \iint_{\gamma>1} e^{-i\eta\cdot\gamma}\big(1-\chi_0(|\eta|)\big)\,|\gamma|^s\,c(x,|\xi|(\eta+\hat{\xi}))\,|\xi|^{-m}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\ &=: \mathcal{B}_{21}+\mathcal{B}_{22}. \end{aligned} \tag{5.48}$$

Define the differential operator $L := (\gamma/|\gamma|^2)\cdot\nabla_\eta$. The term $\mathcal{B}_{21}$ can be estimated as follows,

$$\begin{aligned} |\mathcal{B}_{21}| &\le \int_{|\gamma|\le 1} |\gamma|^s\cdot\Big|\int e^{-i\eta\cdot\gamma}\big(1-\chi_0(|\eta|)\big)\,c(x,|\xi|(\eta+\hat{\xi}))\,|\xi|^{-m}\,\mathrm{d}\eta\Big|\,\mathrm{d}\gamma \\ &\simeq \int_{|\gamma|\le 1} |\gamma|^s\cdot\Big|\int L^n(e^{-i\eta\cdot\gamma})\,\big(1-\chi_0(|\eta|)\big)\,c(x,|\xi|(\eta+\hat{\xi}))\,|\xi|^{-m}\,\mathrm{d}\eta\Big|\,\mathrm{d}\gamma \\ &\lesssim \int_{|\gamma|\le 1} |\gamma|^s|\gamma|^{-n}\cdot\Big|\int e^{-i\eta\cdot\gamma}\,\nabla_\eta^n\Big(\big(1-\chi_0(|\eta|)\big)\,c(x,|\xi|(\eta+\hat{\xi}))\Big)\,|\xi|^{-m}\,\mathrm{d}\eta\Big|\,\mathrm{d}\gamma \\ &\le \int_{|\gamma|\le 1} |\gamma|^{s-n}\int\Big|\nabla_\eta^n\Big(\big(1-\chi_0(|\eta|)\big)\,c(x,|\xi|(\eta+\hat{\xi}))\Big)\Big|\cdot|\xi|^{-m}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\ &\lesssim \int_{|\gamma|\le 1} |\gamma|^{s-n}\int_{|\eta|\notin(\frac{1}{2},\frac{3}{2})} (1+|\xi|\cdot|\eta+\hat{\xi}|)^{m-n}\cdot|\xi|^{n-m}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\ &= \int_{|\gamma|\le 1} |\gamma|^{s-n}\int_{|\eta|\notin(\frac{1}{2},\frac{3}{2})} (|\xi|^{-1}+|\eta+\hat{\xi}|)^{m-n}\,\mathrm{d}\eta\,\mathrm{d}\gamma \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{|\gamma|\leq 1} |\gamma|^{s-n} \int_{|\eta|\notin(\frac{1}{2},\frac{3}{2})} |\eta+\hat{\xi}|^{m-n}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&\leq C < +\infty, \qquad (5.49)
\end{aligned}
$$

for some constant C independent of x, ξ. The last two inequalities in (5.49) take advantage of three facts: $s-n > -n$, $m-n < -n$, and the restriction $|\eta| \notin (1/2, 3/2)$ makes η avoid $-\hat{\xi}$.

To estimate $\mathcal{B}_{22}$, we proceed in a way similar to (5.49), but replacing L^n with L^{n+1},

$$
\begin{aligned}
|\mathcal{B}_{22}| &\lesssim \int_{|\gamma|>1} |\gamma|^{s-1-n} \int \Big|\nabla_\eta^{n+1}\Big(\big(1-\chi_0(|\eta|)\big)\, c(x, |\xi|(\eta+\hat{\xi}))\Big)\Big| \cdot |\xi|^{-m}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&\lesssim \int_{|\gamma|>1} |\gamma|^{s-1-n} \int_{|\eta|\notin(\frac{1}{2},\frac{3}{2})} (|\xi|^{-1}+|\eta+\hat{\xi}|)^{m-1-n}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&\leq \int_{|\gamma|>1} |\gamma|^{s-1-n} \int_{|\eta|\notin(\frac{1}{2},\frac{3}{2})} |\eta+\hat{\xi}|^{m-1-n}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&\leq C < +\infty, \qquad (5.50)
\end{aligned}
$$

for some constant C independent of x, ξ. Also, the last two inequality in (5.50) take advantage of three facts: $s-1-n < -n$, $m-1-n < -n$, and the restriction $|\eta| \notin (1/2, 3/2)$ makes η avoid $-\hat{\xi}$.

Finally, by (5.42), (5.47), (5.48), (5.49) and (5.50), we arrive at

$$
|\big((-\Delta_\xi)^{s/2}c\big)(x,\xi)| \leq C|\xi|^{m-s}, \quad \text{for all} \quad |\xi| \geq 1. \qquad (5.51)
$$

Step 2: For the case where $|\xi| < 1$.

In this step, the $|\xi|$ is set to be less than 1. We differentiate $\big((-\Delta_\xi)^{s/2}c\big)(x,\xi)$ *formally* w.r.t. ξ, and follow the arguments similar to (5.49)-(5.50),

$$
\begin{aligned}
|\partial_{\xi_j}\big((-\Delta_\xi)^{s/2}c\big)(x,\xi)| &\simeq |\partial_{\xi_j} \iint e^{i(\xi-\eta)\cdot\gamma}|\gamma|^s\, c(x,\eta)\,\mathrm{d}\eta\,\mathrm{d}\gamma| \\
&\lesssim |\iint_{|\gamma|\leq 1} L^{1+n}(e^{i(\xi-\eta)\cdot\gamma})|\gamma|^s\gamma_j\, c(x,\eta)\,\mathrm{d}\eta\,\mathrm{d}\gamma| \\
&\quad + |\iint_{|\gamma|>1} L^{2+n}(e^{i(\xi-\eta)\cdot\gamma})|\gamma|^s\gamma_j\, c(x,\eta)\,\mathrm{d}\eta\,\mathrm{d}\gamma|
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \int_{|\gamma|\le 1} |\gamma|^{s-n} \int \langle\eta\rangle^{m-1-n}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&\quad + \int_{|\gamma|>1} |\gamma|^{s-1-n} \int \langle\eta\rangle^{m-2-n}\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&\le C < +\infty,
\end{aligned} \tag{5.52}
$$

where the constant C is independent of x, ξ. Therefore, $\big((-\Delta_\xi)^{s/2}c\big)(x,\xi)$ is continuous w.r.t. ξ in $\mathbb{R}^n$. Moreover, the gradient w.r.t. x and ξ is bounded. Therefore, the $\big((-\Delta_\xi)^{s/2}c\big)(x,\xi)$ is uniformly bounded for all $x \in \mathbb{R}^n$ and all $|\xi| \le 1$. Combining this with (5.51), we arrive at the conclusion. □

By the interchangeability between the $(-\Delta_\xi)^{s/2}$ and ordinary partial differential operators, we can easily obtain the following corollary.

