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Stochastic control and approximation for Boltzmann equation

Zhou, Yulong

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In-attendance:	Dr. Zeng Tieyong Associate Professor, Department of Mathematics, HKBU

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Stochastic Control and Approximation for Boltzmann Equation

ZHOU Yulong

A thesis submitted in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Principal Supervisor: Dr. ZENG Tieyong

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Declaration

I hereby declare that this thesis represents my own work which has been done after registration for the degree of PhD at Hong Kong Baptist University, and has not been previously included in a thesis or dissertation submitted to this or any other institution for a degree, diploma or other qualifications.

I have read the University's current research ethics guidelines, and accept responsibility for the conduct of the procedures in accordance with the University's Committee on the Use of Human & Animal Subjects in Teaching and Research (HASC). I have attempted to identify all the risks related to this research that may arise in conducting this research, obtained the relevant ethical and/or safety approval (where applicable), and acknowledged my obligations and the rights of the participants.

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Abstract

In this thesis we study two problems concerning probability. The first is stochastic control problem, which essentially amounts to find an optimal probability in order to optimize some reward function of probability. The second is to approximate the solution of the Boltzmann equation. Thanks to conservation of mass, the solution can be regarded as a family of probability indexed by time.

In the first part, we prove a dynamic programming principle for stochastic optimal control problem with expectation constraint by measurable selection approach. Since state constraint, drawdown constraint, target constraint, quantile hedging and floor constraint can all be reformulated into expectation constraint, we apply our results to prove the corresponding dynamic programming principles for these five classes of stochastic control problems in a continuous but non-Markovian setting.

In order to solve the Boltzmann equation numerically, in the second part, we propose a new model equation to approximate the Boltzmann equation without angular cutoff. Here the approximate equation incorporates Boltzmann collision operator with angular cutoff and the Landau collision operator. As a first step, we prove the well-posedness theory for our approximate equation. Then in the next step, we show the error estimate between the solutions to the approximate equation and the original equation. Compared to the standard angular cutoff approximation method, our method results in higher order of accuracy.

Keywords: dynamic programming principle, expectation constraint, measurable selection, target problem, quantile hedging, drawdown constraint, floor constraint, homogeneous Boltzmann equation, full-range interactions, hard potentials, high order approximation.

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Chapter 1

Introduction

In a stochastic control problem, we aim to find an optimal control subject to some constraints such the some expected reward is maximized. Essentially, this amounts to find an optimal probability since only the distribution of the paths of the controlled process matters. The Boltzmann equation is a famous model to study the density of rarified gas in kinetic physics. Essentially, the Boltzmann equation is a stochastic partial differential equation. Thanks to conservation of mass, the solution of it can be regarded as a family of probability indexed by time. The numerical methods of solving the Boltzmann equation rely heavily on stochastic analysis. Therefore these two objects are closely related due to these known facts.

This chapter includes three sections. Section 1.1 gives a description of our motivation to derive the dynamic programming principle for stochastic control problem under expectation constraint. In Section 1.2, we first introduce the homogeneous Boltzmann equation with hard potential, and then present our motivation to find a new equation to approximate the Boltzmann equation more accurately, and finally state our main results. In the last section, we give an outline of this thesis.

1.1 Stochastic control

The problem of stochastic optimal control under state constraint has been studied extensively. It is a problem of maximizing an expected reward, under the constraint that the controlled state process has to stay in a given subset of the state space. In most cases, the value function of the problem is characterized by the (viscosity) solutions of the associated partial differential equations (PDEs). This was done via first order PDEs by Soner in [84] for deterministic control problem and in [85] for piecewise deterministic processes. Ishii and Koike [59] completed the boundary conditions for the value function by a new formulation. For stochastic control problems, the associated PDEs are of second order, as one can refer to [60, 62].

The drawdown constraint on wealth process was first considered by Grossman and Zhou [51]. Under drawdown constraint, the controlled process (wealth) must stay above a fraction of its current maximum. Elie and Touzi [35] considered the infinite horizon utility maximization of future consumption under the drawdown constraint. They guessed the candidate solution by passing from the dynamic programming equation to the partial differential equation (PDE).

Target problem is another type of control problem in which there are some constraints on the state variable. In a target problem, we require the state variable must stay in some subset of the state space at a pre-specified terminal time. As in [86], Soner and Touzi proved a geometric type of dynamic programming principle and then used it to identify the reachability set by viscosity solutions of the associated PDEs. There is a rich literature on the variants of target problems; see, e.g. [14, 15, 87].

Quantile hedging problems arise from finance, in which the state variable has only to stay in some given subset with a probability greater than some pre-specified level; see [36] for details.

EI Karoui, Jeanblanc and Lacoste [31] considered the problem of maximizing expected utility under a European or American guarantee. In a European guarantee, the terminal value of the controlled process must be greater than a pre-specified value. Essentially, a European guarantee is a special case of target problem since we may take the target set to be the infinite interval bounded below by the specified value. A guarantee of American type insures that the controlled process satisfies the requirement at any intermediate time during the period considered. More generally, one may consider floor constraints in which the controlled process must stay above a given function over the whole period.

As one will see in section 2.3 of the present thesis, the above mentioned various types of constraints on the controlled process can be reformulated into a same form of expectation constraint, which was called generalized state constraints in [16]. Therefore our focus is on stochastic control problem under expectation constraint.

The dynamic programming principle (DPP) is the bridge to connect the control problem and the associated PDE (HJB equation). For this reason, given a particular stochastic control problem, we are always interested in whether the dynamic programming principle holds or not. Weak dynamic programming principle is sufficient to relate value function to viscosity solutions of the associated PDEs, as illustrated by Bouchard and Touzi in [17]. For generalized state constraint control problem, Bouchard and Nutz in [16] provided a weak dynamic programming principle by the classical method of discretizing the state space and deriving stability under cancatenation. Our goal is to formulate and prove the DPP for a control problem under expectation constraint.

In a stochastic control problem, we aim to find an optimal control. Given a control, the controlled state process actually induces a new probability measure. So to find an optimal control, it is equivalent to seek an optimal probability measure among the probability measures induced by those admissible controls. We emphasize that the transformation from control to probability is not only a technical convenience. Essentially only the distribution of the paths of the controlled process matters since the reward function depends on the controlled path and one wants to maximize the expected reward. Thus by considering those admissible controls, we capture the key story. Recently, in [72] Nutz and van Handel constructed a sublinear expectation by taking supremum over a set of probability measures. The sublinear expectation satisfies the tower property also known as time-consistency. If we regard probabilities there as controls, then the sublinear expectation is the value function of a stochastic control problem. In this point of view, the tower property is essentially DPP. Motivated by this, we prove the DPP for a control problem under expectation constraint by deriving the tower property of a suitably defined sublinear expectation. The main tool used in the proof is measurable selection theorem as in [72]. Employing the ideas in [16], we deal with expectation constraint by auxiliary martingales. The DPP can be used to derive the path-dependent partial differential equation to characterize the value function. By solving the partial differential equation, we may get the value function.

Since we are working in the path space, the admissible probabilities and the value function now can be path-dependent. Then, our results can be used to prove DPP for path-dependent stochastic control problems. Of course, one limitation of our framework is that we are only able to consider continuous processes. However, we hope the result can be generalized into the space of càdlàg processes.

1.2 Boltzmann equation

1.2.1 The Boltzmann equation

Our interest is to consider the numerical method for the spatially homogeneous Boltzmann equation with long-range interaction in the case of hard potentials. By spatial homogeneity, we mean that the unknown function depends only on time and velocity, and is independent of the position variable. Under this circumstance, the Boltzmann equation becomes:

$$\partial_t f = Q(f, f), \tag{1.1}$$

where $f(t, v) \geq 0$ represents the density of collision particles moving with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$. The bilinear Boltzmann collision operator Q is defined as

$$Q(g, h)(v) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma)(g'_* h' - g_* h) d\sigma dv_*.$$

We remark that it only acts on the velocity variable v, reflecting the physical assumption that collisions are localized in space and time. As usual, we take the conventional shorthand $h = h(v)$, $g_* = g(v_*)$, $h' = h(v')$, $g'_* = g(v'_*)$, with v', v'_* being given through formulas

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \ , \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \ , \qquad \sigma \in \mathbb{S}^2 \, . \tag{1.2}$$

The nonnegative function $B(v-v_*,\sigma)$ in the collision operator is called the Boltzmann collision kernel, which depends only on $|v-v_*|$ and $\langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle$. Let us introduce the angle variable θ through $\cos\theta = \langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle$. Without loss of generality, the kernel $B(v-v_*,\sigma)$ is assumed to be supported in the set $0 \le \theta \le \frac{\pi}{2}$, i.e, $\langle \frac{v-v_*}{|v-v_*|}, \sigma \rangle \ge 0$. Note that this is not a restriction, for otherwise, we can replace B by

$$\bar{B}(v-v_*,\sigma) \stackrel{\text{def}}{=} [B(v-v_*,\sigma) + B(v-v_*,-\sigma)]\mathbf{1}_{\langle \frac{v-v_*}{|v-v_*|},\sigma\rangle \ge 0}.$$

As usual, $\mathbf{1}_E$ denotes the characteristic function of a set E.

1.2.2 Assumptions on the Boltzmann collision kernel

Throughout the thesis, the collision kernel satisfies:

- (A-1) The cross-section B has the form of

  $$B(v-v_*,\sigma) = \Phi(|v-v_*|)b(\cos\theta),$$

  where both Φ and b are nonnegative functions.

- (A-2) The angular function b satisfies

  $$K^{-1}\theta^{-1-2s} \le \sin\theta b(\cos\theta) \le K\theta^{-1-2s}, \quad \text{with } 0 < s < 1, \ K \ge 1.$$

- (A-3) The kinetic factor Φ has the form of

  $$\Phi(|v-v_*|) = |v-v_*|^\gamma.$$

- (A-4) The parameter γ verifies that $0 < \gamma \le 2$.

We remark that under assumption (A-2), we have $A_2 \stackrel{\text{def}}{=} \int_{\mathbb{S}^2} b(\cos\theta)\sin^2\theta d\sigma < \infty$.

The solutions of the Boltzmann equation (1.1) enjoy the physical properties: conservation of mass, momentum and kinetic energy. Mathematically, for any time $t \geq 0$, there holds

$$\int_{\mathbb{R}^3} f(t,v)\varphi(v)dv = \int_{\mathbb{R}^3} f(0,v)\varphi(v)dv, \quad \varphi(v) = 1, v, |v|^2.$$

Additionally, the famous Boltzmann's H theorem conveys decrease of entropy, which formally reads

$$-\frac{d}{dt}\int_{\mathbb{R}^3} f\log f dv = -\int_{\mathbb{R}^3} Q(f,f)\log f dv \geq 0.$$

1.2.3 Existent results, motivations and difficulties

The well-posedness theory of the spatially homogeneous Boltzmann equation with angular cutoff, namely when $\int_0^{\pi/2}\sin\theta b(\cos\theta)d\theta < \infty$, had been explored extensively by many researchers. In the case of hard potentials, Arkeryd [11] and Mischler-Wennberg [67] established the existence and uniqueness of the solutions in weighted L^1 space. Then, in [63] Lu-Mouhot extended the results to the space of non-negative measure with finite non-increasing kinetic energy. For the well-posedness of the spatially homogeneous Boltzmann equation without angular cutoff, we refer to [53] and the references therein. As for the regularity theory of the equation, we refer to [69] for the estimates of the gain part of the operator and the propagation of smoothness in the case of angular cutoff and refer to [8], [20], [58] and [83] in the case of long-range interaction.

For any $0 < \epsilon \leq \frac{\sqrt{2}}{2}$, let $b^\epsilon = b\mathbf{1}_{\sin\frac{\theta}{2}\geq\epsilon}$, and Q^ϵ be the operator associated to the angular cutoff kernel $B^\epsilon(v-v_*,\sigma) = |v-v_*|^\gamma b^\epsilon(\cos\theta)$. That is,

$$Q^\epsilon(g,h)(v) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B^\epsilon(v-v_*,\sigma)(g'_*h' - g_*h)d\sigma dv_*.$$

Then the angular cutoff Boltzmann equation

$$\partial_t f = Q^\epsilon(f,f) \tag{1.3}$$

is well-posed (see [55]). And moreover if f and f^ϵ are solutions to the Boltzmann equation (1.1) and the cutoff Boltzmann equation (1.3) with the same initial datum f_0 respectively, then one has

$$f = f^\epsilon + O(\epsilon^{2-2s}).$$

The cutoff Boltzmann operator Q^ϵ omits all grazing collisions and then results in an error of order $2-2s$. We emphasize that the cutoff Boltzmann equation (1.3) is not a good approximation to the Boltzmann equation (1.1) as the singularity parameter s approaches to 1.

The effect of grazing collisions has been studied extensively, and we refer to [23] and [52]. It is proved that the limit of concentrating grazing collisions leads to the Landau collision operator. Mathematically, if denote $b_\epsilon = b\mathbf{1}_{\sin\frac{\theta}{2}\le\epsilon}$, and let Q_ϵ be the operator associated to $B_\epsilon(v-v_*,\sigma) = |v-v_*|^\gamma b_\epsilon(\cos\theta)$, according to [52], we shall have

$$||\epsilon^{2-2s}Q_L(f,f) - Q_\epsilon(f,f)||_{L^1} \lesssim \epsilon^{3-2s}||f||^2_{H^5_{\gamma+12}}. \tag{1.4}$$

Let us recall the Landau collision operator Q_L,

$$Q_L(g,h)(v) \stackrel{\text{def}}{=} \nabla_v\cdot\{\int_{\mathbb{R}^3} a(v-v_*)[g(v_*)\nabla_v h(v) - \nabla_v g(v_*)h(v)]dv_*\},$$

where a is a symmetrical matrix defined by

$$a(v) = \Lambda|v|^{\gamma+2}(I - \frac{v\otimes v}{|v|^2}), \tag{1.5}$$

and Λ is a constant.

This motivates us to compensate the omission of grazing collisions by Landau operator. Specifically, we consider the operator

$$M^\epsilon(g,h) \stackrel{\text{def}}{=} Q^\epsilon(g,h) + \epsilon^{2-2s}Q_L(g,h), \tag{1.6}$$

and propose our approximate equation,

$$\partial_t f = M^\epsilon(f,f). \tag{1.7}$$

If $\tilde{f}^\epsilon$ is the solution to equation (1.7), we will prove

$$f = \tilde{f}^\epsilon + O(\epsilon^{3-2s}). \tag{1.8}$$

That is, by adding Landau operator to the cutoff Boltzmann equation, we increase the order of error from $2-2s$ to $3-2s$. The accuracy of approximation of the Boltzmann equation (1.1) by equation (1.7) remains even if the singularity parameter s goes to 1. Another motivation for studying equation (1.7) is the recent development of numerical methods. We believe that our approximate equation can be solved numerically. In this regard, see next subsection for a detailed discussion. We emphasize that the solutions of our approximate equation (1.7) also have the above mentioned properties, namely, conservation of mass, momentum, energy and entropy dissipation.

