

## DOCTORAL THESIS

### Convergence analysis and applications of two optimization algorithms

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# Abstract

Nowadays, many optimization problems in real applications share a separable structure in the objective and it becomes more and more common to solve these problems by decomposition methods such as fast iterative shrinkage-thresholding algorithm (FISTA), generalized alternating direction method of multipliers (GADMM), and first-order primal-dual algorithm (PD), to name just a few.

In this thesis, we focus on two optimization algorithms for solving convex programs:  $\theta$ -scheme and a preconditioned primal-dual algorithm. For the  $\theta$ -scheme, we first present an elaborative convergence analysis in a Hilbert space and propose a general convergent inexact  $\theta$ -scheme. Second, for unconstrained problems, we prove the convergence of the  $\theta$ -scheme and show a sublinear convergence rate in terms of the objective function. Furthermore, a practical inexact  $\theta$ -scheme is derived to solve  $l_2$ -loss based problems and its convergence is proved. Third, for constrained problems, even though the convergence of the  $\theta$ -scheme is available in the literature, yet its sublinear convergence rate is unknown until we provide one via a variational reformulation of the solution set. Besides, in order to relax the condition imposed on the  $\theta$ -scheme, we propose a new variant and show its convergence. Finally, some preliminary numerical experiments demonstrate the efficiency of the  $\theta$ -scheme and our proposed methods. For the preconditioned primal-dual algorithm, noticing that a practical step size cannot lie in the theoretical region, we show that the range of dual step size can be enlarged by 1/3 at most and at the same time, the convergence and a sublinear convergence rate can be ensured. Therefore, this practical step size can indeed guarantee the convergence. Furthermore, if more regularity conditions are imposed on objective functions, we can obtain a linear convergence rate. Finally, some connection with other methods is revealed.

In future work, we focus on the acceleration of the  $\theta$ -scheme. Some preliminary numerical experiments demonstrate the potential efficiency of our proposed accelerated  $\theta$ -scheme. Therefore, it would be our priority of further study.

**Keywords:** Linear inverse problem, Optimization algorithm, Splitting method,  $\theta$ -scheme, Primal-dual algorithm, Convergence rate, Dual problem

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