Corollary 5.3.1. *For any $m < 0$ and $s \in (0,1)$, we have*

$$
\big((-\Delta_\xi)^{s/2}c\big)(x,\xi) \in S^{m-s} \quad \textit{for any} \quad c(x,\xi) \in S^m.
$$

Proof of Corollary 5.3.1. Write $\tilde{c}(x,\xi) = (-\Delta_\xi)^{s/2}c(x,\xi)$. Then

$$
\begin{aligned}
\partial_x^\alpha \partial_\xi^\beta \tilde{c}(x,\xi) &\simeq \partial_x^\alpha \partial_\xi^\beta \iint e^{i(\xi-\eta)\cdot\gamma} |\gamma|^s\, c(x,\eta)\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&\simeq \partial_x^\alpha \partial_\xi^\beta \int e^{i\xi\cdot\gamma} |\gamma|^s\, \mathcal{F}_{\xi\to\gamma}\{c\}(x,\gamma)\,\mathrm{d}\gamma \\
&\simeq \partial_x^\alpha \int e^{i\xi\cdot\gamma} |\gamma|^s\, \gamma^\beta \mathcal{F}_{\xi\to\gamma}\{c\}(x,\gamma)\,\mathrm{d}\gamma \\
&\simeq \partial_x^\alpha \int e^{i\xi\cdot\gamma} |\gamma|^s\, \mathcal{F}_{\xi\to\gamma}\{\partial_\xi^\beta(c)\}(x,\gamma)\,\mathrm{d}\gamma \\
&\simeq \partial_x^\alpha \iint e^{i(\xi-\eta)\cdot\gamma} |\gamma|^s\, (\partial_\xi^\beta c)(x,\eta)\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&= \iint e^{i(\xi-\eta)\cdot\gamma} |\gamma|^s\, (\partial_x^\alpha \partial_\xi^\beta c)(x,\eta)\,\mathrm{d}\eta\,\mathrm{d}\gamma \\
&= \big((-\Delta_\xi)^{s/2}(\partial_x^\alpha \partial_\xi^\beta c)\big)(x,\xi).
\end{aligned}
$$

Applying Lemma 5.3.2, we obtain

$$
|\partial_x^\alpha \partial_\xi^\beta \tilde{c}(x,\xi)| \le C_{\alpha,\beta} \langle\xi\rangle^\beta.
$$

The proof is complete. □

Recall the definition of the normal vector $\boldsymbol{n}$ around (5.2). The asymptotics associated with the term F_1 is established in the following lemma.

Lemma 5.3.3. *We have*

$$\mathbb{E}(|F_1(\hat{x}, k, \cdot)|^2) \leq Ck^{-4}, \quad \forall\, k > 1, \tag{5.53}$$

for all $\hat{x}\colon \hat{x} \cdot \boldsymbol{n} \geq 0$*, and the constant* C *in* (5.53) *is independent of* $\hat{x}$*,* k*.*

In what follows, we may use $\mathcal{C}(\cdot)$ and its variants, such as $\vec{\mathcal{C}}(\cdot)$, $\mathcal{C}_{a,b}(\cdot)$, etc., to represent some generic smooth scalar/vector function(s), within $C_c^\infty(\mathbb{R}^3)$ or $C_c^\infty(\mathbb{R}^{3\times 4})$, whose particular definition may be change line by line.

Proof of Lemma 5.3.3. Utilizing (4.5) and (4.6), one can show

$$\begin{aligned}
&\mathbb{E}(|F_1(\hat{x}, k, \cdot)|^2) \\
= {} & \mathbb{E}\Big(\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot y} q(y, \cdot) \int_{\mathbb{R}^3} \frac{e^{ik|y-s|}}{4\pi|y-s|} f(s, \cdot)\,\mathrm{d}s\,\mathrm{d}y \\
& \quad \cdot \int_{\mathbb{R}^3} e^{ik\hat{x}\cdot z} \overline{q}(z, \cdot) \int_{\mathbb{R}^3} \frac{e^{-ik|z-t|}}{4\pi|z-t|} \overline{f}(t, \cdot)\,\mathrm{d}t\,\mathrm{d}z \Big) \\
\simeq {} & \int e^{ik\varphi(y,s,z,t)} \Big(\int e^{i(z-y)\cdot\xi} c_q(z, \xi)\,\mathrm{d}\xi \Big) \Big(\int e^{i(t-s)\cdot\eta} c_f(t, \xi)\,\mathrm{d}\eta \Big) \cdot \mathcal{C} \cdot\, \mathrm{d}(s, y, t, z),
\end{aligned} \tag{5.54}$$

where $\varphi(y, s, z, t) := -\hat{x}\cdot(y-z) - |y-s| + |z-t|$, $\mathcal{C} = \mathcal{C}(y, z, s, t) \in C_c^\infty(\mathbb{R}^{3\times 4})$, and the $\mathrm{d}(s, y, t, z)$ is short for $\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}t\,\mathrm{d}z$. We omit the repeated integral symbols and the integral domain in the calculation for simplicity. The term $\mathcal{C}$ in (5.54) is within $C_c^\infty(\mathbb{R}^{3\times 4})$ due to the fact that q and f are compactly supported and $\mathrm{dist}(D_q, D_f) > 0$.

Now we are to differentiate the term $e^{ik\varphi(y,s,z,t)}$ by two certain differential operators, in order to obtain the decay w.r.t. the argument k. To this end, we define two differential operators with C^∞-smooth coefficients,

$$L_1 := \frac{(y-s)\cdot\nabla_s}{ik|y-s|}, \quad L_2 = L_{2,\hat{x}} := \frac{\nabla_y\varphi \cdot \nabla_y}{ik|\nabla_y\varphi|},$$

where $\nabla_y\varphi = \frac{s-y}{|s-y|} - \hat{x}$. The operator $L_{2,\hat{x}}$ depends on $\hat{x}$ because ∇_y does. Due to the fact that $y \in D_q$ while $s \in D_f$, the operator L_1 is well-defined. Recall the definition of the normal vector $\boldsymbol{n}$. It can be verified that there is a positive lower bound of $|\nabla_y \varphi|$ for all $\hat{x} \in \mathbb{S}^2 \colon \hat{x} \cdot \boldsymbol{n} \geq 0$. It can be checked that

$$L_1(e^{ik\varphi(y,s,z,t)}) = L_2(e^{ik\varphi(y,s,z,t)}) = e^{ik\varphi(y,s,z,t)}.$$

By using integration by parts, one can compute

$$\begin{aligned}
&\mathbb{E}(|F_1(\hat{x},k,\cdot)|^2) \\
=& \int \big(L_1^2 L_2^2\big)(e^{ik\varphi(y,s,z,t)}) \cdot \Big(\int e^{i(z-y)\cdot\xi} c_q(z,\xi)\,\mathrm{d}\xi\Big) \\
&\cdot \Big(\int e^{i(t-s)\cdot\eta} c_f(t,\eta)\,\mathrm{d}\eta\Big) \cdot \mathcal{C}(y,z,s,t)\,\mathrm{d}(s,y,t,z) \\
\simeq& \, k^{-4} \int_{\mathcal{D}} e^{ik\varphi(y,s,z,t)} \big[\mathcal{J}_1\,(\mathcal{K}_1\,\mathcal{C} + \vec{\mathcal{K}}_2 \cdot \vec{\mathcal{C}} + \sum_{a,b=1,2,3} \mathcal{K}_{3;a,b}\,\mathcal{C}_{a,b}) \\
&+ \sum_{c=1,2,3} \mathcal{J}_{2;c}\,(\mathcal{K}_1\,\mathcal{C}_c + \vec{\mathcal{K}}_2 \cdot \vec{\mathcal{C}}_c + \sum_{a,b=1,2,3} \mathcal{K}_{3;a,b}\,\mathcal{C}_{a,b,c}) \\
&+ \sum_{a',b'=1,2,3} \mathcal{J}_{3;a',b'}(\mathcal{K}_1\,\mathcal{C}_{a',b'} + \vec{\mathcal{K}}_2 \cdot \vec{\mathcal{C}}_{a',b'} + \sum_{a,b=1,2,3} \mathcal{K}_{3;a,b}\,\mathcal{C}_{a,b,a',b'})\big]\,\mathrm{d}(s,y,t,z), \quad (5.55)
\end{aligned}$$