In the current thesis, we study the well-posedness of equation (1.7) and then give the error analysis of the approximate equation (1.7) and the original Boltzmann equation (1.1). There are two main difficulties in this task. One is to show the existence of a non-negative solution to equation (1.7). We proceed by constructing a sequence of convergent non-negative functions with its limit being the solution. Since we consider hard potentials ($\gamma > 0$), there will be an increase of weight at each iteration. Observing the coefficient before the weight increased term is strictly less 1, we prove that, on a whole level, the increased weight is limited. The other difficulty is related to the estimate of the error function F_R^ϵ as defined in (4.1). Again, weight increase problem happens here and another problem is no sign information of F_R^ϵ. We circumvent the problem of lacking sign information by writing the equation of error function in a suitable way. The weight increase problem is dealt with by carefully separating the integration region such that either the increased weight is eliminated or the coefficient before the weight increased term is controlled as desired.

1.2.4 Existing numerical results and hints on future work

Our approximate equation contains both the angular cut-off Boltzmann operator Q^ϵ and Landau operator Q_L. Numerical methods of the Boltzmann equation and Landau equation have been investigated extensively. The most famous one is Kac's program. Kac started from the Markov process corresponding to collisions only, and tried to prove the limit towards the spatially homogeneous Boltzmann equation. For Kac's program approximating Boltzmann equation, we refer to the recent work [65] and the references therein. In [65], the authors proved the propagation of chaos quantitatively in an abstract framework by proving stability and convergence estimates between linear semigroups. They then applied their results to prove the propagation of chaos of Kac's program in the cases of hard sphere model ($B(v - v_*, \cos\theta) = |v - v_*|$) and true Maxwell molecules ($B(v - v_*, \cos\theta) = b(\cos\theta)$).

As for particle system approximating the Landau equation, one may refer to [43] and the references therein. The authors in [43] proved quantitatively the propagation of chaos for a N-particle continuous drift diffusion process under the cases of Maxwell molecules ($\gamma = 0$) and hard potentials ($0 < \gamma \leq 1$).

As one can see from above, the Boltzmann equation corresponds to the limit of jump processes, while the Lanau equation corresponds to the limit of continuous processes. If we are to numerically solve our approximate equation (1.7), we need some jump-diffusion processes. Actually, the method in [65] is general and robust to deal with mixture of jump and diffusion processes. As shown to be successful in [66], the authors considered the Boltzmann equation for diffusively excited granular media, used jump-diffusion processes to approximate it, and then proved the propagation of chaos. The jump part is the Boltzmann operator with an integrable kernel, while the diffusive part is a Laplace operator. We know that the Landau operator behaves like the Laplace operator, except with some compensation to conserve energy.

In the recent work [40], the authors replaced the small collisions by a small diffusion term to approximate the Kac equation without cutoff, and successfully built a stochastic particle system to approximate the solution of the Kac equation without cutoff. The Kac equation is a one-dimensional case of the Boltzmann equation.

Thanks to the above breakthroughs, our approximate equation (1.7) has great potential to be solved numerically. In our future work, we will build a particle system based on equation (1.7) and prove the propagation of chaos.

1.2.5 Notations and main results

We now give the notations which will be used in the thesis.

- For each integer $N \geq 0$, let us denote the Sobolev space

$$H^N = \Big\{ f(v) : \sum_{|\alpha| \leq N} \|\partial_v^\alpha f\|_{L^2} < +\infty \Big\},$$

  where the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\partial_v^\alpha = \partial_{v_1}^{\alpha_1} \partial_{v_2}^{\alpha_2} \partial_{v_3}^{\alpha_3}$.

- For any numbers $m, l \in \mathbb{R}$, let us denote the weighted Sobolev space

$$H_l^m = \Big\{ f(v) : \|\langle D\rangle^m \langle \cdot \rangle^l f\|_{L^2} < +\infty \Big\},$$

  where $\langle v\rangle \stackrel{\text{def}}{=} (1+|v|^2)^{\frac{1}{2}}$, and with a symbol $a(\xi)$, the pesudo-differential operator $a(D)$ is defined by

$$(a(D)f)(v) \stackrel{\text{def}}{=} (2\pi)^{-3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(v-u)\cdot\xi} a(\xi) f(u) du d\xi.$$

- We also adopt the standard notations

$$\|f\|_{L_q^p} = \big(\int_{\mathbb{R}^3} |f(v)|^p \langle v\rangle^{qp} dv \big)^{\frac{1}{p}}, \quad \|f\|_{L \log L} = \int_{\mathbb{R}^3} |f| \log(1 + |f|) dv.$$

- For the ease of notation, let us define a new norm $||\cdot||_{\epsilon,m,l}$ for any $\epsilon, l > 0$ and $m \in \mathbb{N}$ as:

$$||f||_{\epsilon,m,l}^2 \stackrel{\text{def}}{=} ||f||_{H_l^{m+s}}^2 + \epsilon^{2-2s} ||f||_{H_l^{m+1}}^2,$$

  If $m = 0$, we simply write $||\cdot||_{\epsilon,l}$ instead of $||\cdot||_{\epsilon,0,l}$. If $m = l = 0$, we simply write $||\cdot||_\epsilon$ instead of $||\cdot||_{\epsilon,0}$. Then for any $\epsilon \leq 1$, $||\cdot||_{H_l^{m+s}} \leq ||\cdot||_{\epsilon,m,l} \leq 2||\cdot||_{H_l^{m+1}}$.

- Let us define the symbol $W^\epsilon(\xi)$ by

$$W^\epsilon(\xi) = \langle \xi \rangle^s \mathbf{1}_{|\xi| \leq \frac{1}{\epsilon}} + \epsilon^{-s} \mathbf{1}_{|\xi| > \frac{1}{\epsilon}},$$

which comes from the coercivity estimate of the cutoff Boltzmann operator Q^ϵ.

- For any $f, g \in L^2(\mathbb{R}^3)$, we denote by $\langle f, g \rangle$ the inner product of f and g.

- The notation $a \lesssim b$ means that a uniform constant C exists, which may be different across different lines, such that $a \leq Cb$. If both $a \lesssim b$ and $b \lesssim a$, we simply write $a \sim b$.

We do not bother to distinguish a function and its value at a point. For example, we do not distinguish weight function $\langle \cdot \rangle^l$ and the value $\langle v \rangle^l$ it takes at a point v.

We recall Young's inequality for use in future. For $a, b \geq 0$ and $p, q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$, there holds

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{1.9}$$

As a result, for any $\eta > 0$, we have the basic inequality

$$ab \leq \eta a^p + (p\eta)^{-\frac{q}{p}} \frac{b^q}{q}. \tag{1.10}$$

We also recall the Gronwall's inequality. For any $a, b \in \mathbb{R}$, and a function y defined on $\mathbb{R}_+$ satisfying

$$\frac{dy}{dt} \leq a + by(t),$$

then

$$y(t) \leq y(0)e^{bt} + \frac{a}{b}(e^{bt} - 1). \tag{1.11}$$

There is also an integral type of Gronwall's inequality. Let y, α, β be functions defined on $\mathbb{R}_+$. If β is nonnegative and for any $t > a \geq 0$, y satisfies

$$y(t) \leq \alpha(t) + \int_a^t \beta(r) y(r) dr,$$

then

$$y(t) \leq \alpha(t) + \int_a^t \alpha(r)\beta(r)\exp\big(\int_r^t \beta(u)du\big)dr. \tag{1.12}$$

If, in addition, the function α is non-decreasing, then

$$y(t) \leq \alpha(t)\exp\big(\int_a^t \beta(r)dr\big). \tag{1.13}$$

Before stating our main results, let us give the definition of ϕ which is related to the weight function:

$$\begin{cases} \phi(0,l) = 2l+5; \\ \phi(s,l) = \dfrac{(2l+4)(2+s)-2l}{s}; \\ \phi(1,l) = \max\{\phi(s,x(l)), y(l)\}; \\ \phi(m,l) = \max\{u(m,l), \phi(m-1,z(l)), \qquad m \geq 2, \end{cases} \tag{1.14}$$

where

$$\begin{cases} x(l) = \dfrac{2l+7}{s} - \dfrac{1-s}{s}(l+\dfrac{\gamma}{2}); \\ y(l) = \dfrac{3x(l)-(s+2)l}{1-s}; \\ z(l) = 2l+7+\dfrac{l+7}{s}; \\ u(m,l) = (m+2)z(l)-(m+1)l. \end{cases} \tag{1.15}$$

We begin with a well-posedness result of our approximate equation and propagation of moments and smoothness of the solution to it.

Theorem 1.1. *Let $\phi : \mathbb{N}\times\mathbb{R}_+ \to \mathbb{R}$ be the function defined as in (1.14). Let $N \in \mathbb{N}$ and $l \geq 0$. If $f_0 \in L^1_q \cap H^N_l$ with $q \geq \phi(N,l)$, then* (1.7) *admits a non-negative and unique solution f^ϵ in $L^\infty([0,\infty); L^1_q \cap H^N_l)$ and moreover there exists a constant C, depending only on $||f_0||_{L^1_q}$ and $||f_0||_{H^N_l}$, such that for any $t \geq 0$ and ϵ small enough,*

$$||f^\epsilon(t)||_{L^1_q} \leq C(||f_0||_{L^1_q}), \tag{1.16}$$

and

$$||f^\epsilon(t)||^2_{H^N_l} + \int_t^{t+1} ||f^\epsilon(r)||^2_{\epsilon,N,l+\gamma/2}dr \leq C(||f_0||_{L^1_q}, ||f_0||_{H^N_l}). \tag{1.17}$$

Remark 1.1. The result of Theorem 1.1 is also true when $\epsilon = 0$, which corresponds to the propagation of moments and smoothness of solution to the original Boltzmann equation (1.1).

The next two theorems describe the error between solutions of the Boltzmann equation and our approximate equation.

Theorem 1.2. *Let $l \geq 0$ such that $(\frac{4}{\pi})^{2l-2s}(l-s) \geq \frac{2^{4-2s}\pi K}{A_2}$ and $2l \geq \frac{s}{1-s}(\gamma+2)+\gamma$. Suppose $f_0 \in L^1_q \cap H^5_{2l+\gamma+12}$ with $q \geq \phi(5, 2l+\gamma+12)$. Let f and f^ϵ be solutions to the Boltzmann equation* (1.1) *and the approximate equation* (1.7) *with the same initial datum f_0 respectively, then we have for any $t \geq 0$,*

$$||f(t) - f^\epsilon(t)||_{L^1_{2l}} \leq C(f_0, t)\epsilon^{3-2s}, \tag{1.18}$$

where $C(f_0,t)$ is a constant depending only on $||f_0||_{L^1_q}, ||f_0||_{H^5_{2l+\gamma+12}}$ and time t.

Let us introduce the definition of ψ:

$$\begin{cases} \psi(0,l) = 2l+\gamma+17, \\ \psi(m,l) = l+\gamma+10, \qquad m \geq 1, \end{cases} \tag{1.19}$$

and φ:

$$\begin{cases} \varphi(0,l) = \phi(5, 2l+\gamma+17), \\ \varphi(m,l) = \max\{\varphi(m-1, z(l)), \rho(m,l)\}, \qquad m \geq 1, \end{cases} \tag{1.20}$$

where $\rho(m,l) = (m+7)\psi(m-1, z(l)) - (m+6)(l+\gamma+10)$. Then we have:

Theorem 1.3. *Let $N \in \mathbb{N}$ and $l \geq 0$ such that $(\frac{4}{\pi})^{2l+5-2s}(2l+5-2s) \geq \frac{2^{5-2s}\pi K}{A_2}$ and $2l+5 \geq \frac{s}{1-s}(\gamma+2)+\gamma$. Let $\psi, \varphi : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}$ be functions defined as in (1.19) and (1.20). Suppose $f_0 \in L^1_q \cap H^{N+5}_{\psi(N,l)}$ with $q \geq \varphi(N,l)$. Let f and f^ϵ be solutions to the Boltzmann equation* (1.1) *and the approximate equation* (1.7) *with the same initial datum f_0 respectively, then we have for any $t \geq 0$,*

$$||f(t) - f^\epsilon(t)||_{H^N_l} \leq C(f_0, t)\epsilon^{3-2s}, \tag{1.21}$$

where $C(f_0,t)$ is a constant depending only on $||f_0||_{L^1_q}, ||f_0||_{H^{N+5}_{\psi(N,l)}}$ and time t.

Note that the error estimates depends on time, that is, it is a local result. For a very general initial datum, it seems very hard to get a global error estimate. But when the initial datum is near Maxwellian, one can get a uniform error estimate.

1.3 Plan of the thesis

In chapter 2, the DPP for stochastic control problem under expectation constraint is given and proved. Applications to control problems under the above (in section 1.1) mentioned specific constraints (state constraint, floor constraint, drawdown constraint, target problem, quantile hedging) are then reported.

In chapter 3, we state three estimates (upper bound, coercivity, commutator) of the operator M^{ϵ}, and then establish the well-posedness theory of our approximate equation, namely, uniqueness and existence of non-negative solution.

Chapter 4 proves the high order convergence of solutions between the Boltzmann equation and our approximate equation.

In the last chapter, we make a summary and give some future work.

Chapter 2

Dynamic Programming Principle

The current chapter is organized as follows. In the first section, our main result is presented and proved following usual notations and necessary preliminaries. Section 2.2 is devoted to apply our main result to prove DPP for stochastic control problem under expectation constraint. Immediate applications to control problems under the specific constraints (state constraint, floor constraint, drawdown constraint, target problem, quantile hedging) are reported in section 2.3. In the last section, we give some facts about the continuous function space, which are useful to measurability of some functions used in section 2.3.

2.1 Main result

2.1.1 Notations

We begin with some notations which will be used only in the current chapter. We fix the dimension $d \in \mathbb{N}\backslash\{0\}$ and a time horizon $T \in (0, \infty)$. Let $\Omega = \{\omega \in C([0,T];\mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space of continuous functions equipped with the topology of uniform convergence, P_0 the Wiener measure on Ω, and B the canonical process $B_t(\omega) = \omega_t$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be the filtration generated by B and set $\mathcal{F} = \mathcal{F}_T$. Additionally, we use $\mathfrak{P}(\Omega)$ to denote the set of all probability measures on $(\Omega, \mathcal{F})$, equipped with the weak convergence topology. Note that both Ω and $\mathfrak{P}(\Omega)$ are Borel spaces.

In the following, we use $E^P[\,]$ to denote taking expectation with respect to probability P, and if $P = P_0$ which is the Wiener measure, we will simply write $E[\,]$.

To avoid cumbersome notation, it will be useful to define integrals for all measurable functions ξ with values in the extended real line $\bar{\mathbb{R}} = [-\infty, \infty]$. Namely, we set

$$E^P[\xi] \stackrel{\text{def}}{=} E^P[\xi^+] - E^P[\xi^-]$$

if $E^P[\xi^+]$ or $E^P[\xi^-]$ is finite, and use the convention

$$E^P[\xi] \stackrel{\text{def}}{=} -\infty \text{ if } E^P[\xi^+] = E^P[\xi^-] = +\infty.$$

We adopt the convention that the supremum (resp. infimum) over an empty set is $-\infty$ (resp. ∞), i.e. $\sup \emptyset = -\infty$ (resp. $\inf \emptyset = \infty$).

We use $\mathcal{T}$ to denote the collection of all $\mathbb{F}-$stopping times taking values in $[0, T]$.