where the integral domain $\mathcal{D} \subset \mathbb{R}^{3\times 4}$ is bounded and

$$\begin{aligned}
\mathcal{J}_1 &:= \int e^{i(t-s)\cdot\eta}\, c_f(t,\eta)\,\mathrm{d}\eta, & \mathcal{K}_1 &:= \int e^{i(z-y)\cdot\xi}\, c_q(z,\xi)\,\mathrm{d}\xi, \\
\vec{\mathcal{J}}_2 &:= \nabla_s \int e^{i(t-s)\cdot\eta}\, c_f(t,\eta)\,\mathrm{d}\eta, & \vec{\mathcal{K}}_2 &:= \nabla_y \int e^{i(z-y)\cdot\xi}\, c_q(z,\xi)\,\mathrm{d}\xi, \\
\mathcal{J}_{3;a,b} &:= \partial^2_{s_a,s_b} \int e^{i(t-s)\cdot\eta}\, c_f(t,\eta)\,\mathrm{d}\eta, & \mathcal{K}_{3;a,b} &:= \partial^2_{y_a,y_b} \int e^{i(z-y)\cdot\xi}\, c_q(z,\xi)\,\mathrm{d}\xi,
\end{aligned}$$

and $\mathcal{J}_{2;c}$ (resp. $\mathcal{K}_{2;c}$) is the cth component of the vector $\vec{\mathcal{J}}_2$ (resp. $\vec{\mathcal{K}}_2$).

For the case where $s \neq t$, these three quantities, $\mathcal{J}_1$, $\vec{\mathcal{J}}_2$ and $\mathcal{J}_{3;a,b}$, can be estimated as follows,

$$\begin{aligned}
|\mathcal{J}_1| &= |\int e^{i(t-s)\cdot\eta}\, c_f(t,\eta)\,\mathrm{d}\eta| = |s-t|^{-2} \cdot |\int \Delta_\eta(e^{i(s-t)\cdot\eta})\, c_f(t,\eta)\,\mathrm{d}\eta| \\
&= |s-t|^{-2} \cdot |\int e^{i(t-s)\cdot\eta} (\Delta_\eta c_f)(t,\eta)\,\mathrm{d}\eta| \leq |s-t|^{-2} \int |(\Delta_\eta c_f)(t,\eta)|\,\mathrm{d}\eta \\
&\lesssim |s-t|^{-2} \int \langle\eta\rangle^{-m_f-2}\,\mathrm{d}\eta \lesssim |s-t|^{-2}, \qquad (5.56)
\end{aligned}$$

and

$$
\begin{aligned}
|\vec{\mathcal{J}}_{2;c}| &= |\partial_{s_c} \int e^{i(t-s)\cdot\eta}\, c_f(t,\eta)\,\mathrm{d}\eta| = |\int e^{i(t-s)\cdot\eta}\cdot c_f(t,\eta)\eta_c\,\mathrm{d}\eta| \\
&= |s-t|^{-2}\cdot|\int \Delta_\eta(e^{i(t-s)\cdot\eta})\, c_f(t,\eta)\eta_c\,\mathrm{d}\eta| = |s-t|^{-2}\cdot|\int e^{i(t-s)\cdot\eta}\Delta_\eta(c_f(t,\eta)\eta_c)\,\mathrm{d}\eta| \\
&\lesssim |s-t|^{-2}\int\langle\eta\rangle^{-m_f+1-2}\,\mathrm{d}\eta \lesssim |s-t|^{-2}, \qquad (5.57)
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\mathcal{J}_{3;a,b} &\simeq \int e^{i(t-s)\cdot\eta}\cdot c_f(t,\eta)\eta_a\eta_b\,\mathrm{d}\eta \simeq |s-t|^{-2}\int \Delta_\eta(e^{i(t-s)\cdot\eta})\cdot c_f(t,\eta)\eta_a\eta_b\,\mathrm{d}\eta \\
&= |s-t|^{-2}\int e^{i(t-s)\cdot\eta}\cdot\Delta_\eta(c_f(t,\eta)\eta_a\eta_b)\,\mathrm{d}\eta. \qquad (5.58)
\end{aligned}
$$

Now, if we further differentiate the term $e^{i(t-s)\cdot\eta}$ in (5.58) by $\frac{i(s-t)}{|s-t|^2}\nabla_\eta$ and then transfer the operator ∇_η onto $\Delta_\eta(c_f(t,\eta)\eta_a\eta_b)$ by using integration by parts, we would arrive at

$$
|\mathcal{J}_{3;a,b}| \lesssim |s-t|^{-3}\int|\nabla_\eta\Delta_\eta(c_f(t,\eta)\eta_a\eta_b)|\,\mathrm{d}\eta \le |s-t|^{-3}\int\langle\eta\rangle^{-m_f-1}\,\mathrm{d}\eta.
$$

The term $\int\langle\eta\rangle^{-m_f-1}\,\mathrm{d}\eta$ is absolutely integrable now, but the term $|s-t|^{-3}$ is not integrable at the hyperplane $s=t$ in $\mathbb{R}^3$. To circumvent this dilemma, the fractional Laplacian is appealed. By using Lemmas 5.3.1 and 5.3.2, we can continue (5.58) as

$$
\begin{aligned}
|\mathcal{J}_{3;a,b}| &\simeq |s-t|^{-2}\cdot\big||s-t|^{-s}\int(-\Delta_\eta)^{s/2}(e^{i(t-s)\cdot\eta})\cdot\Delta_\eta(c_f(t,\eta)\eta_j\eta_\ell)\,\mathrm{d}\eta\big| \\
&= |s-t|^{-2-s}\cdot|\int e^{i(t-s)\cdot\eta}\cdot(-\Delta_\eta)^{s/2}\big(\Delta_\eta(c_f(t,\eta)\eta_j\eta_\ell)\big)\,\mathrm{d}\eta| \\
&\lesssim |s-t|^{-2-s}\int\langle\eta\rangle^{-m_f+2-2-s}\,\mathrm{d}\eta = |s-t|^{-2-s}\int\langle\eta\rangle^{-m_f-s}\,\mathrm{d}\eta, \qquad (5.59)
\end{aligned}
$$

where the number s is chosen to satisfy $0<3-m_f<s<1$. Therefore, we have

$$
\begin{cases}
-m_f - s < -3, & (5.60a)\\
-2 - s > -3. & (5.60b)
\end{cases}
$$

Thanks to the condition (5.60a), we can continue (5.59) as

$$|\mathcal{J}_{3;a,b}| \lesssim |s-t|^{-2-s} \int \langle \eta \rangle^{-m_f - s} \,\mathrm{d}\eta \lesssim |s-t|^{-2-s}. \tag{5.61}$$

Using similar arguments, we can also conclude

$$\begin{cases} |\mathcal{K}_1|,\ |\vec{\mathcal{K}}_2| \lesssim |y-z|^{-2}, \\ \quad |\mathcal{K}_{3;a,b}| \lesssim |y-z|^{-2-s}. \end{cases} \tag{5.62}$$

Now, combining (5.55), (5.56), (5.57), (5.61) and (5.62), we arrive at

$$\begin{aligned} & \mathbb{E}(|F_1(\hat{x},k,\cdot)|^2) \\ \lesssim\ & k^{-4} \int_{\mathcal{D}} (|\mathcal{J}_1| + |\vec{\mathcal{J}}_2| + \sum_{a',b'=1,2,3} |\mathcal{J}_{3;a',b'}|) \cdot (|\mathcal{K}_1| + |\vec{\mathcal{K}}_2| + \sum_{a,b=1,2,3} |\mathcal{K}_{3;a,b}|) \,\mathrm{d}(s,y,t,z) \\ \lesssim\ & k^{-4} \int_{\mathcal{D}} |s-t|^{-2-s} \cdot |y-z|^{-2-s} \,\mathrm{d}(s,y,t,z) \\ \lesssim\ & k^{-4} \int_{\widetilde{\mathcal{D}}} |s-t|^{-2-s} \,\mathrm{d}s \,\mathrm{d}t \cdot \int_{\widetilde{\mathcal{D}}} |y-z|^{-2-s} \,\mathrm{d}y \,\mathrm{d}z \end{aligned} \tag{5.63}$$

for some large enough bounded domain $\widetilde{\mathcal{D}} \subset \mathbb{R}^{3\times 2}$ satisfying $\mathcal{D} \subset \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}}$. Note that the integral (5.63) should be understood as singular integral because of the presence of the singularities happening when $s = t$ and $y = z$. By (5.63) and (5.60b), we can finally conclude (5.53). The proof is complete. □

5.3.2 Asymptotics of F_2

The following lemma is necessary for the estimates of $F_2(\hat{x},k,\omega)$.

Lemma 5.3.4. *Assume that $\epsilon > 0$. For $\forall s \in \mathbb{R}$, $\forall k \in \mathbb{R}$ and $\forall \hat{x} \in \mathbb{S}^{n-1}$, we have*

$$\|e^{-ik\hat{x}\cdot(\cdot)}\varphi\|_{H^s_{-1/2-\epsilon}} \leq C_{s,\varphi} \langle k \rangle^s, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n),$$

where the constant $C_{s,\varphi}$ depends on s and φ, but is independent of $\hat{x}$, k.

Proof. By the Plancherel theorem and Peetre's inequality, one can have

$$\begin{aligned}
\|e^{-ik\hat{x}\cdot(\cdot)}\varphi\|^2_{H^s_{-1/2-\epsilon}} &= \int \langle x\rangle^{-1-2\epsilon}|(I-\Delta)^{s/2}\big(e^{-ik\hat{x}\cdot(\cdot)}\varphi\big)(x)|^2\,\mathrm{d}x \\
&\leq \int |(I-\Delta)^{s/2}\big(e^{-ik\hat{x}\cdot(\cdot)}\varphi\big)(x)|^2\,\mathrm{d}x \\
&\simeq \int \langle\xi\rangle^{2s}|\mathcal{F}\{e^{-ik\hat{x}\cdot(\cdot)}\varphi\}(\xi)|^2\,\mathrm{d}\xi = \int \langle\xi\rangle^{2s}|\widehat{\varphi}(\xi+k\hat{x})|^2\,\mathrm{d}\xi \\
&= \int \langle\xi-k\hat{x}\rangle^{2s}|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi \leq \langle k\rangle^{2s}\int \langle\xi\rangle^{2|s|}|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi.
\end{aligned}$$

The $\widehat{\varphi}$ is rapidly decaying because $\varphi \in C_c^\infty(\mathbb{R}^n)$. Thus, the integral $\int\langle\xi\rangle^{2|s|}|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi$ is a finite number depending on s, φ. The proof is done. □

Lemma 5.3.5. *For every $s \in (\frac{3-m_q}{2}, \frac{1}{2})$, there exists a subset $\Omega_s \subset \Omega$: $\mathbb{P}(\Omega_s) = 0$ such that for $\forall\,\omega \in \Omega\backslash\Omega_s$, the inequality*

$$|F_2(\hat{x},k,\omega)| \leq C_s(\omega)k^{5s-2} \tag{5.64}$$

holds uniformly for $\forall\hat{x} \in \mathbb{S}^2$ and for $\forall k > 1$, where the $C_s(\omega)$ is finite almost surely.

Proof. We define χ_q (resp. χ_f) as a function in $C_c^\infty(\mathbb{R}^3)$ with $\chi_q(x) \equiv 1$ (resp. $\chi_f(x) \equiv 1$) for $\forall x \in \operatorname{supp} q$ (resp. $\forall x \in \operatorname{supp} f$). From (5.39), Theorems 5.2.1, 5.2.2 and Lemma 5.3.4, one can compute

$$\begin{aligned}
|F_2(\hat{x},k,\omega)| &\leq \sum_{j\geq 2}\Big|\int_{\mathbb{R}^3} e^{-ik\hat{x}\cdot z}\chi_q(z)\big[(q\mathcal{R}_k)^j f\big](z)\,\mathrm{d}z\Big| \\
&\leq \|e^{-ik\hat{x}\cdot(\cdot)}\chi_q\|_{H^s_{-1/2-\epsilon}}\sum_{j\geq 2}\|(q\mathcal{R}_k)^j(f\cdot\chi_q)\|_{H^{-s}_{1/2+\epsilon}} \\
&\leq C_s\cdot\langle k\rangle^s\cdot C_{\epsilon,s}(\omega)\sum_{j\geq 2}k^{-j(1-2s)}\|f\cdot\chi_q\|_{H^{-s}_{1/2+\epsilon}} \\
&\leq C_{\epsilon,s}(\omega)\cdot\langle k\rangle^s\cdot k^{-2(1-2s)}\|f\cdot\chi_q\|_{H^{-s}_{1/2+\epsilon}} \\
&\leq C_{\epsilon,s}(\omega)k^{5s-2}\|\chi_q\|_{H^s_{-1/2-\epsilon}},
\end{aligned} \tag{5.65}$$

with a random variable $C_{\epsilon,s}(\omega)$ which is finite almost surely. The last inequality in (5.65) utilizes the fact that $f(\cdot,\omega)$ is microlocally isotropic of order m_f so that Theorem 5.2.2 holds for $f(\cdot,\omega)$. Let $\epsilon = 1/2$ in (5.65), we arrive at (5.64). □

5.4 The recovery of the source

In this section, we focus on the recovery of $\mu_f(x)$ associated with the random source term. In the recovering procedure, only the passive scattering measurement is used. Thus, the α in (5.1) is set to be 0, and the random sample ω is fixed. The data set $\mathcal{M}_f(\omega)$ is used to achieve the unique recovery.

Lemma 5.4.1 is useful in the recovery procedure, and it is originated from [11].

Lemma 5.4.1. *For any stochastic process* $\{g(k,\omega)\}_{k\in\mathbb{R}_+}$ *satisfying*

$$\int_1^{+\infty} k^{m-1}\mathbb{E}(|g(k,\cdot)|)\,\mathrm{d}k < +\infty,$$

there holds

$$\lim_{K\to+\infty}\frac{1}{K}\int_K^{2K} k^m g(k,\omega)\,\mathrm{d}k = 0, \quad a.s.\ \omega\in\Omega.$$

Proof of Lemma 5.4.1. By $\int_1^{+\infty} k^{m-1}\mathbb{E}(|g(k,\cdot)|)\,\mathrm{d}k < +\infty$ and Fubini's Theorem, we know

$$\int_1^{+\infty} k^{m-1}|g(k,\omega)|\,\mathrm{d}k < +\infty, \qquad \text{a.s. } \omega\in\Omega. \tag{5.66}$$

Define $g_K(k,w) := \frac{\chi_{(K,2K)}(k)}{2K}k^m g(k,\omega)$, where $\chi_{(K,2K)}(k)$ is the characteristic function of the interval $(K,2K)$. For almost surely every fixed ω, we have

$$g_K(\cdot,\omega)\to 0 \quad \text{as} \quad K\to+\infty,$$

Moreover, the function series $\{g_K(k,\omega)\}_K$ is dominated, in the argument k, by the function $k^{m-1}g(k,\omega)$. Thus, from (5.66) and the dominated convergence theorem, we can conclude

$$\lim_{K\to+\infty}\int_1^{+\infty} g_K(k,\omega)\,\mathrm{d}k = 0 \ \text{ a.s. } \omega\in\Omega.$$

The proof is complete. □

Now we are ready to recovery $\mu_f(x)$.