2.1.2 Preliminaries

In this subsection, we recall some facts about conditional probability distribution from [17, Ch.1] and several basic definitions from the theory of analytic sets from [1, Ch.7].

For any $P \in \mathfrak{P}(\Omega), \tau \in \mathcal{T}$, there exists a regular conditional probability distribution $\{P^\omega_\tau\}_{\omega \in \Omega}$ of P given $\mathcal{F}_\tau$. That is, $P^\omega_\tau \in \mathfrak{P}(\Omega)$ for each ω, while $\omega \in \Omega \to P^\omega_\tau(A) \in [0, 1]$ is $\mathcal{F}_\tau$-measurable for any $A \in \mathcal{F}$ and

$$E^{P^\omega_\tau}[\xi] = E^P[\xi | \mathcal{F}_\tau](\omega) \text{ for } P\text{-a.e. } \omega \in \Omega,$$

whenever ξ is $\mathcal{F}$-measurable and bounded. Moreover, P^ω_τ can be chosen to be concentrated on the set of paths that coincide with ω up to time $\tau(\omega)$. In other words, $P^\omega_\tau(\Omega^\omega_\tau) = 1$ for any $\omega \in \Omega$, where $\Omega^\omega_\tau \stackrel{\text{def}}{=} \{\omega' \in \Omega : \omega' = \omega \text{ on } [0, \tau(\omega)]\}$. Further, given $P \in \mathfrak{P}(\Omega)$ and a family $(Q^\omega)_{\omega \in \Omega}$ such that $\omega \in \Omega \to Q^\omega \in \mathfrak{P}(\Omega)$ is $\mathcal{F}_\tau$-mearurable with $Q^\omega(\Omega^\omega_\tau) = 1$ for all $\omega \in \Omega$, one can define a concatenated probability measure $P \otimes_\tau Q^\cdot$ by

$$P \otimes_\tau Q^\cdot(A) \stackrel{\text{def}}{=} \int_\Omega Q^\omega(A) P(d\omega), \quad \forall A \in \mathcal{F}.$$

A subset of a Borel space is called analytic if it is the image of a Borel subset of another (uncountable) Borel space under a Borel-measurable function. For a Borel space X, denote $AN(X)$ the collection of all its analytic sets. The σ-field $\mathcal{A}_X$ generated by $AN(X)$ is called the analytic σ-field. Given a σ-field $\mathcal{G}$ of X, the universal completion of $\mathcal{G}$ is the σ-field $\mathcal{G}^* = \cap_{P\in\mathfrak{P}(X,\mathcal{G})}\mathcal{G}^P$, where $\mathfrak{P}(X,\mathcal{G})$ is the collection of all probability measures on $(X,\mathcal{G})$ and $\mathcal{G}^P$ is the completion of $\mathcal{G}$ under P. Let $\mathcal{B}_X$ be the Borel σ-field of X, then $\mathcal{B}_X^*$ is called the universal σ-field. For any $P \in \mathfrak{P}(X,\mathcal{B}_X)$, we have

$$\mathcal{B}_X \subseteq AN(X) \subseteq \mathcal{A}_X \subseteq \mathcal{B}_X^* \subseteq \mathcal{B}_X^P.$$

An $\bar{\mathbb{R}}$-valued function f is called:

- Borel-measurable if $\{f \geq c\} \in \mathcal{B}_X$ for any $c \in \mathbb{R}$;
- upper semianalytic if $\{f \geq c\} \in AN(X)$ for any $c \in \mathbb{R}$;
- lower semianalytic if $\{f \leq c\} \in AN(X)$ for any $c \in \mathbb{R}$;
- analytically measurable if $\{f \geq c\} \in \mathcal{A}_X$ for any $c \in \mathbb{R}$;
- universally measurable if $\{f \geq c\} \in \mathcal{B}_X^*$ for any $c \in \mathbb{R}$.

2.1.3 Problem formulation and main results

For each $(t,\omega) \in [0,T]\times\Omega$, we are given a set $\mathcal{P}(t,\omega) \subseteq \mathfrak{P}(\Omega)$. These sets are adapted in that

$$\mathcal{P}(t,\omega) = \mathcal{P}(t,\tilde{\omega}) \ \text{ if } \ \omega = \tilde{\omega} \ \text{ on } \ [0,t].$$

We assume for any $(t,\omega) \in [0,T]\times\Omega, \mathcal{P}(t,\omega) \neq \emptyset$, and for any $P \in \mathcal{P}(t,\omega), P(\Omega_t^\omega) = 1$. If τ is a stopping time, we denote $\mathcal{P}(\tau,\omega) = \mathcal{P}(\tau(\omega),\omega)$.

Given two functions $\xi, \eta : \Omega \to \bar{\mathbb{R}}$. Assume that ξ is upper semianalytic, while η is lower semianalytic. Define $\mathcal{P}(\eta,m) = \{P \in \mathfrak{P}(\Omega) : E^P[\eta] \leq m\}$, and set $\mathcal{P}(t,\omega,m) = \mathcal{P}(t,\omega) \cap \mathcal{P}(\eta,m)$. We consider the following value function

$$V(t,\omega,m) = \sup_{P\in\mathcal{P}(t,\omega,m)} E^P[\xi].$$

Employing the ideas in [16], we deal with expectation constraint $E^P[\eta] \leq m$ by auxiliary martingales. For each $P \in \mathcal{P}(t, \omega, m)$, let $\mathcal{M}^+_{t,\omega,m}(P)$ be the collection of all processes $[t, T] \times \Omega \to \bar{\mathbb{R}}$ such that

(1) $M_t \leq m$;

(2) $E^P[M_{s_2}|\mathcal{F}_{s_1}] \leq M_{s_1}$ for P-a.e., for any $s_1, s_2 \in [t, T]$ with $s_1 \leq s_2$;

(3) $M_T \geq \eta$ for P-a.e..

Let us use $\mathcal{T}^t$ to denote the set of all stopping times taking values in $[t, T]$. Given a stopping time $\tau \in \mathcal{T}^t, \omega \in \Omega, m \in \mathbb{R}, P \in \mathcal{P}(t, \omega, m), M \in \mathcal{M}^+_{t,\omega,m}(P)$, we denote $\mathcal{P}(\tau, \omega, M_\tau) = \mathcal{P}(\tau(\omega), \omega, M_\tau(\omega))$, and similarly $V(\tau, \omega, M_\tau) = V(\tau(\omega), \omega, M_\tau(\omega))$.

The following conditions are needed for our main result.

Assumption 2.1. *Let $(t, \bar{\omega}) \in [0, T] \times \Omega$ and $\tau \in \mathcal{T}^t$, for any $P \in \mathcal{P}(t, \bar{\omega})$, we assume*

(1) Measurability: The graph $[[\mathcal{P}]] := \{(t, \omega, Q) : (t, \omega) \in [0, T] \times \Omega, Q \in \mathcal{P}(t, \omega)\}$ is an analytic subset of $[0, T] \times \Omega \times \mathfrak{P}(\Omega)$.

(2) Invariance: There is a family of regular conditional probability distribution $(P^\omega_\tau)_{\omega \in \Omega}$ of P given $\mathcal{F}_\tau$ such that $P^\omega_\tau \in \mathcal{P}(\tau, \omega)$ for P-a.e. $\omega \in \Omega$.

(3) Stability under pasting: Let $(Q^\omega)_{\omega \in \Omega}$ be such that $\omega \to Q^\omega$ is $\mathcal{F}_\tau-$measurable and $Q^\omega \in \mathcal{P}(\tau, \omega)$ for P-a.e. $\omega \in \Omega$, then $P \otimes_\tau Q^\cdot \in \mathcal{P}(t, \bar{\omega})$.

Remark 2.1. Assumption 2.1 here is the same as Assumption 2.1 of [12]. In that paper, the authors gived two situations (regarding to $G-$expectations and random $G-$expectations) under which Assumption 2.1 is satisfied.

Our main result is the following theorem.

Theorem 2.1. *Under Assumption 2.1,*

$$\begin{aligned} V(t, \bar{\omega}, m) &= \sup_{P \in \mathcal{P}(t,\bar{\omega},m)} \sup_{M \in \mathcal{M}^+_{t,\bar{\omega},m}(P)} E^P[V(\tau, \cdot, M_\tau)] \\ &= \sup_{P \in \mathcal{P}(t,\bar{\omega},m)} \inf_{M \in \mathcal{M}^+_{t,\bar{\omega},m}(P)} E^P[V(\tau, \cdot, M_\tau)]. \end{aligned} \tag{2.1}$$

In order to prove Theorem 2.1, we state three lemmas.

Lemma 2.1. *If ξ is upper semianalytic, the function $P \in \mathfrak{P}(\Omega) \to E^P[\xi] \in \bar{\mathbb{R}}$ is upper semianalytic.*

Proof. See [1, Corollary.7.48.1, p.148] for details. $\square$

Lemma 2.2. *Let $D = \{(t,\omega,m,P) : (t,\omega,m) \in [0,T]\times\Omega\times\mathbb{R}, P \in \mathcal{P}(t,\omega,m)\}$, then D is an analytic subset of $[0,T]\times\Omega\times\mathbb{R}\times\mathfrak{P}(\Omega)$.*

Proof. The function $L(m,P) \overset{\text{def}}{=} E^P[\eta] - m$ is lower semianalytic by Lemma 2.1. Observe that

$$D = \{(t,\omega,m,P) : (t,\omega,m) \in [0,T]\times\Omega\times\mathbb{R}, P \in \mathcal{P}(t,\omega)\} \cap \{(t,\omega,m,P) : L(m,P) \leq 0\}.$$

The first part is analytic by Assumption 2.1(1), and the second part is analytic thanks to lower semianalytic property of L. $\square$

For ease of notation, set $X = [0,T]\times\Omega\times\mathbb{R}$, and define $\mathrm{proj}_X(D) = \{(t,\omega,m) : (t,\omega,m,P) \in D\}$. Then we have $\mathrm{proj}_X(D) = \{(t,\omega,m) : \mathcal{P}(t,\omega,m) \neq \emptyset\}$.

Lemma 2.3. *The function $V : proj_X(D) \to \bar{\mathbb{R}}$ is upper semianalytic. Moreover, for every $\epsilon > 0$, there exists an analytically measurable function $\varphi_\epsilon : proj_X(D) \to \mathfrak{P}(\Omega)$ such that for every $(t,\omega,m) \in proj_X(D), (t,\omega,m,\varphi_\epsilon(t,\omega,m)) \in D$ and*

$$E^{\varphi_\epsilon(t,\omega,m)}[\xi] \geq \begin{cases} V(t,\omega,m) - \epsilon & V(t,\omega,m) < \infty; \\ \epsilon^{-1} & V(t,\omega,m) = \infty. \end{cases}$$

Proof. Based on Lemma 2.1 and Lemma 2.2, Lemma 2.3 is a consequence of [1, proposition 7.47, p.179] and [1, proposition 7.50, p.148]. $\square$

The rest of this subsection devotes to a proof of Theorem 2.1.

Proof of Theorem 2.1 (Step 1) Let us firstly prove one direction of the above relationship (2.1), namely,

$$V(t,\bar{\omega},m) \leq \sup_{P\in\mathcal{P}(t,\bar{\omega},m)} \inf_{M\in\mathcal{M}^+_{t,\bar{\omega},m}(P)} E^P[V(\tau,\cdot,M_\tau)]. \tag{2.2}$$

Fix $P \in \mathcal{P}(t, \bar{\omega}, m)$ and $M \in \mathcal{M}^+_{t,\bar{\omega},m}(P)$. By Assumption 2.1(2), there is a family of regular conditional probability distribution (P^ω_τ) of P given $\mathcal{F}_\tau$ such that $P^\omega_\tau \in \mathcal{P}(\tau, \omega)$ for P-a.e. $\omega \in \Omega$. We assert that $P^\omega_\tau \in \mathcal{P}(\tau, \omega, M_\tau)$ for P-a.e. $\omega \in \Omega$. To see this, for P-a.e. $\omega \in \Omega$,

$$\begin{aligned} E^{P^\omega_\tau}[\eta] &= E^P[\eta|\mathcal{F}_\tau](\omega) \\ &\leq E^P[M_T|\mathcal{F}_\tau](\omega) \\ &\leq M_\tau(\omega). \end{aligned} \tag{2.3}$$

Then for P-a.e. $\omega \in \Omega$, we have

$$\begin{aligned} E^P[\xi|\mathcal{F}_\tau](\omega) &= E^{P^\omega_\tau}[\xi] \\ &\leq V(\tau, \omega, M_\tau). \end{aligned} \tag{2.4}$$

Taking $P(d\omega)$-expectations on both sides, we get

$$E^P[\xi] \leq E^P[V(\tau, \omega, M_\tau)].$$

Since $M \in \mathcal{M}^+_{t,\bar{\omega},m}(P)$ is arbitrary, we have

$$E^P[\xi] \leq \inf_{M \in \mathcal{M}^+_{t,\bar{\omega},m}(P)} E^P[V(\tau, \omega, M_\tau)].$$

The inequality (2.2) follows by taking the supremum over $\mathcal{P}(t, \bar{\omega}, m)$.

(Step 2) Now we are ready to prove the other direction:

$$V(t, \bar{\omega}, m) \geq \sup_{P \in \mathcal{P}(t,\bar{\omega},m)} \sup_{M \in \mathcal{M}^+_{t,\bar{\omega},m}(P)} E^P[V(\tau, \cdot, M_\tau)]. \tag{2.5}$$

Fix $\epsilon > 0, P \in \mathcal{P}(t, \bar{\omega}, m)$, and take an arbitrary $M \in \mathcal{M}^+_{t,\bar{\omega},m}(P)$. As the composition of universally measurable functions is universally measurable, the map

$$\omega \in \Omega \to \varphi_\epsilon(\tau(\omega), \omega, M_\tau(\omega)) \in \mathfrak{P}(\Omega)$$

is $\mathcal{F}^*_\tau$-measurable by the universally measurable extension of Galmarino's test. (See lemma 2.5 in [72] for details.) Then there exists an $\mathcal{F}_\tau$-measurable kernel $Q_\epsilon : \Omega \to \mathfrak{P}(\Omega)$ such that $Q^\omega_\epsilon = \varphi_\epsilon(\tau(\omega), \omega, M_\tau(\omega))$ for P-a.e. $\omega \in \Omega$. Again by Assumption

2.1(2) and equation (2.3), we have $\mathcal{P}(\tau,\omega,M_\tau)\neq\emptyset$ for P-a.e. $\omega\in\Omega$. Thus for P-a.e. $\omega\in\Omega$, $Q_\epsilon^\omega\in\mathcal{P}(\tau,\omega,M_\tau)$ and

$$E^{Q_\epsilon^\omega}[\xi]\geq\begin{cases}V(\tau,\omega,M_\tau)-\epsilon & V(\tau,\omega,M_\tau)<\infty;\\ \epsilon^{-1} & V(\tau,\omega,M_\tau)=\infty.\end{cases}$$

Then $P\otimes_\tau Q_\epsilon^{\cdot}\in\mathcal{P}(t,\bar{\omega})$ by Assumption 2.1(3), we assert further that $P\otimes_\tau Q_\epsilon^{\cdot}\in\mathcal{P}(t,\bar{\omega},m)$. To see this,

$$\begin{aligned}E^{P\otimes_\tau Q_\epsilon^{\cdot}}[\eta]&=E^P[E^{Q_\epsilon^{\cdot}}[\eta]]\\&\leq E^P[M_\tau]\\&\leq E^P[M_t|\mathcal{F}_t]\\&\leq m.\end{aligned}\tag{2.6}$$

By further derivation, we have

$$\begin{aligned}E^P[V(\tau,\omega,M_\tau)\wedge\epsilon^{-1}]&\leq E^P[E^{Q_\epsilon^\omega}[\xi]]+\epsilon\\&=E^{P\otimes_\tau Q_\epsilon^{\cdot}}[\xi]+\epsilon\\&\leq\sup_{P'\in\mathcal{P}(t,\bar{\omega},m)}E^{P'}[\xi]+\epsilon\\&=V(t,\bar{\omega},m)+\epsilon.\end{aligned}\tag{2.7}$$

Let ϵ tend to 0, we obtain

$$E^P[V(\tau,\omega,M_\tau)]\leq V(t,\bar{\omega},m).$$

Since $M\in\mathcal{M}^+_{t,\bar{\omega},m}(P)$ is arbitrary, we have

$$\sup_{M\in\mathcal{M}^+_{t,\bar{\omega},m}(P)}E^P[V(\tau,\omega,M_\tau)]\leq V(t,\bar{\omega},m).$$

Finally $P\in\mathcal{P}(t,\bar{\omega},m)$ is arbitrary, we can obtain (2.5) by taking supremum over $\mathcal{P}(t,\bar{\omega},m)$.

2.2 General stochastic control problem under expectation constraint

In this section, we apply our result in the previous section to derive DPP for a stochastic control problem under expectation constraint.

We first formulate the stochastic control problem. Let $\Omega' = \{\omega' \in C([0,T];\mathbb{R}^n) : \omega'_0 = 0\}$ for some $n \in \mathbb{N} \setminus \{0\}$, which may be different from the dimension d of Ω. Consider the probability space $(\Omega', \mathcal{F}', P'_0)$ similarly defined as $(\Omega, \mathcal{F}, P_0)$ in the beginning of section 2. That is, P'_0 is the Wiener measure on $(\Omega', \mathcal{F}')$.

For each $(t,\omega) \in [0,T] \times \Omega$, we are given a non-empty set $\mathcal{U}(t,\omega)$ whose elements are regarded as controls starting with past path ω at time t. We assume $\mathcal{U}(t,\omega)$ depends on ω only up to time t. That is,

$$\mathcal{U}(t,\omega) = \mathcal{U}(t,\tilde{\omega}) \text{ if } \omega = \tilde{\omega} \text{ on } [0,t].$$

For each $(t,\omega) \in [0,T] \times \Omega$ and $\nu \in \mathcal{U}(t,\omega)$, we are given a continuous process $X^{t,\omega,\nu} : [0,T] \times \Omega' \to \mathbb{R}^d$ satisfying $X^{t,\omega,\nu}_s(\omega') = \omega_s$ for all $s \in [0,t]$ and $\omega' \in \Omega'$.

Fix two given functions $f, g : \Omega \to \bar{\mathbb{R}}$. Suppose that f is upper semianalytic, while g is lower semianalytic. For each $(t,\omega,m) \in [0,T] \times \Omega \times \mathbb{R}$, we define the set of admissible controls at constraint level m as

$$\mathcal{U}(t,\omega,m) := \{\nu \in \mathcal{U}(t,\omega) : E[g(X^{t,\omega,\nu})] \leq m\}, \tag{2.8}$$

and the value function as

$$V(t,\omega,m) = \sup_{\nu \in \mathcal{U}(t,\omega,m)} E[f(X^{t,\omega,\nu})]. \tag{2.9}$$

Note that the expectations in (2.8) and (2.9) are taken with respect to P'_0.

We assume for each $\nu \in \mathcal{U}(t,\omega)$, the controlled process $X^{t,\omega,\nu} : (\Omega', \mathcal{F}') \to (\Omega, \mathcal{F})$ is measurable. That is, for every $A \in \mathcal{F}$, we have $(X^{t,\omega,\nu})^{-1}(A) \in \mathcal{F}'$. Now the process $X^{t,\omega,\nu}$ induces a probability measure $P_{t,\omega,\nu}$ on $(\Omega, \mathcal{F})$ by

$$P_{t,\omega,\nu}(A) = P'_0((X^{t,\omega,\nu})^{-1}(A)), \quad A \in \mathcal{F}.$$

Let us call $P_{t,\omega,\nu}$ the probability induced by ν for any $\nu \in \mathcal{U}(t,\omega)$, and define $\mathcal{P}(t,\omega) = \{P_{t,\omega,\nu} \in \mathfrak{P}(\Omega) : \nu \in \mathcal{U}(t,\omega)\}$ to be the set of probabilities induced by elements in $\mathcal{U}(t,\omega)$. Then we have

$$\mathcal{P}(t,\omega) = \mathcal{P}(t,\tilde{\omega}) \text{ if } \omega = \tilde{\omega} \text{ on } [0,t].$$

Additionally, we have $\mathcal{P}(t,\omega) \neq \emptyset$ for any $(t,\omega) \in [0,T] \times \Omega$, and $P(\Omega_t^\omega) = 1$ for any $P \in \mathcal{P}(t,\omega)$. Then by the definition of $P_{t,\omega,\nu}$, we have

$$E^{P_{t,\omega,\nu}}[f] = E[f(X^{t,\omega,\nu})] \text{ and } E^{P_{t,\omega,\nu}}[g] = E[g(X^{t,\omega,\nu})].$$

Finally, the set $\mathcal{P}(t,\omega,m)$ of admissible probability laws is defined as $\mathcal{P}(t,\omega,m) = \{P \in \mathcal{P}(t,\omega) : E^P[g] \leq m\}$. Then we will have

$$\begin{aligned} V(t,\omega,m) &= \sup_{\nu \in \mathcal{U}(t,\omega,m)} E[f(X^{t,\omega,\nu})] \\ &= \sup_{P \in \mathcal{P}(t,\omega,m)} E^P[f]. \end{aligned} \tag{2.10}$$

A control problem with an expectation constraint of the form (2.8) is not amenable to dynamic programming if we just consider a fixed level m. As in [16], we will formulate the constraint dynamically by using auxiliary martingales. For each $\nu \in \mathcal{U}(t,\omega,m)$, let $\mathcal{M}^+_{t,\omega,m}(\nu)$ be the collection of all processes $[t,T] \times \Omega \to \bar{\mathbb{R}}$ such that

(1) $M_t \leq m$;

(2) $M_T(X^{t,\omega,\nu}) \geq g(X^{t,\omega,\nu})$ for P_0'-a.e.;

(3) $M(X^{t,\omega,\nu})$ is a sub-martingale under P_0'.

We aim to prove the following dynamic programming principle:

$$\begin{aligned} V(t,\omega,m) &= \sup_{\nu \in \mathcal{U}(t,\omega,m)} \sup_{M \in \mathcal{M}^+_{t,\omega,m}(\nu)} E[V(\tau, X^{t,\omega,\nu}, M_\tau)] \\ &= \sup_{\nu \in \mathcal{U}(t,\omega,m)} \inf_{M \in \mathcal{M}^+_{t,\omega,m}(\nu)} E[V(\tau, X^{t,\omega,\nu}, M_\tau)]. \end{aligned} \tag{2.11}$$

Theorem 2.2. *If the family of sets $\mathcal{P}(t,\omega)$ induced by $\mathcal{U}(t,\omega)$ satisfies Assumption 2.1, then the dynamic programming principle (2.11) holds true.*

Proof. It is easy to check $\mathcal{P}(t,\omega,m) = \{P_{t,\omega,\nu} \in \mathfrak{P}(\Omega) : \nu \in \mathcal{U}(t,\omega,m)\}$ and $\mathcal{M}^+_{t,\omega,m}(\nu) = \mathcal{M}^+_{t,\omega,m}(P_{t,\omega,\nu})$, so (2.11) follows directly from Theorem 2.1. $\square$

The DPP in (2.11) involves both supremum and infimum over $\mathcal{M}^+_{t,\omega,m}(\nu)$ in general, because the martingale set $\mathcal{M}^+_{t,\omega,m}(\nu)$ depends on control ν. But in a special case which shall be established in the following proposition, we can get rid of the supremum and infimum over $\mathcal{M}^+_{t,\omega,m}(\nu)$.

Proposition 2.1. *If $\mathcal{M}^+_{t,\omega,m} := \cap_{\nu\in\mathcal{U}(t,\omega,m)}\mathcal{M}^+_{t,\omega,m}(\nu)$ is non-empty, then for any $M \in \mathcal{M}^+_{t,\omega,m}$, there holds*

$$V(t,\omega,m) = \sup_{\nu\in\mathcal{U}(t,\omega,m)} E[V(\tau, X^{t,\omega,\nu}, M_\tau)]. \tag{2.12}$$

Proof. By the definition of supremum and infimum, equation (2.12) is valid. $\square$

2.3 Specific applications

In this section, we apply our result in section 2.2 to specific stochastic control problems under various constraints. As promised in our introduction, we reformulate each of those constraints (state, floor, drawdown, target, quantile hedging) to expectation constraints. Thus the dynamic programming principles for these problems are derived as a direct consequence of the DPP for general stochastic control problem under expectation constraint.

Throughout this section, we will work in following setting. Let $\Omega' = \{\omega' \in C([0,T];\mathbb{R}^n) : \omega'_0 = 0\}$ for some $n \in \mathbb{N} \setminus \{0\}$, and consider the probability space $(\Omega', \mathcal{F}', P'_0)$. For each $(t,\omega) \in [0,T] \times \Omega$, we are given a non-empty set $\mathcal{U}(t,\omega)$ at time t starting with past path ω. The elements in $\mathcal{U}(t,\omega)$ are seen as controls without constraint. The set Ω contains all $\mathbb{R}^d-$valued continuous functions on $[0,T]$ starting from a same point. For the moment, we do not specify Ω in terms of its dimension d and initial point, since it will be different under different constraints.

We assume as before $\mathcal{U}(t,\omega)$ depends on ω only up to time t. That is,

$$\mathcal{U}(t,\omega) = \mathcal{U}(t,\tilde{\omega}) \text{ if } \omega = \tilde{\omega} \text{ on } [0,t].$$

For each $(t,\omega) \in [0,T] \times \Omega$ and $\nu \in \mathcal{U}(t,\omega)$, we are given a continuous process $X^{t,\omega,\nu} : [0,T] \times \Omega' \to \mathbb{R}^d$ satisfying $X_s^{t,\omega,\nu}(\omega') = \omega_s$ for all $s \in [0,t]$ and $\omega' \in \Omega'$.

Given an upper semianalytic reward function $f : \Omega \to \bar{\mathbb{R}}$, we may now construct control problems under the five types of constraints and transform them into control problems under expectation constraints one by one.

We put the first three (state, floor, drawdown) in one subsection since they are imposed on the whole path of the controlled process, and leave the other two (target problem, quantile hedging) in another in that they are only concerned with the terminal value of the controlled process.

2.3.1 State, floor, drawdown constraints

In this subsection, we deal with stochastic control problems under state, floor, drawdown constraints.

Case 1: State Constraint. Fix $d \in \mathbb{N} \setminus \{0\}$, and let $\Omega = \{\omega \in C([0,T];\mathbb{R}^d) : \omega_0 = 0\}$. We fix an open set $\mathcal{O} \subseteq \mathbb{R}^d$ and consider the stochastic control problem subject to the constraint that the controlled state process has to remain in $\mathcal{O}$. That is, we study the control problem with state constraint (CPSC):

$$\bar{V}_1(t,\omega) = \sup_{\nu \in \bar{\mathcal{U}}_1(t,\omega)} E[f(X^{t,\omega,\nu})],$$

where

$$\bar{\mathcal{U}}_1(t,\omega) = \{\nu \in \mathcal{U}(t,\omega) : X_s^{t,\omega,\nu}(\omega') \in \mathcal{O} \ \text{ for all } \ s \in [0,T], \ P_0'\text{-a.s. } \omega' \in \Omega'\}.$$

Of course, we must require $0 \in \mathcal{O}$. Assume that

$$\bar{\mathcal{U}}_1(t,\omega) \neq \emptyset \ \text{ for } \ (t,\omega) \in [0,T] \times \mathcal{O}. \tag{2.13}$$

In condition (2.13), $(t,\omega) \in [0,T] \times \mathcal{O}$ means that $\omega_s \in \mathcal{O}$ for $s \in [0,t]$. We now transform the above state constraint to expectation constraint. Let $\Omega(\mathcal{O}) = \{\omega : \omega_s \in \mathcal{O} \ \text{ for all } \ s \in [0,T]\}$, and set function g_1 by

$$g_1(\omega) = \begin{cases} 0 & \omega \in \Omega(\mathcal{O}); \\ 1 & \text{otherwise}. \end{cases} \tag{2.14}$$

Based on this pair of f and g_1, we can define as before

$$\mathcal{U}_1(t,\omega,m) = \{\nu \in \mathcal{U}(t,\omega) : E[g_1(X^{t,\omega,\nu})] \le m\},$$

and study the transformed problem (TPSC):

$$V_1(t,\omega,m) = \sup_{\nu \in \mathcal{U}_1(t,\omega,m)} E[f(X^{t,\omega,\nu})].$$

By the definition of g_1, we have

$$X_s^{t,\omega,\nu} \in \mathcal{O} \text{ for all } s \in [0,T], \ P_0'\text{-a.s.} \quad \Longleftrightarrow \quad E[g_1(X^{t,\omega,\nu})] \le 0,$$

and therefore

$$\bar{\mathcal{U}}_1(t,\omega) = \mathcal{U}_1(t,\omega,0) \text{ and } \bar{V}_1(t,\omega) = V_1(t,\omega,0). \tag{2.15}$$

Case 2: Floor Constraint. Let $d = 1$ in this case and $\Omega = \{\omega \in C([0,T];\mathbb{R}) : \omega_0 = 0\}$. Fix $k \in \Omega$ as the floor, and require the controlled process no less than the continuous function k. Namely, we study the control problem under floor constraint (CPFC):

$$\bar{V}_2(t,\omega) = \sup_{\nu \in \bar{\mathcal{U}}_2(t,\omega)} E[f(X^{t,\omega,\nu})],$$

where

$$\bar{\mathcal{U}}_2(t,\omega) = \{\nu \in \mathcal{U}(t,\omega) : X_s^{t,\omega,\nu}(\omega') \ge k_s \ \text{ for all } \ s \in [0,T], \ P_0'\text{-a.s. } \omega' \in \Omega'\}.$$

Now define $\Omega_t^k = \{\omega \in \Omega : \ \omega_s \ge k_s \text{ for all } \ s \in [0,t]\}$. Assume that for any $t \in [0,T]$ and $\omega \in \Omega_t^k$,

$$\bar{\mathcal{U}}_2(t,\omega) \ne \emptyset.$$

Let $\Omega(k) = \{\omega \in \Omega : \ \omega_s \geq k_s \text{ for all } \ s \in [0,T]\}$, and set function g_2 by

$$g_2(\omega) = \begin{cases} 0 & \omega \in \Omega(k); \\ 1 & \text{otherwise.} \end{cases} \tag{2.16}$$