Proof of Theorem 5.1.1. This proof depends on Lemma 5.3.3, which requires $\hat{x}\cdot\boldsymbol{n}\geq 0$. Hence, we assume that $\hat{x}\cdot\boldsymbol{n}\geq 0$ unless otherwise stated.

Recall the definition of F_p ($p = 0, 1, 2$) in (5.39). As already mentioned at the beginning of Section 5.3, we study the data of the following form

$$\begin{aligned} &\frac{1}{K}\int_K^{2K} k^{m_f} 16\pi^2 \overline{u^\infty(\hat{x}, k, \omega)} u^\infty(\hat{x}, k+\tau, \omega)\,\mathrm{d}k \\ = &\sum_{p,q=0}^{2} \frac{1}{K}\int_K^{2K} k^{m_f} \overline{F_p(\hat{x}, k, \omega)} F_q(\hat{x}, k+\tau, \omega)\,\mathrm{d}k \\ =: &\sum_{p,q=0}^{2} I_{p,q}(\hat{x}, K, \tau, \omega). \end{aligned} \tag{5.67}$$

According to Corollary 4.4 in [11], for $\forall \tau \geq 0$ and for $\forall \hat{x} \in \mathbb{S}^2$, there exists $\Omega_{\tau,\hat{x}}^{0,0} \subset \Omega$, with $\mathbb{P}(\Omega_{\tau,\hat{x}}^{0,0}) = 0$, such that

$$\forall \omega \in \Omega\backslash\Omega_{\tau,\hat{x}}^{0,0}, \quad \lim_{K\to+\infty} I_{0,0}(\hat{x}, K, \tau, \omega) = (2\pi)^{3/2}\widehat{\mu}_f(\tau\hat{x}), \tag{5.68}$$

which also implies that

$$\forall \omega \in \Omega\backslash\Omega_{\tau,\hat{x}}^{0,0}, \quad \lim_{K\to+\infty} \frac{1}{K}\int_K^{2K} k^{m_f} |F_0(\hat{x}, k, \omega)|^2\,\mathrm{d}k = (2\pi)^{3/2}\widehat{\mu}_f(0). \tag{5.69}$$

Readers should note that the definition of the far-field patterns are slightly different in the sign of k between [11] and this chapter, see (5.38)-(5.39) in this chapter and (50) in [11]. This explains why the conjugation operations between (5.67) in this chapter and (65) in [11] are placed on different terms.

Now we are to estimate higher order terms. The Cauchy-Schwarz inequality gives

$$|I_{p,q}| \leq \big(\frac{1}{K}\int_K^{2K} k^{m_f} |F_p(\hat{x}, k, \omega)|^2\,\mathrm{d}k\big)^{\frac{1}{2}} \cdot \big(\frac{1}{K}\int_K^{2K} k^{m_f} |F_q(\hat{x}, k+\tau, \omega)|^2\,\mathrm{d}k\big)^{\frac{1}{2}}. \tag{5.70}$$

Recall that $m_f < 3$. From Lemma 5.3.3 we have

$$\int_1^{+\infty} k^{m_f-1}\mathbb{E}(|F_1(\hat{x}, k, \cdot)|^2)\,\mathrm{d}k \lesssim \int_1^{+\infty} k^{m_f-1}k^{-4}\,\mathrm{d}k \leq \int_1^{+\infty} k^{-2}\,\mathrm{d}k = 1. \tag{5.71}$$

By (5.71) and Lemma 5.4.1, we conclude that

$$\lim_{K\to+\infty} \frac{1}{K}\int_K^{2K} k^{m_f} |F_1(\hat{x}, k, \omega)|^2\,\mathrm{d}k = 0 \quad \text{a.s. } \omega \in \Omega. \tag{5.72}$$

For every $s \in ((3-m_q)/2, 1/2)$, Lemma 5.3.5 gives

$$\frac{1}{K}\int_K^{2K} k^{m_f}|F_2(\hat{x},k,\omega)|^2\,\mathrm{d}k \le \frac{C_s(\omega)}{K}\int_K^{2K} k^{m_f}k^{2(5s-2)}\,\mathrm{d}k \le \frac{C_s(\omega)}{K^{4-m_f-10s}}. \tag{5.73}$$

Recall that $11/4 < m_q < m_f < 5m_q - 11$. We know $(3-m_q)/2 < (4-m_f)/10$.

Choosing any $s \in \big((3-m_q)/2, (4-m_f)/10\big)$, we have $4-m_f-10s > 0$. Combining this with (5.73), we conclude that

$$\lim_{K\to+\infty}\frac{1}{K}\int_K^{2K} k^{m_f}|F_2(\hat{x},k,\omega)|^2\,\mathrm{d}k = 0 \quad \text{a.s. } \omega\in\Omega. \tag{5.74}$$

Formula (5.74) easily implies that

$$\lim_{K\to+\infty}\frac{1}{K}\int_K^{2K} k^{m_f}|F_2(\hat{x},k+\tau,\omega)|^2\,\mathrm{d}k = 0 \quad \text{a.s. } \omega\in\Omega, \tag{5.75}$$

for every fixed $\tau\in\mathbb{R}$.

Write $\mathcal{A} := \{(p,q)\,;\,0\le p,q\le 2\}\backslash\{(0,0)\}$. By (5.70), (5.69), (5.72) and (5.75) we have that, for $\forall\tau\ge 0$ and for $\forall\hat{x}\in\mathbb{S}^2$ there exists $\Omega_{\tau,\hat{x}}^{p,q}\subset\Omega : \mathbb{P}(\Omega_{\tau,\hat{x}}^{p,q}) = 0$, $\Omega_{\tau,\hat{x}}^{p,q}$ depending on τ and $\hat{x}$, such that

$$\forall(p,q)\in\mathcal{A},\quad \forall\omega\in\Omega\backslash\Omega_{\tau,\hat{x}}^{p,q},\quad \lim_{K\to+\infty} I_{p,q}(\hat{x},K,\tau,\omega) = 0. \tag{5.76}$$

Write $\Omega_{\tau\hat{x}} := \cup_{(p,q)\in\mathcal{A}\cup\{(0,0)\}}\Omega_{\tau,\hat{x}}^{p,q}$, thus $\mathbb{P}(\Omega_{\tau\hat{x}}) = 0$. Then (5.76) gives

$$\forall\omega\in\Omega\backslash\Omega_{\tau\hat{x}},\quad \forall(p,q)\in\mathcal{A},\quad \lim_{K\to+\infty} I_{p,q}(\hat{x},K,\tau,\omega) = 0. \tag{5.77}$$

Combining (5.67), (5.68) and (5.77), we arrive at the following statement:

$$\begin{aligned} &\forall\, y\in\mathbb{R}^3,\ \exists\,\Omega_y\subset\Omega\colon \mathbb{P}(\Omega_y) = 0,\ \text{s.t. } \forall\omega\in\Omega\backslash\Omega_y,\ \text{we have} \\ &\lim_{K\to+\infty}\frac{1}{K}\int_K^{2K} k^{m_f}16\pi^2\overline{u^\infty(\hat{x},k,\omega)}u^\infty(\hat{x},k+\tau,\omega)\,\mathrm{d}k = (2\pi)^{3/2}\widehat{\mu}_f(\tau\hat{x}). \end{aligned} \tag{5.78}$$