Based on this pair of f and g_2, we can define as before

$$\mathcal{U}_2(t,\omega,m) = \{\nu \in \mathcal{U}(t,\omega) : E[g_2(X^{t,\omega,\nu})] \leq m\},$$

and study the transformed problem (TPFC):

$$V_2(t,\omega,m) = \sup_{\nu \in \mathcal{U}_2(t,\omega,m)} E[f(X^{t,\omega,\nu})].$$

By the definition of g_2, we have

$$X_s^{t,\omega,\nu} \geq k_s \ \text{ for all } \ s \in [0,T], \ P_0'\text{-a.s.} \quad \Longleftrightarrow \quad E[g_2(X^{t,\omega,\nu})] \leq 0,$$

and therefore

$$\bar{\mathcal{U}}_2(t,\omega) = \mathcal{U}_2(t,\omega,0) \text{ and } \bar{V}_2(t,\omega) = V_2(t,\omega,0). \tag{2.17}$$

Case 3: Drawdown Constraint. Let $d = 1$ in this case. Fix $x \geq 0$, and let $\Omega = \{\omega \in C([0,T];\mathbb{R}) : \omega_0 = x\}$. Fix $\alpha \in [0,1]$, the controlled process is required to satisfy the maximum drawdown condition (with drawdown no less than $1-\alpha$). Precisely, we study the control problem under drawdown constraint (CPDC):

$$\bar{V}_3(t,\omega) = \sup_{\nu \in \bar{\mathcal{U}}_3(t,\omega)} E[f(X^{t,\omega,\nu})],$$

where

$$\begin{aligned} \bar{\mathcal{U}}_3(t,\omega) \quad &= \quad \{\nu \in \mathcal{U}(t,\omega) : X_s^{t,\omega,\nu}(\omega') \geq \alpha X_s^{t,\omega,\nu,*}(\omega') \\ &\qquad \text{for all } s \in [0,T], P_0'\text{-a.s. } \omega' \in \Omega'\}, \end{aligned}$$

where $X_s^{t,\omega,\nu,*} \overset{\text{def}}{=} \sup_{0 \leq r \leq s} X_r^{t,\omega,\nu}$. Given any $\omega \in \Omega$, we define its corresponding current maximum function as

$$\omega_s^* = \sup_{0 \leq r \leq s} \omega_r.$$

Now define $\Omega_t^* = \{\omega \in \Omega : \ \omega_s \geq \alpha \omega_s^* \text{ for all } \ s \in [0,t]\}$. Assume that for any $t \in [0,T]$ and $\omega \in \Omega_t^*$,

$$\bar{\mathcal{U}}_3(t,\omega) \neq \emptyset.$$

Let $\Omega(*) = \{\omega \in \Omega : \ \omega_s \geq \alpha \omega_s^* \text{ for all } \ s \in [0,T]\}$, and set function g_3 by

$$g_3(\omega) = \begin{cases} 0 & \omega \in \Omega(*); \\ 1 & \text{otherwise.} \end{cases} \tag{2.18}$$

Based on this pair of f and g_3, we can define as before

$$\mathcal{U}_3(t,\omega,m) = \{\nu \in \mathcal{U}(t,\omega) : E[g_3(X^{t,\omega,\nu})] \leq m\},$$

and study the transformed problem (TPDC):

$$V_3(t,\omega,m) = \sup_{\nu \in \mathcal{U}_3(t,\omega,m)} E[f(X^{t,\omega,\nu})].$$

By the definition of g_3, we have

$$X_s^{t,\omega,\nu} \geq X_s^{t,\omega,\nu,*} \ \text{ for all } \ s \in [0,T], \ P_0'\text{-a.s.} \quad \Longleftrightarrow \quad E[g_3(X^{t,\omega,\nu})] \leq 0,$$

and therefore

$$\bar{\mathcal{U}}_3(t,\omega) = \mathcal{U}_3(t,\omega,0) \text{ and } \bar{V}_3(t,\omega) = V_3(t,\omega,0). \tag{2.19}$$

Based on functions g_i $(i = 1,2,3)$, for each $\nu \in \mathcal{U}_i(t,\omega,m)$, $\mathcal{M}^+_{t,\omega,m,i}(v)$ is defined accordingly as in the previous section, so is $\mathcal{M}^+_{t,\omega,m,i}$.

Remark 2.2. In case 1 and case 2, we assume the starting point of the controlled process is origin. If the controlled process does not start from origin, we can make translation such that it starts from origin without any loss of generality. However, case 3 is somewhat different. If a path satisfies the maximum drawdown condition, after a translation, it may not satisfy the condition anymore. For example, path ω^1 defined by $\omega_t^1 = 2 - t$ on $[0,1]$ satisfies $\omega_t^1 \geq \frac{1}{2}\omega_t^{1,*}$ on $[0,1]$, but path $\omega^2 \stackrel{\text{def}}{=} \omega^1 - 1$ does not satisfy $\omega_t^2 \geq \frac{1}{2}\omega_t^{2,*}$ on $[0,1]$. So we allow the controlled process to start from any point $x \geq 0$.

In order to apply Theorem 2.2, functions g_i ($i = 1, 2, 3$) need to be lower semianalytic. We leave this task in the last section of this chapter.

Theorem 2.3. *If the family of sets $\mathcal{P}(t, \omega)$ induced by $\mathcal{U}(t, \omega)$ satisfies Assumption 2.1, then the corresponding dynamic programming principles for (CPSC), (CPFC) and (CPDC) hold true:*

$$\bar{V}_i(t, \omega) = \sup_{\nu \in \bar{\mathcal{U}}_i(t,\omega)} E[\bar{V}_i(\tau, X^{t,\omega,\nu})] \text{ for } i = 1, 2, 3. \tag{2.20}$$

Proof. By Theorem 2.2, the dynamic programming principles (2.11) for (TPSC) (TPFC) and (TPDC) hold true. In particular, take $m = 0$, if $\nu \in \mathcal{U}_i(t, \omega, 0)$, we have for P_0'-a.e., $g_i(X^{t,\omega,\nu}) = 0$. Therefore the constant process $0 \in \mathcal{M}^+_{t,\omega,m,i}$, and from Proposition 2.1 we obtain,

$$V_i(t, \omega, 0) = \sup_{\nu \in \mathcal{U}_i(t,\omega,0)} E[V_i(\tau, X^{t,\omega,\nu}, 0)]. \tag{2.21}$$

Therefore, (2.20) follows from (2.21) due to (2.15), (2.17) and (2.19). □

2.3.2 Target problem, quantile hedging

In this subsection, we see how Theorem 2.2 can be applied to prove dynamic programming principle for control problems under the remaining two types of constraints. We will work in the the same common setting as in the previous subsection. We consider quantile hedging first and regard target problem as a special case of quantile hedging.

Case 4: Quantile Hedging. Fix $d \in \mathbb{N} \setminus \{0\}$, and let $\Omega = \{\omega \in C([0, T]; \mathbb{R}^d) : \omega_0 = 0\}$. In a quantile hedging problem we are satisfied if the probability of the controlled process $X^{t,\omega,v}$ staying in target G is greater than a pre-specified value $m \in [0, 1]$. Note that $m = 1$ corresponds to target problem. Fix an analytic set $G \subseteq \mathbb{R}^d$, consider the control problem under quantile hedging constraint (CPQC):

$$\bar{V}_4(t, \omega, m) = \sup_{\nu \in \bar{\mathcal{U}}_4(t,\omega,m)} E[f(X^{t,\omega,\nu})],$$

where

$$\bar{\mathcal{U}}_4(t,\omega,m) = \{\nu \in \mathcal{U}(t,\omega) : P_0'(X_T^{t,\omega,\nu} \in G) \geq m\}.$$

We now transform the above quantile hedging constraint to expectation constraint. In this case, we set g_4 as

$$g_4(\omega) = \begin{cases} 0 & \omega_T \in G; \\ 1 & \text{otherwise.} \end{cases}$$

Based on this pair of f and g_4, we can define as before

$$\mathcal{U}_4(t,\omega,m) = \{\nu \in \mathcal{U}(t,\omega) : E[g_4(X^{t,\omega,\nu})] \leq m\},$$

and study the transformed problem (TPQC):

$$V_4(t,\omega,m) = \sup_{\nu \in \mathcal{U}_4(t,\omega,m)} E[f(X^{t,\omega,\nu})].$$

By the definition of g_4, we have

$$P_0'(X_T^{t,\omega,\nu} \in G) \geq m \iff E[g_4(X^{t,\omega,\nu})] \leq 1-m,$$

and therefore

$$\bar{\mathcal{U}}_4(t,\omega,m) = \mathcal{U}_4(t,\omega,1-m) \text{ and } \bar{V}_4(t,\omega,m) = V_4(t,\omega,1-m). \tag{2.22}$$

For each $\nu \in \bar{\mathcal{U}}_4(t,\omega,m)$, let $\bar{\mathcal{M}}^-_{t,\omega,m}(v)$ be the collection of all processes $[t,T]\times\Omega \to \bar{\mathbb{R}}$ such that

(1) $M_t \leq m$;

(2) $M_T(X^{t,\omega,\nu}) \leq 1 - g_4(X^{t,\omega,\nu})$ for P_0'-a.e.;

(3) $M(X^{t,\omega,\nu})$ is a sub-martingale under P_0'.

Then we have $\bar{\mathcal{M}}^-_{t,\omega,m}(v) = 1 - \mathcal{M}^+_{t,\omega,1-m,4}(v) \stackrel{\text{def}}{=} \{1 - M : M \in \mathcal{M}^+_{t,\omega,1-m,4}(v)\}$, where $\mathcal{M}^+_{t,\omega,1-m,4}(v)$ is defined as in section 2.2 by using function g_4 for $\nu \in \mathcal{U}_4(t,\omega,1-m)$.

Theorem 2.4. *If the family of sets $\mathcal{P}(t,\omega)$ induced by $\mathcal{U}(t,\omega)$ satisfies Assumption 2.1, then the following dynamic programming principle for (CPHC) holds true:*

$$\begin{aligned}\bar{V}_4(t,\omega,m) &= \sup_{\nu\in\bar{\mathcal{U}}_4(t,\omega,m)} \sup_{M\in\bar{\mathcal{M}}^-_{t,\omega,m}(v)} E[\bar{V}_4(\tau,X^{t,\omega,\nu},M_\tau)] \\ &= \sup_{\nu\in\bar{\mathcal{U}}_4(t,\omega,m)} \inf_{M\in\bar{\mathcal{M}}^-_{t,\omega,m}(v)} E[\bar{V}_4(\tau,X^{t,\omega,\nu},M_\tau)].\end{aligned} \tag{2.23}$$

Proof. By Theorem 2.2, the dynamic programming principles (2.11) for (TPQC) holds true. That is, we have

$$\begin{aligned}V_4(t,\omega,1-m) &= \sup_{\nu\in\mathcal{U}_4(t,\omega,1-m)} \sup_{M\in\mathcal{M}^+_{t,\omega,1-m,4}(v)} E[V_4(\tau,X^{t,\omega,\nu},M_\tau)] \\ &= \sup_{\nu\in\mathcal{U}_4(t,\omega,1-m)} \inf_{M\in\mathcal{M}^+_{t,\omega,1-m,4}(v)} E[V_4(\tau,X^{t,\omega,\nu},M_\tau)].\end{aligned} \tag{2.24}$$

Thanks to $\bar{\mathcal{M}}^-_{t,\omega,m}(v) = 1-\mathcal{M}^+_{t,\omega,1-m,4}(v)$, we obtain

$$\sup_{M\in\mathcal{M}^+_{t,\omega,1-m,4}(v)} E[V_4(\tau,X^{t,\omega,\nu},M_\tau)] = \sup_{M\in\bar{\mathcal{M}}^-_{t,\omega,m}(v)} E[V_4(\tau,X^{t,\omega,\nu},1-M_\tau)],$$

and

$$\inf_{M\in\mathcal{M}^+_{t,\omega,1-m,4}(v)} E[V_4(\tau,X^{t,\omega,\nu},M_\tau)] = \inf_{M\in\bar{\mathcal{M}}^-_{t,\omega,m}(v)} E[V_4(\tau,X^{t,\omega,\nu},1-M_\tau)].$$

Together with (2.22), we arrive at (2.23). □

If $m=1$, then the constant process $1\in\cap_{\nu\in\bar{\mathcal{U}}_4(t,\omega,1)}\bar{\mathcal{M}}^-_{t,\omega,1}(v)$, thus we have the DPP for target problem in the following proposition.

Proposition 2.2. *If the family of sets $\mathcal{P}(t,\omega)$ induced by $\mathcal{U}(t,\omega)$ satisfies Assumption 2.1, then the following dynamic programming principle for target problem holds true:*

$$\bar{V}_4(t,\omega,1) = \sup_{\nu\in\bar{\mathcal{U}}_4(t,\omega,1)} E[\bar{V}_4(\tau,X^{t,\omega,\nu},1)]. \tag{2.25}$$

Remark 2.3. Proposition 2.2 reduces to a geometric type of dynamic programming principle when f is chosen to be a constant function. Specifically, if we define the reachability set:

$$D(t) \stackrel{\text{def}}{=} \{\omega\in\Omega : \exists\ \nu\in\mathcal{U}(t,\omega),\ \text{such that } X^{t,\omega,\nu}_T\in G\ P'_0\text{-a.s.}\},$$

the geometric type of dynamic programming principle states that

$$D(t) = \{\omega \in \Omega : \exists\ \nu \in \mathcal{U}(t,\omega),\ \text{such that } X^{t,\omega,\nu} \in D(\tau)\ P_0'\text{-a.s.}\}. \tag{2.26}$$

If we set $f \equiv 1$, then the above relationship (2.26) can be derived from (2.25).

2.4 Some facts about the continuous function s-pace

Functions g_i $(i = 1, 2, 3)$ defined in (2.14), (2.16) and (2.18) are Borel-measurable thanks to the following three lemmas.

Lemma 2.4. *Let* $\Omega = \{\omega \in C([0,T];\mathbb{R}^d) : \omega_0 = 0\}$ *and* $\Omega(\mathcal{O}) = \{\omega : \omega_s \in \mathcal{O}\ \ \textit{for all}\ \ s \in [0,T]\}$, *then* $\Omega(\mathcal{O})$ *is a Borel-measurable subset of* Ω.

Proof. Define $h(\omega) \stackrel{\text{def}}{=} \inf_{t\in[0,T]} d(\omega_t, \mathbb{R}^d \setminus \mathcal{O})$, where $d(x, \mathbb{R}^d \setminus \mathcal{O}) = \inf_{y\in\mathbb{R}^d\setminus\mathcal{O}} ||x-y||$ with $||\cdot||$ being the Euclidean norm. Then h is a continuous hence measurable function on Ω. Observing that

$$\Omega(\mathcal{O}) = \{\omega \in \Omega : h(\omega) > 0\}, \tag{2.27}$$

we conclude $\Omega(\mathcal{O})$ is Borel-measurable. □

Then function g_1 is measurable as a consequence of Lemma 2.4.