To prove Theorem 5.1.1, the logical order between y and ω should be exchanged. We achieve this by mimicking **Step 2** in the proof of Lemma 4.2 in [35]. Denote the usual Lebesgue measure on $\mathbb{R}^3$ as $\mathbb{L}$ and the product measure $\mathbb{L}\times\mathbb{P}$ as μ, and construct the product measure space $\mathbb{M} := (\mathbb{R}^3 \times \Omega, \mathcal{G}, \mu)$ in the canonical way, where $\mathcal{G}$ is the corresponding complete σ-algebra. Define

$$Z(y, \omega) := \lim_{K \to +\infty} \frac{1}{K} \int_K^{2K} k^{m_f} 16\pi^2 \overline{u^\infty(\hat{y}, k, \omega)} u^\infty(\hat{y}, k + |y|, \omega) \, \mathrm{d}k - (2\pi)^{3/2} \widehat{\mu}_f(y).$$

Write $\mathcal{A} := \{(y, \omega) \in \mathbb{R}^3 \times \Omega \,;\, Z(y, \omega) \neq 0\}$. Then $\mathcal{A}$ is a subset of $\mathbb{M}$. Set $\chi_{\mathcal{A}}$ as the characteristic function of $\mathcal{A}$ in $\mathbb{M}$. By (5.78) we obtain

$$\int_{R^3} \big(\int_\Omega \chi_{\mathcal{A}}(y, \omega) \, \mathrm{d}\mathbb{P}(\omega) \big) \, \mathrm{d}\mathbb{L}(y) = 0. \tag{5.79}$$

By (5.79) and Corollary 7 in Section 20.1 in [45], we obtain

$$\int_{\mathbb{M}} \chi_{\mathcal{A}}(y, \omega) \, \mathrm{d}\mu = \int_\Omega \big(\int_{R^3} \chi_{\mathcal{A}}(y, \omega) \, \mathrm{d}\mathbb{L}(y) \big) \, \mathrm{d}\mathbb{P}(\omega) = 0. \tag{5.80}$$

Because $\chi_{\mathcal{A}}(y, \omega)$ is nonnegative, (5.80) implies

$$\exists\, \Omega_0 \colon \mathbb{P}(\Omega_0) = 0, \text{ s.t. } \forall\, \omega \in \Omega \backslash \Omega_0, \int_{R^3} \chi_{\mathcal{A}}(y, \omega) \, \mathrm{d}\mathbb{L}(y) = 0. \tag{5.81}$$

Formula (5.81) further implies for every $\omega \in \Omega \backslash \Omega_0$,

$$\exists\, S_\omega \subset \mathbb{R}^3 \colon \mathbb{L}(S_\omega) = 0, \text{ s.t. } \forall\, y \in \mathbb{R}^3 \backslash S_\omega, \; Z(y, \omega) = 0. \tag{5.82}$$

Now Theorem 5.1.1 is proved by (5.82) for the case where $\hat{x} \cdot \boldsymbol{n} \geq 0$.

Note that μ_f is real-valued, so $\widehat{\mu}_f(\tau\hat{x}) = \overline{\widehat{\mu}_f(-\tau\hat{x})}$ when $\hat{x} \cdot \boldsymbol{n} < 0$. The proof is complete. □

5.5 The recovery of the potential

This section is devoted to the recovery of $\mu_q(x)$ associated with the the random potential. The data set $\mathcal{M}_q(\omega)$ is utilized to achieve the recovery. Throughout this section, the α in (5.1) is set to be 1.

Proof of Theorem 5.1.2. Similar to Proof of Theorem 5.1.1, the case where $\hat{x} \cdot \boldsymbol{n} < 0$ can be proved by utilizing the fact that μ_q is real-valued. In what follows, we assume that $\hat{x} \cdot \boldsymbol{n} \geq 0$ unless otherwise stated.

From (5.38) we have

$$
\begin{aligned}
&\frac{1}{K} \int_K^{2K} k^{m_q} 16\pi^2 \overline{u^\infty(\hat{x}, k, -\hat{x}, \omega)} u^\infty(\hat{x}, k+\tau, -\hat{x}, \omega) \,\mathrm{d}k \\
= &\sum_{p,q=0}^{2} \frac{1}{K} \int_K^{2K} k^{m_q} \sum_{p=0}^{2} [\overline{F_p(\hat{x}, k, \omega)} + \overline{G_p(\hat{x}, k, \omega)}] \\
&\cdot \sum_{q=0}^{2} [F_q(\hat{x}, k+\tau, \omega) + G_q(\hat{x}, k+\tau, \omega)] \,\mathrm{d}k \\
=: &\sum_{p,q=0,1,2} \big[I'_{p,q}(\hat{x}, K, \tau, \omega) + J_{p,q}(\hat{x}, K, \tau, \omega) \\
&+ L^1_{p,q}(\hat{x}, K, \tau, \omega) + L^2_{p,q}(\hat{x}, K, \tau, \omega) \big].
\end{aligned} \tag{5.83}
$$

where

$$
\begin{cases}
I'_{p,q}(\hat{x}, K, \tau, \omega) := \dfrac{1}{K} \displaystyle\int_K^{2K} k^{m_q} \overline{F_p(\hat{x}, k, \omega)} F_q(\hat{x}, k+\tau, \omega) \,\mathrm{d}k \\
J_{p,q}(\hat{x}, K, \tau, \omega) := \dfrac{1}{K} \displaystyle\int_K^{2K} k^{m_q} \overline{G_p(\hat{x}, k, \omega)} G_q(\hat{x}, k+\tau, \omega) \,\mathrm{d}k \\
L^1_{p,q}(\hat{x}, K, \tau, \omega) := \dfrac{1}{K} \displaystyle\int_K^{2K} k^{m_q} \overline{F_p(\hat{x}, k, \omega)} G_q(\hat{x}, k+\tau, \omega) \,\mathrm{d}k \\
L^2_{p,q}(\hat{x}, K, \tau, \omega) := \dfrac{1}{K} \displaystyle\int_K^{2K} k^{m_q} \overline{G_p(\hat{x}, k, \omega)} F_q(\hat{x}, k+\tau, \omega) \,\mathrm{d}k.
\end{cases} \tag{5.84}
$$

Note that the $I'_{p,q}$ is different from the $I_{p,q}$, defined in (5.67), in that the power of k in the definition of $I'_{p,q}$ is m_q while that of $I_{p,q}$ is m_f.

It is shown in [11] that there exists $\Omega_J \subset \Omega \colon \mathbb{P}(\Omega_J) = 0$ such that

$$
\forall \omega \in \Omega \backslash \Omega_J, \quad \lim_{K \to +\infty} J_{0,0}(\hat{x}, K, \tau, \omega) = (2\pi)^{3/2} \widehat{\mu}_q(2\tau\hat{x}), \tag{5.85}
$$

$$
\forall \omega \in \Omega \backslash \Omega_J, \quad \lim_{K \to +\infty} J_{p,q}(\hat{x}, K, \tau, \omega) = 0, \quad (p,q) \in \mathcal{A}. \tag{5.86}
$$

We conclude that there exists $\Omega_{I'} \subset \Omega \colon \mathbb{P}(\Omega_{I'}) = 0$ such that

$$
\forall \omega \in \Omega \backslash \Omega_{I'}, \quad \lim_{K \to +\infty} \sum_{p,q=0}^{2} I'_{p,q}(\hat{x}, K, \tau, \omega) = 0. \tag{5.87}
$$

The reason for (5.87) to hold is that

$$\big|\sum_{p,q=0}^{2} I'_{p,q}(\hat{x}, K, \tau, \omega)\big| \leq \frac{1}{K^{m_f - m_q}} \sum_{p,q=0}^{2} \Big[\Big(\frac{1}{K}\int_{K}^{2K} k^{m_f}|F_p(\hat{x}, k, \omega)|^2 \,\mathrm{d}k\Big)^{\frac{1}{2}} \cdot \Big(\frac{1}{K}\int_{K}^{2K} k^{m_f}|F_p(\hat{x}, k+\tau, \omega)|^2 \,\mathrm{d}k\Big)^{\frac{1}{2}}\Big]. \tag{5.88}$$

Mimicking **Step 2** in the proof of Lemma 4.2 in [35], from (5.69), (5.72) and (5.74)-(5.75), we can conclude that there exists $\Omega_0 \colon \mathbb{P}(\Omega_0) = 0$ such that for every $\omega \in \Omega\backslash\Omega_0$, there exists $S_\omega \subset \mathbb{R}^3 \colon \mathbb{L}(S_\omega) = 0$ such that when $\forall y \in \mathbb{R}^3\backslash S_\omega$, we have

$$\begin{cases} \lim\limits_{K\to+\infty} \dfrac{1}{K}\displaystyle\int_{K}^{2K} k^{m_f}|F_0(\hat{y}, k, \omega)|^2 \,\mathrm{d}k = (2\pi)^{3/2}\widehat{\mu}_f(0), & \text{(5.89a)} \\ \lim\limits_{K\to+\infty} \dfrac{1}{K}\displaystyle\int_{K}^{2K} k^{m_f}|F_j(\hat{y}, k, \omega)|^2 \,\mathrm{d}k = 0, \quad (j = 1, 2), & \text{(5.89b)} \\ \lim\limits_{K\to+\infty} \dfrac{1}{K}\displaystyle\int_{K}^{2K} k^{m_f}|F_2(\hat{y}, k+|y|, \omega)|^2 \,\mathrm{d}k = 0. & \text{(5.89c)} \end{cases}$$

Combining (5.88)-(5.89), we arrive at (5.87).

Now we analyze $\sum_{p,q=0}^{2} L^1_{p,q}(\hat{x}, K, \tau, \omega)$,

$$\big|\sum_{p,q=0}^{2} L^1_{p,q}(\hat{x}, K, \tau, \omega)\big| \leq \frac{1}{K^{m_f - m_q}} \sum_{p,q=0}^{2} \Big[\Big(\frac{1}{K}\int_{K}^{2K} k^{m_f}|F_p(\hat{x}, k, \omega)|^2 \,\mathrm{d}k\Big)^{\frac{1}{2}} \cdot \Big(\frac{1}{K}\int_{K}^{2K} k^{m_f}|G_p(\hat{x}, k+\tau, \omega)|^2 \,\mathrm{d}k\Big)^{\frac{1}{2}}\Big]. \tag{5.90}$$

By (5.85), (5.86), (5.89), (5.90) and similar arguments in **Step 2** in the proof of Lemma 4.2 in [35] again, we conclude that

$$\lim_{K\to+\infty} \big|\sum_{p,q=0}^{2} L^1_{p,q}(\hat{x}, K, \tau, \omega)\big| \lesssim \lim_{K\to+\infty} \frac{|\widehat{\mu}_f(0)|}{K^{m_f - m_q}} = 0, \quad \text{a.s.} \tag{5.91}$$

Similarly, we can have

$$\lim_{K\to+\infty} \big|\sum_{p,q=0}^{2} L^2_{p,q}(\hat{x}, K, \tau, \omega)\big| = 0, \quad \text{a.s.} \tag{5.92}$$

Combining (5.83), (5.85)-(5.87) and (5.91)-(5.92), we arrive at

$$\lim_{K\to+\infty}\frac{1}{K}\int_K^{2K} k^{m_q}16\pi^2\overline{u^\infty(\hat{x},k,-\hat{x},\omega)}u^\infty(\hat{x},k+\tau,-\hat{x},\omega)\,\mathrm{d}k=(2\pi)^{3/2}\widehat{\mu}_q(2\tau\hat{x}).$$

The proof is complete. □

Chapter 6

Discussion

6.1 Concluding remarks

In Chapter 3, we are concerned with a random Schrödinger system with Gaussian white noise source. First, the well-posedness of the direct problem is studied. Then, the variance function of the random source is recovered by using a single realization of the passive scattering measurements. By utilizing active scattering measurements under a single realization of the random sample, the potential is further recovered. Finally, with the help of multiple realizations of the random sample, the expectation of the random source is recovered. The major novelty of this chapter is that on the one hand, both the random source and the potential are unknown, and on the other hand, both passive and active measurements are used to recover all of the unknowns.

Chapter 4 considers the same system as in Chapter 3, except that the source term is replaced by a generalized Gaussian random field of microlocally isotropic type (see Definition 4.2.1). In this case, the well-posedness of the direct problem is also studied. Then, the rough strength of the random source is recovered by using a single realization of passive scattering measurements.

It is well-known that a practically nonzero scattered wave of a Schrödinger system in $\mathbb{R}^3$ should decay at the rate of r^{-1}, where the r is the distance away from the origin. In the first two topics, we have formulated our direct problem in $L^2_{-1/2-\epsilon}(\mathbb{R}^3)$ where the ϵ can be chosen to be any positive real number. Thus the decaying rate is guaranteed. However, the regularity of the solution are not taken into consideration.

A different formulation of the direct problem, which takes the regularity of the solution into consideration, is needed. Moreover, in Chapters 3 and 4, only the source is assumed to be random. We are particularly interested in the case that both the source and potential are random.

In Chapter 5, both the source and potential are assumed to be random and unknown, and we investigated the regularity issue of the solution of direct problem. In Chapter 5, the well-posedness of the direct problem is studied. Then, the rough strength of the random source is recovered by using a single realization of the passive scattering measurements. By further utilizing active scattering measurements under a single realization of the random sample, the potential is recovered by backscattering data. In order to prove the unique recovery, we studied the asymptotics of higher order terms by using pseudodifferential operators and microlocal analysis, both of which are very powerful tools in mathematical analysis.

6.2 Future works

In the previous three topics, the randomness is assumed to be independent of time. However, time-related random inverse problems are also very interesting and are worthy of studying. In our future work, we plan to study stochastic inverse problems.

Another interesting topic is to study the scenario where the problem setting is changed from Schrödinger system to Helmholtz system, Maxwell system and elastic system. In these settings, the Born expansion may not apply when the wave number is large. Therefore, new framework should be proposed to solve the corresponding inverse problems.

Bibliography

[1] R. Albanese and P. B. Monk, *The inverse source problem for Maxwell's equations*, Inverse Problems, **22** (2006), pp. 1023–1035.

[2] S. Alinhac and P. Gérard, *Pseudo-differential operators and the Nash-Moser theorem*, Grad. Stud. Math., Vol. 82, Amer. Math. Soc., Providence, 2007.

[3] G. Bao, C. Chen, and P. Li, *Inverse random source scattering problems in several dimensions*, SIAM/ASA J. Uncertain. Quantif., **4** (2016), pp. 1–25.

[4] G. Bao, J. Lin, and F. Triki, *A multi-frequency inverse source problem*, J. Differential Equations, **249** (2010), pp. 3443–3465.

[5] E. Blåsten, *Nonradiating sources and transmission eigenfunctions vanish at corners and edges*, *SIAM J. Math. Anal.*, **50** (2018), pp. 6255–6270.