Remark 2.4. The above lemma is still valid if we replace $\mathcal{O}$ by a closed set $F \subset \mathbb{R}^d$ containing 0. To see this, let $\{r_i\}_{i=1}^{\infty}$ be the sequence of all rational numbers in $[0,1]$. For each i, define $\Omega(F, r_i) = \{\omega \in \Omega : \omega_{r_i} \in F\}$. Clearly $\Omega(F, r_i) \in \mathcal{F}_{r_i} \subset \mathcal{F}$. It is easily to check

$$\Omega(F) = \cap_{i=1}^{\infty}\Omega(F, r_i) \in \mathcal{F},$$

from denseness of rational numbers and continuity property of any element in Ω.

In the rest of this section, let $\Omega = C([0,1];\mathbb{R})$, which is the space of continuous functions on the unit interval $[0,1]$. It is a metric space with the metric defined by

$$d(\omega^1, \omega^2) = \sup_{0 \le t \le 1} |\omega^1(t) - \omega^2(t)|.$$

Let $n \ge 1$ be an integer, and set $t_i^n = \frac{i}{n}$ for $i = 0, 1, \cdots, n$. Let L^n be space of piecewise linear functions with rational values at turning points $\{t_i^n\}_{0 \le i \le n}$, precisely,

$$\begin{aligned} L^n &= \{\omega \in \Omega : \omega(t_i^n) \in \mathbb{Q} \text{ for any } 0 \le i \le n, \\ &\qquad \omega \text{ is linear on } [t_i^n, t_{i+1}^n] \text{ for any } 0 \le i \le n-1\}. \end{aligned}$$

Note that L^n is countable, and thus $L = \cup_{n \ge 1} L^n$ is also countable.

We claim that L is dense in Ω. Fix an $\omega \in \Omega$, then ω is uniformly continuous. For any $\epsilon > 0$, there exists $\delta > 0$, such that, if $|s - t| \le \delta$, then $|\omega(s) - \omega(t)| \le \frac{\epsilon}{5}$. Now choose n large enough such that $\frac{1}{n} \le \delta$. By the denseness of rational numbers in real numbers, we can find $n+1$ rational numbers $\{r_i\}_{0 \le i \le n}$ such that,

$$\sup_{0 \le i \le n} |\omega(t_i^n) - r_i| \le \frac{\epsilon}{5}.$$

Moreover, for any $0 \le i \le n-1$, there holds

$$|r_{i+1} - r_i| = |r_{i+1} - \omega(t_{i+1}^n)| + |\omega(t_{i+1}^n) - \omega(t_i^n)| + |\omega(t_i^n) - r_i| \le \frac{3}{5}\epsilon.$$

We denote $\omega^n \in L^n$ as the piecewise linear function with $\omega^n(t_i^n) = r_i$ for any $0 \le i \le n$. Then for $t \in [t_i^n, t_{i+1}^n]$, we have,

$$\begin{aligned} |\omega(t) - \omega^n(t)| &= |\omega(t) - \omega(t_i^n)| + |\omega(t_i^n) - \omega^n(t_i^n)| + |\omega^n(t_i^n) - \omega^n(t)| \\ &\le \frac{2}{5}\epsilon + |r_{i+1} - r_i| \\ &\le \epsilon. \end{aligned}$$

That is, $d(\omega, \omega^n) \le \epsilon$, which implies the denseness of L in Ω.

Fix an element $k \in \Omega$, and consider the paths bounded below by k,

$$U = \{\omega \in \Omega : \omega(t) - k(t) \ge 0, \text{ for any } t \in [0,1]\}. \tag{2.28}$$

Lemma 2.5. *The set U defined in (2.28) is a Borel-measurable subset of Ω.*

Proof. We collect all the elements in L^n which are greater than k at any points, namely, define

$$L^n_u = L^n \cap U.$$

Again, L^n_u is countable, and thus

$$F_u = \cap_{k=1}^{\infty} \cup_{n=1}^{\infty} \cup_{\omega \in L^n_u} B(\omega, \frac{1}{k})$$

is measurable. Here $B(\omega, r) = \{\tilde{\omega} \in \Omega : d(\tilde{\omega}, \omega) < r\}$ is the open ball with radius $r > 0$.

Now we prove $U = F_u$. Suppose $\omega \in U$, thanks to denseness of $L = \cup_{n \geq 1} L^n$, for any $\epsilon > 0$, there exists $n \geq 1$ and $\omega^n \in L^n$, such that, $d(\omega^n, \omega) \leq \frac{\epsilon}{3}$. Take an arbitrary rational number $r \in (\frac{\epsilon}{3}, \frac{2\epsilon}{3})$, and consider the function $\omega^{n,r} \stackrel{\text{def}}{=} \omega^n + r$. It is obvious that $\omega^{n,r} \in L^n$. We also have $\omega^{n,r} \in U$, due to

$$\begin{aligned} \omega^{n,r}(t) - k(t) &= \omega^{n,r}(t) - \omega^n(t) + \omega^n(t) - \omega(t) + \omega(t) - k(t) \\ &\geq r - \frac{\epsilon}{3} \\ &\geq 0. \end{aligned}$$

Thus, $\omega^{n,r} \in L^n_u$. Further, $d(\omega, \omega^{n,r}) \leq d(\omega, \omega^n) + d(\omega^n, \omega^{n,r}) \leq \epsilon$, which implies $U \subseteq F_u$.

On the other hand, suppose $\omega \in F_u$ but $\omega \notin U$. Then there exists $t \in [0, 1]$, such that, $\omega(t) < k(t)$. By the definition of F_u, there exist $n \geq 1$ and $\omega^n \in L^n_u$, such that, $d(\omega, \omega^n) \leq \frac{k(t) - \omega(t)}{2}$. Thus we arrive at

$$\omega^n(t) - k(t) = \omega^n(t) - \omega(t) + \omega(t) - k(t) \leq -\frac{k(t) - \omega(t)}{2}.$$

This is a contradiction with $\omega^n \in L^n_u$, and so $F_u \subseteq U$. $\square$

Now let us turn to drawdown constraints. Given any $\omega \in \Omega$, we define its corresponding current maximum function as

$$\omega^*(t) = \sup_{0 \leq s \leq t} \omega(s).$$

Fix an $\alpha \in [0,1]$, consider the following set of functions satisfying a drawdown condition,

$$\Omega_\alpha^+ = \{\omega \in \Omega : \omega(0) \geq 0, \text{ and } \omega(t) \geq \alpha\omega^*(t), \text{ for any } t \in [0,1]\}. \tag{2.29}$$

Note that if $\omega \in \Omega_\alpha^+$, then $\omega(t) \geq 0$ for any $t \in [0,1]$.

Lemma 2.6. *The set Ω_α^+ defined in (2.29) is a Borel-measurable subset of Ω.*

Proof. Set

$$L_\alpha^n = L^n \cap \Omega_\alpha^+.$$

The same as before, L_α^n is countable, and thus

$$F_\alpha = \cap_{k=1}^\infty \cup_{n=1}^\infty \cup_{\omega \in L_\alpha^n} B(\omega, \frac{1}{k})$$

is measurable.

In the following, we prove $\Omega_\alpha^+ = F_\alpha$. Suppose $\omega \in \Omega_\alpha^+$, then ω is uniformly continuous. For any $\epsilon > 0$, there exists $\delta > 0$, such that, if $|s-t| \leq \delta$, then $|\omega(s) - \omega(t)| \leq \frac{\epsilon}{5}$. Now choose n large enough such that $\frac{1}{n} \leq \delta$. By the denseness of rational numbers in real numbers, we can find $n+1$ non-negative rational numbers $\{r_i\}_{0 \leq i \leq n}$ such that,

$$\sup_{0 \leq i \leq n} |\omega(t_i^n) - r_i| \leq \frac{\epsilon}{5} \text{ and } \inf_{0 \leq i \leq n} (r_i - \alpha r_i^*) \geq 0,$$

where $r_i^* = \max_{0 \leq j \leq i} r_j$. This can be done as follows. For $0 \leq i \leq n$, choose an arbitrary rational number r_i on the open interval $(\omega(t_i^n) + \frac{i}{n+1}\frac{\epsilon}{5}, \omega(t_i^n) + \frac{i+1}{n+1}\frac{\epsilon}{5})$. Moreover, for any $0 \leq i \leq n-1$, there holds

$$|r_{i+1} - r_i| = |r_{i+1} - \omega(t_{i+1}^n)| + |\omega(t_{i+1}^n) - \omega(t_i^n)| + |\omega(t_i^n) - r_i| \leq \frac{3}{5}\epsilon.$$

We denote $\omega^n \in L^n$ as the piecewise linear function with $\omega^n(t_i^n) = r_i$ for any $0 \leq i \leq n$. It is not difficult to check that $\omega^n \in L_\alpha^n$. Then for $t \in [t_i^n, t_{i+1}^n]$, we have,

$$\begin{aligned} |\omega(t) - \omega^n(t)| &= |\omega(t) - \omega(t_i^n)| + |\omega(t_i^n) - \omega^n(t_i^n)| + |\omega^n(t_i^n) - \omega^n(t)| \\ &\leq \frac{2}{5}\epsilon + |r_{i+1} - r_i| \\ &\leq \epsilon. \end{aligned}$$

That is, $d(\omega, \omega^n) \le \epsilon$, which implies $\Omega_\alpha^+ \subseteq F_\alpha$.

On the other hand, suppose $\omega \in F_\alpha$ but $\omega \notin \Omega_\alpha^+$. Since $\omega \in F_\alpha$, we must have $\omega(0) \ge 0$. Due to the assumption $\omega \notin \Omega_\alpha^+$, then there exists $t \in (0, 1]$, such that, $\omega(t) < \alpha\omega^*(t)$. By the definition of F_α, there exist $n \ge 1$ and $\omega^n \in L_\alpha^n$, such that, $d(\omega, \omega^n) \le \frac{\alpha\omega^*(t)-\omega(t)}{4}$. Thus we arrive at

$$\begin{aligned}
\omega^n(t) - \alpha\omega^{n,*}(t) &= \omega^n(t) - \omega(t) + \omega(t) - \alpha\omega^*(t) + \alpha(\omega^*(t) - \omega^{n,*}(t)) \\
&\le 2d(\omega, \omega^n) + \omega(t) - \alpha\omega^*(t) \\
&\le -\frac{\alpha\omega^*(t) - \omega(t)}{2} \\
&< 0,
\end{aligned}$$

where we used the fact $d(\omega^{1,*}, \omega^{2,*}) \le d(\omega^1, \omega^2)$. This is a contradiction with $\omega^n \in L_\alpha^n$, and so $F_\alpha \subseteq \Omega_\alpha^+$. □

Let $\Omega_x = \{\omega \in \Omega : \omega_0 = x\}$. Thus measurability of functions g_2 and g_3 is obvious by noting Ω_x is a measurable set of Ω. In detail, by Lemma 2.5, set $\Omega_0 \cap U$ is a measurable subset of Ω, so a measurable subset of Ω_0. Therefore function g_2 is Borel-measurable. Based on Lemma 2.6, set $\Omega_x \cap \Omega_\alpha^+$ is a measurable subset of Ω, so a measurable subset of Ω_x, which results in measurability of g_3.

Chapter 3

Well-posedness of the Approximate Equation

In the current chapter, we first state three estimates (upper bound, coercivity, commutator) of the operator M^ϵ. Next, we construct the well-posedness theory of our approximate equation, namely, uniqueness and existence of non-negative solution.

3.1 Estimates of the collision operators

In this section, we state three estimates of the operator M^ϵ, as defined in (1.6), which will be used frequently in the rest of the thesis. We begin with upper bound of the collision operator M^ϵ.

Theorem 3.1. *Suppose the kernel B satisfies the Assumption (A-1)-(A-4), and Q^ϵ is the collision operator associated to the collision kernel B^ϵ. Let $w_1, w_2 \in \mathbb{R}$ with $w_1 + w_2 \geq \gamma + 2$, $a_1, a_2 \geq 0$ with $a_1 + a_2 = 2s$ and $b_1, b_2 \geq 0$ with $b_1 + b_2 = 2$. Then for suitable functions g, h and f, the estimate below holds uniformly in ϵ:*

$$|\langle M^\epsilon(g,h), f\rangle| \lesssim ||g||_{L^1_w}(||h||_{H^{a_1}_{w_1}}||f||_{H^{a_2}_{w_2}} + \epsilon^{2-2s}||h||_{H^{b_1}_{w_1}}||f||_{H^{b_2}_{w_2}}), \tag{3.1}$$

where $w = \gamma + 2 + (-w_1)^+ + (-w_2)^+$.

Proof. For the cutoff Boltzmann operator Q^ϵ, as in [54], for any $w_1, w_2 \in \mathbb{R}$ with $w_1 + w_2 \geq \gamma + 2$, there holds

$$|\langle Q^\epsilon(g,h), f\rangle| \lesssim ||g||_{L^1_{\gamma+2s+(-w_1)^+ + (-w_2)^+}}||h||_{H^{a_1}_{w_1}}||f||_{H^{a_2}_{w_2}}. \tag{3.2}$$

Again from [54], we have

$$|\langle Q_L(g,h),f\rangle| \lesssim ||g||_{L^1_{\gamma+2+(-w_1)^+ +(-w_2)^+}} ||h||_{H^{b_1}_{w_1}} ||f||_{H^{b_2}_{w_2}}. \tag{3.3}$$

Patching together the above two estimates, the estimate (3.1) follows accordingly. □

We now turn to coercivity estimate of the operator.

Theorem 3.2. *Suppose the collision kernel B satisfies the Assumption (A-1)-(A-4), and Q^ϵ is the collision operator associated to the collision kernel B^ϵ. Suppose function g is nonnegative and satisfies*

$$||g||_{L^1_2} + ||g||_{LlogL} < \infty, \tag{3.4}$$

then there exists constants $C_1(g)$ and $C_2(g)$ depending only on $||g||_{L^1_1}$ and $||g||_{LlogL}$ such that

$$-\langle M^\epsilon(g,f),f\rangle \geq C_1(g)||f||^2_{\epsilon,\gamma/2} - C_2(g)||f||^2_{L^2_{\gamma/2}}. \tag{3.5}$$

Proof. For the cutoff Boltzmann operator Q^ϵ, with a similar argument as in [1], one has

$$-\langle Q^\epsilon(g,f),f\rangle \geq C_1(g)||W^\epsilon(D)f||^2_{L^2_{\gamma/2}} - C_2(g)||f||^2_{L^2_{\gamma/2}}.$$

For the Landau operator Q_L, by [26], there holds

$$-\langle Q_L(g,f),f\rangle_v \geq C_1(g)||f||^2_{H^1_{\gamma/2}} - C_2(g)||f||^2_{L^2_{\gamma/2}}. \tag{3.6}$$

By noting that

$$||W^\epsilon(D)f||^2_{L^2_{\gamma/2}} + \epsilon^{2-2s}||f||^2_{H^1_{\gamma/2}} \sim ||f||^2_{\epsilon,\gamma/2},$$

the coercivity estimate (3.5) follows immediately. □

In the last, we move to commutator estimates. We first give the commutator estimate of the cutoff Boltzmann operator Q^ϵ as a lemma.