[6] E. Blåsten and H. Liu, *Scattering by curvatures, radiationless sources, transmission eigenfunctions and inverse scattering problems*, arXiv:1808.01425, 2018.

[7] P. Blomgren, G. Papanicolaou, and H. Zhao, *Super-resolution in time-reversal acoustics*, The Journal of the Acoustical Society of America, **111** (2002), pp. 230–248.

[8] L. Borcea, G. Papanicolaou, and C. Tsogka, *Adaptive interferometric imaging in clutter and optimal illumination*, Inverse Problems, **22** (2006), pp. 1405.

[9] L. Borcea, G. Papanicolaou, C. Tsogka, and J. Berryman, *Imaging and time reversal in random media*, Inverse Problems, **18** (2002), pp. 1247.

[10] A. L. Bukhgeim, *Recovering a potential from Cauchy data in the two-dimensional case*, Journal of Inverse and Ill-posed Problems, **16** (2008), pp. 19–33.

[11] P. Caro, T. Helin, and M. Lassas, *Inverse scattering for a random potential*, Anal. Appl., **Jan.** (2019), in press, DOI: 10.1142/S0219530519500015.

[12] C. Clason and M. Klibanov, *The quasi-reversibility method for thermoacoustic tomography in a heterogeneous medium*, SIAM J. Sci. Comput., **30** (2007), pp. 1–23.

[13] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd Edition, Springer Science & Business Media, New York, 2013.

[14] A. J. Devaney, E. A. Marengo, and M. Li, *Inverse Source Problem in Nonhomogeneous Background Media*, SIAM J. Appl. Math., **67** (2007), pp. 1353–1378.

[15] R. M. Dudley, *Real Analysis and Probability*, 2nd Edition, Cambridge Stud. Adv. Math., Vol. 7, Cambridge Univ. Press, Cambridge, 2002.

[16] G. Eskin, *Lectures on Linear Partial Differential equations*, Grad. Stud. Math., Vol. 123, AMS, Providence, 2011.

[17] L. Grafakos and S. Oh, *The kato-ponce inequality*, Comm. Partial Differential Equations, **39** (2014), pp. 1128–1157.

[18] R. Griesmaier and J. Sylvester, *Uncertainty principles for three-dimensional inverse source problems*, SIAM J. Appl. Math., **77** (2017), pp. 2066–2092.

[19] D. J. Griffiths, *Introduction to Quantum Mechanics*, Cambridge Univ. Press, Cambridge, 2016.

[20] K. H. Hauer, L. Kühn, and R. Potthast, *On uniqueness and non-uniqueness for current reconstruction from magnetic fields*, *Inverse Problems*, **21** (2005), pp. 955–967.

[21] T. Helin, M. Lassas, L. Oksanen, and T. Saksala, *Correlation based passive imaging with a white noise source*, J. Math. Pures Appl., **116** (2018), pp. 1–29.

[22] L. Hörmander, *The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis*, Second Edition, Springer, Berlin, 1990.

[23] M. Ikehata, *Reconstruction of a source domain from the Cauchy data*, Inverse Problems, **15** (1999), pp. 637–645.

[24] V. Isakov, *Inverse Source Problems*, Mathematical Surveys and Monographs, 34. Amer. Math. Soc., Providence, 1990.

[25] V. Isakov and S. Lu, *Increasing stability in the inverse source problem with attenuation and many frequencies*, SIAM J. Appl. Math., **78** (2018), pp. 1–18.

[26] M. Klibanov, *Thermoacoustic tomography with an arbitrary elliptic operator*, Inverse Problems, **29** (2013), 025014.

[27] C. Knox and A. Moradifam, *Determining both the source of a wave and its speed in a medium from boundary measurements*, arXiv:1803.06750, 2018.

[28] S. Kusiak and J. Sylvester, *The scattering support*, Comm. Pure Appl. Math., **56** (2003), pp. 1525–1548.

[29] S. Kusiak and J. Sylvester, *The Convex Scattering Support in a Background Medium*, SIAM Journal on Mathematical Analysis, **36** (2005), pp. 1142–1158.

[30] S. Kusuoka, *The support property of a Gaussian white noise and its applications*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **29** (1982), pp. 387–400.

[31] M. Lassas, L. Päivärinta, and E. Saksman, *Inverse problem for a random potential*, Contemp. Math., **362**, Amer. Math. Soc., Providence, RI, 2004.

[32] M. Lassas, L. Päivärinta and E. Saksman, *Inverse Scattering Problem for a Two Dimensional Random Potential*, Comm. Math. Phys., **279** (2008), pp. 669–703.

[33] J. Li, T. Helin, P. Li, *Inverse random source problems for time-harmonic acoustic and elastic waves*, arXiv:1811.12478, 2018.

[34] J. Li, P. Li, *Inverse Elastic Scattering for a Random Source*, arXiv:1812.09646, 2018.

[35] J. Li, H. Liu and S. Ma, *Determining a random Schrödinger equation with unknown source and potential*, arXiv:1811.00880, 2018.

[36] J. Li, H. Liu, Z. Shang, and H. Sun, *Two Single-Shot Methods for Locating Multiple Electromagnetic Scatterers*, SIAM J. Appl. Math., **73** (2013), pp. 1721–1746.

[37] H. Liu and X. Liu, *Recovery of an embedded obstacle and its surrounding medium from formally determined scattering data*, Inverse Problems, **33** (2017), 065001.

[38] H. Liu and G. Uhlmann, *Determining both sound speed and internal source in thermo- and photo-acoustic tomography*, Inverse Problems, **31** (2015), 105005.

[39] Q. Lü and X. Zhang, *Global uniqueness for an inverse stochastic hyperbolic problem with three unknowns*, Comm. Pure Appl. Math., **68** (2015), pp. 948–963.

[40] E. Marengo and A. Devaney, *The inverse source problem of electromagnetics: linear inversion formulation and minimum energy solution*, IEEE Transactions on Antennas and Propagation, **47** (1999), pp. 410–412.

[41] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge Univ. Press, Cambridge, 2000.

[42] J. V. Michalowicz, J. M. Nichols, F. Bucholtz and C. C. Olson, *An Isserlis' theorem for mixed Gaussian variables: application to the auto-bispectral density*, J. Stat. Phys., **136** (2009), pp. 89–102.

[43] C. Pozrikidis, *The Fractional Laplacian*, Chapman & Hall/CRC, New York, 2016.

[44] S. M. Ross, *Introduction to Probability Models*, 9th Edition, Academic Press, New York, 2014.

[45] H. L. Royden and P. M. Fitzpatrick, *Real Analysis, Fourth Edition*, Prentice Hall, Upper Saddle River, 2010.

[46] Y. A. Rozanov, *Markov random fields*, Springer, New York, 1982.

[47] J. Sylvester and J. Kelly, *A scattering support for broadband sparse far field measurements*, Inverse Problems, **21** (2005), pp. 759–771.

[48] X. Wang, Y. Guo, D. Zhang and H. Liu, *Fourier method for recovering acoustic sources from multi-frequency far-field data*, Inverse Problems, **33** (2017), 035001.

[49] M. W. Wong, *An introduction to pseudo-differential operators*, 3rd Edition, World Scientific Pub. Co. Pte. Ltd, Hackensack, 2014.

[50] G. Yuan, *Determination of two kinds of sources simultaneously for a stochastic wave equation*, Inverse Problems, **31** (2015), 085003.

[51] D. Zhang and Y. Guo, *Fourier method for solving the multi-frequency inverse source problem for the Helmholtz equation*, Inverse Problems **31** (2015), 035007.

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