Lemma 3.1. *Suppose the kernel B satisfies the Assumption (A-1)-(A-4), and Q^ϵ is the collision operator associated to the collision kernel B^ϵ. Let $N_2, N_3 \in \mathbb{R}$ and $l \geq 0$ with $N_2 + N_3 \geq l + \gamma$, and let $N_1 = |N_2| + |N_3| + \max\{|l-1|, |l-2|\}$. Then for suitable functions g, h and f, the estimate below holds uniformly in ϵ:*

$$|\langle Q^\epsilon(g, h\langle v\rangle^l) - Q^\epsilon(g,h)\langle v\rangle^l, f\rangle| \lesssim ||g||_{L^1_{N_1}} ||h||_{H^s_{N_2}} ||f||_{L^2_{N_3}}. \tag{3.7}$$

Proof. One may refer to [20] for a proof. □

The next lemma is the commutator estimate of the Landau operator Q_L.

Lemma 3.2. *Let $N_2, N_3 \in \mathbb{R}$ and $l \geq 0$ with $N_2 + N_3 \geq l + \gamma$. Then for suitable functions g, h and f, the estimate below holds true:*

$$|\langle Q_L(g, h\langle v\rangle^l) - Q_L(g,h)\langle v\rangle^l, f\rangle| \leq \Lambda C(l) ||g||_{L^1_{\gamma+3}} ||h||_{H^1_{N_2}} ||f||_{L^2_{N_3}}, \tag{3.8}$$

where $C(l) = \max\{2l^2 + 12l, 20l - 2l^2\}$.

Proof. We define as usual the following quantities in 3-dimension:

$$b_i(z) = \sum_{j=1}^3 \partial_j a_{ij}(z) = -2\Lambda |z|^\gamma z_i, \qquad c(z) = \sum_{i,j=1}^3 \partial_{ij} a_{ij}(z) = -2\Lambda(\gamma+3)|z|^\gamma.$$

Hence the Landau operator Q_L can be rewritten as:

$$Q_L(g,h) = \sum_{i,j=1}^3 (a_{ij} * g)\partial_{ij} h - (c * g)h = \sum_{i=1}^3 \partial_i [\sum_{j=1}^3 (a_{ij} * g)\partial_j h - (b_i * g)h].$$

Thus

$$\begin{aligned} D(g,h,f;l) &\stackrel{\text{def}}{=} \langle Q_L(g, h\langle v\rangle^l) - Q_L(g,h)\langle v\rangle^l, f\rangle \\ &= \sum_{i,j=1}^3 \langle a_{ij} * g, f\partial_{ij}(h\langle v\rangle^l) - f\langle v\rangle^l \partial_{ij} h\rangle. \end{aligned}$$

It is easy to check

$$\partial_{ij}(h\langle v\rangle^l) - \langle v\rangle^l \partial_{ij} h = l\langle v\rangle^{l-2}(v_i\partial_j h + v_j \partial_i h) + l\langle v\rangle^{l-2}[(l-2)\frac{v_i v_j}{\langle v\rangle^2} + \delta_{ij}]h.$$

Thus we have

$$\begin{aligned}
D(g,h,f;l) &= l\int_{\mathbb{R}^6} g_* f\langle v\rangle^{l-2}[\sum_{i,j} a_{ij}(v-v_*)(v_i\partial_j h + v_j\partial_i h)]dvdv_* \\
&\quad +l(l-2)\int_{\mathbb{R}^6} g_* hf\langle v\rangle^{l-2}\frac{\sum_{i,j} a_{ij}(v-v_*)v_iv_j}{\langle v\rangle^2}dvdv_* \\
&\quad +l\int_{\mathbb{R}^6} g_* hf\langle v\rangle^{l-2}\sum_i a_{ii}(v-v_*)dvdv_*.
\end{aligned}$$

Considering the following facts

$$\begin{aligned}
\sum_{i,j=1}^{3} a_{ij}(v-v_*)v_i\partial_j h &= \sum_{i,j=1}^{3} a_{ij}(v-v_*)v_j\partial_i h \\
&= (\nabla h)^T a(v-v_*)v \\
&= (\nabla h)^T a(v-v_*)v_*,
\end{aligned}$$

and

$$\sum_{i,j=1}^{3} a_{ij}v_iv_j = \Lambda|v-v_*|^\gamma(|v|^2|v_*|^2-(v\cdot v_*)^2),$$

and

$$\sum_i a_{ii} = 2\Lambda|v-v_*|^{\gamma+2},$$

we get

$$\begin{aligned}
D(g,h,f;l) &= 2l\int_{\mathbb{R}^6} g_* f\langle v\rangle^{l-2}(\nabla h)^T a(v-v_*)v_* dvdv_* \\
&\quad +\Lambda l(l-2)\int_{\mathbb{R}^6} g_* hf\langle v\rangle^{l-2}\frac{|v-v_*|^\gamma(|v|^2|v_*|^2-(v\cdot v_*)^2)}{\langle v\rangle^2}dvdv_* \\
&\quad +2\Lambda l\int_{\mathbb{R}^6} g_* hf\langle v\rangle^{l-2}|v-v_*|^{\gamma+2}dvdv_* \\
&\stackrel{\text{def}}{=} \mathfrak{I}_1+\mathfrak{I}_2+\mathfrak{I}_3.
\end{aligned}$$

Thanks to

$$|a(v-v_*)v_*| \le 4\Lambda\langle v_*\rangle^{\gamma+3}\langle v\rangle^{\gamma+2},$$

we have

$$|\mathfrak{I}_1| \le 8\Lambda l||g||_{L^1_{\gamma+3}}||h||_{H^1_{N_2}}||f||_{L^2_{N_3}},$$

provided $N_2+N_3 \geq l+\gamma$. Similarly, if $N_2+N_3 \geq l+\gamma$, there holds

$$|\mathfrak{I}_3| \leq 8\Lambda l ||g||_{L^1_{\gamma+2}} ||h||_{L^2_{N_2}} ||f||_{L^2_{N_3}}.$$

With the help of the fact

$$\frac{|v-v_*|^\gamma(|v|^2|v_*|^2-(v\cdot v_*)^2)}{\langle v\rangle^2} \leq 2\langle v_*\rangle^{\gamma+2}\langle v\rangle^\gamma,$$

we have

$$|\mathfrak{I}_2| \leq 2\Lambda l|l-2| ||g||_{L^1_{\gamma+2}} ||h||_{L^2_{N_2}} ||f||_{L^2_{N_3}},$$

provided $N_2+N_3 \geq l-2+\gamma$. Putting together the above estimates, if $N_2+N_3 \geq l+\gamma$, the inequality

$$|D(g,h,f;l)| \leq \Lambda \max\{2l^2+12l, 20l-2l^2\} ||g||_{L^1_{\gamma+3}} ||h||_{H^1_{N_2}} ||f||_{L^2_{N_3}}$$

follows accordingly. □

In the end of this section, we state the commutator estimate of the operator M^ϵ.

Theorem 3.3. *Suppose the collision kernel B satisfies the Assumption (A-1)-(A-4), and Q^ϵ is the collision operator associated to the collision kernel B^ϵ. Let $N_2, N_3 \in \mathbb{R}$ and $l \geq 0$ with $N_2+N_3 \geq l+\gamma$, and let $N_1 = \max\{|N_2|+|N_3|+\max\{|l-1|, |l-2|\}, \gamma+3\}$. Then for suitable functions g, h and f, the estimate below holds uniformly in ϵ:*

$$|\langle M^\epsilon(g, h\langle v\rangle^l) - M^\epsilon(g,h)\langle v\rangle^l, f\rangle| \lesssim ||g||_{L^1_{N_1}}(||h||_{H^s_{N_2}} + \epsilon^{2-2s}||h||_{H^1_{N_2}})||f||_{L^2_{N_3}}. \tag{3.9}$$

Proof. The commutator estimate (3.9) follows from Lemma 3.1 and Lemma 3.2. □

3.2 Well-posedness of approximate equation (1.7): existence and uniqueness

In the present section, we will show that (1.7) admits a non-negative, unique and smooth solution if the initial data is regular enough. To do that, we separate the proof into three steps. In the first step, we prove that the linear equation to (1.7) admits a non-negative and regular solution. Then in the next step, by using Picard iteration scheme, we get a local well-posedness result. In the final step, we improve the well-posedness result to be global with the help of the a priori propagation estimates of the solution, which is derived from the symmetric property of the collision operator.

3.2.1 Well-posedness of linear equation to (1.7)

Throughout this subsection, $\epsilon > 0$ is a fixed but small enough number. In the following, given a known function g, we construct a non-negative solution to the linear equation:

$$\begin{cases} \partial_t f = Q^\epsilon(g, f) + \epsilon^{2-2s} Q_L(g, f) \\ f|_{t=0} = f_0. \end{cases} \tag{3.10}$$

Let us define two operators:

$$Q^{\epsilon+}(g, h) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B^\epsilon(v - v_*, \sigma) g'_* h' d\sigma dv_*,$$

$$Q^{\epsilon-}(g, h) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B^\epsilon(v - v_*, \sigma) g_* h d\sigma dv_* = \mathcal{L}(g)h.$$

Then we have $Q^\epsilon = Q^{\epsilon+} - Q^{\epsilon-}$, so we call $Q^{\epsilon+}$ the gain operator and $Q^{\epsilon-}$ the loss operator.

We first give a proposition, which shall be used in the current subsection, the next subsection and also the next chapter.

Proposition 3.1. *Let $p \geq 2, n = \frac{v-v_*}{|v-v_*|}, u = \frac{v+v_*}{|v+v_*|}, h = \sqrt{|v|^2|v_*|^2 - (v \cdot v_*)^2}, j = \frac{u-(u\cdot n)n}{|u-(u\cdot n)n|}$, and $E(\theta) = \langle v\rangle^2 \cos^2 \frac{\theta}{2} + \langle v_*\rangle^2 \sin^2 \frac{\theta}{2}$. Suppose ω is the vector such that $\sigma = \cos\theta n + \sin\theta\omega$, then there holds*

$$\begin{aligned} \langle v'\rangle^{2p} - \langle v\rangle^{2p} \leq& -\langle v\rangle^{2p}(1 - \cos^{2p} \frac{\theta}{2}) + \langle v_*\rangle^{2p} \sin^{2p} \frac{\theta}{2} \\ &+ p(E(\theta))^{p-1} h(j \cdot \omega) \sin\theta \\ &+ (\frac{1}{2} \max\{2^{p-3}, 1\} p(p-1) + 2^{p-1}) \langle v_*\rangle^{2p-2} \langle v\rangle^{2p-2} \sin^2\theta. \end{aligned} \tag{3.11}$$

Proof. It is not difficult to check $\langle v'\rangle^2 = E(\theta) + h(j\cdot\omega)\sin\theta$. By Taylor expansion, we have

$$\begin{aligned}\langle v'\rangle^{2p} &= (E(\theta))^p + p(E(\theta))^{p-1}h(j\cdot\omega)\sin\theta \\ &\quad + p(p-1)(h(j\cdot\omega)\sin\theta)^2\int_0^1(1-\kappa)(E(\theta)+\kappa h(j\cdot\omega)\sin\theta)^{p-2}d\kappa. \\ &\stackrel{\text{def}}{=} \mathfrak{M}_1+\mathfrak{M}_2+\mathfrak{M}_3.\end{aligned}$$

For the last term $\mathfrak{M}_3$, we have for any $\kappa\in[0,1]$:

$$\begin{aligned}E(\theta)+\kappa h(j\cdot\omega)\sin\theta &\le (\langle v\rangle^2+\langle v_*\rangle^2)(1-\frac{1-\kappa}{4}\sin^2\theta) \\ &\le \langle v\rangle^2+\langle v_*\rangle^2.\end{aligned}$$

Together with $h^2\le\langle v\rangle^2\langle v_*\rangle^2$, we arrive at

$$\begin{aligned}\mathfrak{M}_3 &\le p(p-1)\langle v\rangle^2\langle v_*\rangle^2(\langle v\rangle^2+\langle v_*\rangle^2)^{p-2}\sin^2\theta\int_0^1(1-\kappa)d\kappa \\ &\le \frac{1}{2}\max\{2^{p-3},1\}p(p-1)\langle v\rangle^{2p-2}\langle v_*\rangle^{2p-2}\sin^2\theta.\end{aligned}$$

For the term $\mathfrak{M}_1$, we have

$$\begin{aligned}&(\langle v\rangle^2\cos^2\frac{\theta}{2}+\langle v_*\rangle^2\sin^2\frac{\theta}{2})^p \\ \le& \sum_{k=1}^{k_p}\binom{p}{k}\{\langle v\rangle^{2k}\cos^{2k}\frac{\theta}{2}\langle v_*\rangle^{2(p-k)}\sin^{2(p-k)}\frac{\theta}{2}+\langle v\rangle^{2(p-k)}\cos^{2(p-k)}\frac{\theta}{2}\langle v_*\rangle^{2k}\sin^{2k}\frac{\theta}{2}\} \\ \le& \langle v\rangle^{2p}\cos^{2p}\frac{\theta}{2}+\langle v_*\rangle^{2p}\sin^{2p}\frac{\theta}{2}+2^p\langle v\rangle^{2p-2}\langle v_*\rangle^{2p-2}\sin^2\frac{\theta}{2}.\end{aligned}$$

Combining $\mathfrak{M}_1,\mathfrak{M}_2,\mathfrak{M}_3$, we arrive at (3.11). □

We begin with an equation which shall be used to construct solution to the linear equation (3.10).

Lemma 3.3. *Let $g,h\ge0$ be suitable functions. Suppose f^ϵ is the solution to the following equation*

$$\begin{cases}\partial_t f = Q^{\epsilon+}(g,h) - Q^{\epsilon-}(g,f) + \epsilon^{2-2s}Q_L(g,f) \\ f|_{t=0} = f_0 \ge 0.\end{cases} \tag{3.12}$$

Then $f^\epsilon(t)\ge0$ for any $t\ge0$.

Proof. Denote $f_-^\epsilon = \min\{0, f^\epsilon\} \le 0$, then we have $f_-^\epsilon|_{t=0} = 0$, and

$$\frac{d}{dt}(\frac{1}{2}||f_-^\epsilon||_{L^2}^2) + \int_{\mathbb{R}^3} \mathcal{L}(g)(f_-^\epsilon)^2 dv = \int_{\mathbb{R}^3} Q^{\epsilon+}(g,h) f_-^\epsilon dv + \epsilon^{2-2s}\langle Q_L(g,f^\epsilon), f_-^\epsilon\rangle.$$

Since $g, h \ge 0$ and $f_-^\epsilon \le 0$, it is clear that

$$\int_{\mathbb{R}^3} Q^{\epsilon+}(g,h) f_-^\epsilon dv \le 0.$$

By the definition of Q_L, we have

$$\begin{aligned}\langle Q_L(g,f^\epsilon), f_-^\epsilon\rangle &= -\int_{\mathbb{R}^6} g_*(\nabla f_-^\epsilon)^T a(v-v_*)\nabla f_-^\epsilon dv dv_* \\ &\quad +\Lambda(\gamma+3)\int_{\mathbb{R}^6} |v-v_*|^\gamma g_*(f_-^\epsilon)^2 dv dv_* \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2.\end{aligned}$$

Since a is a positive semi-definite matrix, we have $\mathfrak{I}_1 \le 0$. By assumption (A-2), there holds $\int_{\mathbb{S}^2} b^\epsilon(\cos\theta)d\sigma \sim \frac{\epsilon^{-2s}}{s}$. Therefore, there exists $\epsilon_* > 0$ such that, for any $0 < \epsilon \le \epsilon_*$,

$$\epsilon^{2-2s}\mathfrak{I}_2 \le \frac{1}{2}\int_{\mathbb{R}^3} \mathcal{L}(g)(f_-^\epsilon)^2 dv.$$

Finally, we arrive at

$$\frac{d}{dt}(\frac{1}{2}||f_-^\epsilon||_{L^2}^2) + \frac{1}{2}\int_{\mathbb{R}^3} \mathcal{L}(g)(f_-^\epsilon)^2 dv \le 0.$$

Thus $||f_-^\epsilon(t)||_{L^2} = 0$ for any $t \ge 0$, which implies $f^\epsilon(t) \ge 0$ for any $t \ge 0$. □

Now we are ready to construct a solution to the linear equation (3.10).

Lemma 3.4. *Let $l \ge 4, T > 0$ be real numbers. Suppose the non-negative datum $f_0 \in H^5_{l+3\gamma/2+10} \cap L^1_{l+5\gamma/2+16}$ with $||f_0||_{L^1} > 0$. Suppose $g(t,v)$ is a non-negative function satisfying*

$$M = \sup_{0\le t\le T} ||g(t)||_{H^5_{2l+3\gamma+22}\cap L^1_{l+5\gamma/2+16}} + \int_0^T ||g(t)||_{L^1_{l+7\gamma/2+16}} dt < \infty,$$

and

$$m = \inf_{0\le t\le T} ||g(t)||_{L^1} > 0,$$

then (3.10) *has a unique non-negative solution* f *in* $L^\infty([0,T];L^1_{l+5\gamma/2+16}\cap H^5_{l+\gamma+10})$ $\bigcap L^1([0,T];L^1_{l+7\gamma/2+16})$.

Proof. Let us define a sequence of functions $\{f^n\}_{n\in\mathbb{N}}$ by

$$\begin{cases} f^0(t)=f_0, \quad \text{for any } t\geq 0; \\ \partial_t f^n = Q^{\epsilon+}(g,f^{n-1})-Q^{\epsilon-}(g,f^n)+\epsilon^{2-2s}Q_L(g,f^n), \quad n\geq 1 \\ f^n|_{t=0}=f_0. \end{cases} \tag{3.13}$$

According to Lemma 3.3, we have $f^n\geq 0$.

Step 1: (Uniform Upper Bound)

Step 1.1: (Uniform Upper Bound in L^1_l)

In this step, we employ the energy method to get a uniform upper bound of L^1_l norm of $\{f^n\}_n$ with respect to n. Applying the basic inequality (1.10), for any $\eta>0$, there holds

$$\begin{aligned} |v-v_*|^\gamma \leq (|v|^2+2|v||v_*|+|v_*|^2)^{\frac{\gamma}{2}} &\leq ((1+\eta)|v|^2+(1+\frac{1}{\eta})|v_*|^2)^{\frac{\gamma}{2}} \\ &\leq (1+\eta)^{\frac{\gamma}{2}}\langle v\rangle^\gamma+(1+\frac{1}{\eta})^{\frac{\gamma}{2}}\langle v_*\rangle^\gamma. \end{aligned}$$

Also one has

$$\langle v'\rangle^l \leq (1+|v|^2+|v_*|^2)^{\frac{l}{2}} \leq \langle v_*\rangle^l+2^l\langle v\rangle^{l-2}\langle v_*\rangle^{l-2}+\langle v\rangle^l. \tag{3.14}$$

Thanks to the above two facts, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} Q^{\epsilon+}(g,f^{n-1})(v)\langle v\rangle^l dv &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b^\epsilon(\cos\theta)|v-v_*|^\gamma g_* f^{n-1}\langle v'\rangle^l dv dv_* d\sigma \\ &\leq (1+\eta)^{\frac{\gamma}{2}}A^\epsilon||g||_{L^1}||f^{n-1}||_{L^1_{l+\gamma}} \\ &\quad +(1+\frac{1}{\eta})^{\frac{\gamma}{2}}A^\epsilon||g||_{L^1_{l+\gamma}}||f^{n-1}||_{L^1} \\ &\quad +C(l,\gamma,\eta)A^\epsilon||g||_{L^1_l}||f^{n-1}||_{L^1_l}, \end{aligned} \tag{3.15}$$

where $A^\epsilon=\int_{\mathbb{S}^2}b^\epsilon(\cos\theta)d\sigma$. It is easy to check

$$\begin{aligned} \langle v\rangle^\gamma &= (1+|v-v_*+v_*|^2)^{\frac{\gamma}{2}} \leq (1+(1+\frac{1}{\eta})|v_*|^2+(1+\eta)|v-v_*|^2)^{\frac{\gamma}{2}} \\ &\leq (1+\frac{1}{\eta})^{\gamma/2}\langle v_*\rangle^\gamma+(1+\eta)^{\gamma/2}|v-v_*|^\gamma. \end{aligned}$$

That is, for any $\eta > 0$, there holds

$$|v - v_*|^\gamma \geq \frac{\langle v\rangle^\gamma}{(1+\eta)^{\gamma/2}} - \eta^{-\gamma/2}\langle v_*\rangle^\gamma. \tag{3.16}$$

Then we obtain

$$\int_{\mathbb{R}^3} Q^{\epsilon-}(g, f^n)(v)\langle v\rangle^l dv \geq \frac{A^\epsilon}{(1+\eta)^{\gamma/2}}||g||_{L^1}||f^n||_{L^1_{l+\gamma}} - \eta^{-\gamma/2}A^\epsilon||g||_{L^1_\gamma}||f^n||_{L^1_l}. \tag{3.17}$$

For the Landau operator, referring to [26], there holds

$$\begin{aligned}\int_{\mathbb{R}^3} Q_L(g, f^n)(v)\langle v\rangle^l dv &\leq l\Lambda \int_{\mathbb{R}^3} g_* f^n |v - v_*|^\gamma \langle v\rangle^{l-2}(-2|v|^2 + l|v_*|^2)dvdv_* \\ &\leq -l\Lambda||g||_{L^1}||f^n||_{L^1_{l+\gamma}} + (4l+2)l\Lambda||g||_{L^1_4}||f^n||_{L^1_l}.\end{aligned} \tag{3.18}$$

Putting together the above inequalities, we get

$$\begin{aligned}\frac{d}{dt}||f^n||_{L^1_l} \quad \leq \quad & -(\frac{A^\epsilon}{(1+\eta)^{\gamma/2}} + \epsilon^{2-2s}l\Lambda)||g||_{L^1}||f^n||_{L^1_{l+\gamma}} \\ & + (1+\eta)^{\frac{\gamma}{2}}A^\epsilon||g||_{L^1}||f^{n-1}||_{L^1_{l+\gamma}} + (1+\frac{1}{\eta})^{\frac{\gamma}{2}}A^\epsilon||g||_{L^1_{l+\gamma}}||f^{n-1}||_{L^1} \\ & + C(l,\gamma,\eta)A^\epsilon||g||_{L^1_l}||f^{n-1}||_{L^1_l} + \eta^{-\gamma/2}A^\epsilon||g||_{L^1_\gamma}||f^n||_{L^1_l} \\ & + \epsilon^{2-2s}(4l+2)l\Lambda||g||_{L^1_4}||f^n||_{L^1_l}.\end{aligned}$$

Observing that

$$\lim_{\eta\downarrow 0}\{(1+\eta)^{\frac{\gamma}{2}}A^\epsilon - \frac{A^\epsilon}{(1+\eta)^{\gamma/2}}\} = 0,$$

we can take a positive η such that,

$$(1+\eta)^{\frac{\gamma}{2}}A^\epsilon \leq \frac{A^\epsilon}{(1+\eta)^{\gamma/2}} + \frac{1}{2}\epsilon^{2-2s}l\Lambda.$$

With such a small η, let us denote $a = \frac{A^\epsilon}{(1+\eta)^{\gamma/2}} + \frac{1}{2}\epsilon^{2-2s}l\Lambda$, $\delta = \frac{1}{2}\epsilon^{2-2s}l\Lambda$, $K_1 = (1+\frac{1}{\eta})^{\frac{\gamma}{2}}A^\epsilon$, $K_2 = \sup_{0\leq s\leq T} C(l,\gamma,\eta)A^\epsilon||g(s)||_{L^1_l}$, $K_3 = \sup_{0\leq s\leq T}\{\eta^{-\gamma/2}A^\epsilon||g(s)||_{L^1_\gamma} + \epsilon^{2-2s}(4l+2)l\Lambda||g(s)||_{L^1_4}\}$. Therefore, we arrive at a neater inequality on the interval $[0,T]$,

$$\begin{aligned}\frac{d}{dt}||f^n||_{L^1_l} + (a+\delta)||g||_{L^1}||f^n||_{L^1_{l+\gamma}} \quad \leq \quad & a||g||_{L^1}||f^{n-1}||_{L^1_{l+\gamma}} \\ & + (K_1||g||_{L^1_{l+\gamma}} + K_2)||f^{n-1}||_{L^1_l} + K_3||f^n||_{L^1_l}.\end{aligned}$$

By defining $y^n(t) = e^{-K_3 t}||f^n(t)||_{L^1_l}$ and $x^n(t) = \int_0^t e^{-K_3 s}||g(s)||_{L^1}||f^n(s)||_{L^1_{l+\gamma}} ds$ for any $0 \leq t \leq T$ and $n \geq 0$, we derive that

$$y^n(t) + (a+\delta)x^n(t) \leq ||f_0||_{L^1_l} + ax^{n-1}(t) + \int_0^t (K_1||g(s)||_{L^1_{l+\gamma}} + K_2)y^{n-1}(s)ds.$$

Now denote $S^n(t) = \sum_{i=0}^n (\frac{a}{a+\delta})^i y^{n-i}(t)$ for $n \geq 0$, by recursive derivation and noting that $y^0(t) \leq ||f_0||_{L^1_l}$ and $x^0(t) \leq M\frac{1-e^{-K_3 t}}{K_3}||f_0||_{L^1_{l+\gamma}}$, we obtain

$$\begin{aligned} S^n(t) + (a+\delta)x^n(t) &\leq \sum_{i=0}^{n-1}(\frac{a}{a+\delta})^i||f_0||_{L^1_l} + (\frac{a}{a+\delta})^n y^0(t) + (\frac{a}{a+\delta})^{n-1}ax^0(t) \\ &\quad + \int_0^t (K_1||g(t_{n-1})||_{L^1_{l+\gamma}} + K_2)S^{n-1}(t_{n-1})dt_{n-1} \\ &\leq (\frac{a}{\delta}+1)||f_0||_{L^1_l} + aM\frac{1-e^{-K_3 t}}{K_3}||f_0||_{L^1_{l+\gamma}}(\frac{a}{a+\delta})^{n-1} \\ &\quad + \int_0^t (K_1||g(t_{n-1})||_{L^1_{l+\gamma}} + K_2)S^{n-1}(t_{n-1})dt_{n-1}. \end{aligned}$$

By further recursive derivation, we have

$$\begin{aligned} &S^n(t) + (a+\delta)x^n(t) \\ \leq\ & (\frac{a}{\delta}+1)||f_0||_{L^1_l}\sum_{i=0}^{n-1}\frac{(\int_0^t (K_1||g(s)||_{L^1_{l+\gamma}} + K_2)ds)^i}{i!} \\ &+aM\frac{1-e^{-K_3 t}}{K_3}||f_0||_{L^1_{l+\gamma}}(\frac{a}{a+\delta})^{n-1}\sum_{i=0}^{n-1}\frac{(\frac{a+\delta}{a}\int_0^t (K_1||g(s)||_{L^1_{l+\gamma}} + K_2)ds)^i}{i!} \\ &+\int_0^t\int_0^{t_{n-1}}\cdots\int_0^{t_1} S^0(t_0)\prod_{i=0}^{n-1}(K_1||g(t_i)||_{L^1_{l+\gamma}} + K_2)dt_{n-1}dt_{n-2}\cdots dt_0 \\ \leq\ & (\frac{a}{\delta}+1)||f_0||_{L^1_l}\exp(\int_0^t (K_1||g(s)||_{L^1_{l+\gamma}} + K_2)ds) \\ &+aM\frac{1-e^{-K_3 t}}{K_3}||f_0||_{L^1_{l+\gamma}}(\frac{a}{a+\delta})^{n-1}\exp(\frac{a+\delta}{a}\int_0^t (K_1||g(s)||_{L^1_{l+\gamma}} + K_2)ds). \end{aligned}$$

Noting that

$$||f^n(t)||_{L^1_l} \leq e^{K_3 t}S^n(t),$$

and

$$\int_0^t ||f^n(s)||_{L^1_{l+\gamma}} ds \leq m^{-1}e^{K_3 t}x^n(t),$$

and recalling the definition of constants K_1, K_2, K_3, we obtain

$$\begin{aligned}&\sup_n(||f^n(t)||_{L^1_l}+\int_0^t||f^n(s)||_{L^1_{l+\gamma}}ds)\\ &\leq C(||f_0||_{L^1_{l+\gamma}},t,\sup_{0\leq s\leq t}||g(s)||_{L^1_l},\int_0^t||g(s)||_{L^1_{l+\gamma}}ds).\end{aligned}\tag{3.19}$$

Step 1.2: (Uniform Upper Bound in L^2_l)

In this step, we show the uniform upper bound of L^2_l norm of $\{f^n\}_n$ with respect to n. It is not difficult to see that

$$\begin{aligned}\frac{d}{dt}(\frac{1}{2}||f^n||^2_{L^2_l}) &= \langle Q^{\epsilon+}(g,f^{n-1})-Q^{\epsilon-}(g,f^n)+\epsilon^{2-2s}Q_L(g,f^n),f^n\langle v\rangle^{2l}\rangle\\ &\overset{\text{def}}{=}\mathfrak{I}_1-\mathfrak{I}_2+\epsilon^{2-2s}\mathfrak{I}_3.\end{aligned}\tag{3.20}$$

By Cauchy-Schwartz inequality, there holds

$$\begin{aligned}\mathfrak{I}_1 &= \int B^\epsilon g_* f^{n-1}f^n(v')\langle v'\rangle^{2l}dvdv_*d\sigma\\ &\lesssim (\int B^\epsilon g_*(f^{n-1})^2\langle v'\rangle^{2l}dvdv_*d\sigma)^{1/2}\times(\int B^\epsilon g_*(f^n)^2(v')\langle v'\rangle^{2l}dv'dv_*d\sigma)^{1/2}\\ &\lesssim (A^\epsilon||g||_{L^1_{2l+\gamma}}||f^{n-1}||^2_{L^2_{l+\gamma/2}})^{1/2}\times(A^\epsilon||g||_{L^1}||f^n||^2_{L^2_{l+\gamma/2}})^{1/2}\\ &\lesssim A^\epsilon||g||_{L^1_{2l+\gamma}}||f^{n-1}||_{L^2_{l+\gamma/2}}||f^n||_{L^2_{l+\gamma/2}},\end{aligned}\tag{3.21}$$

where we have used the estimate (3.14) and the usual change of variable $v\to v'$. By direct calculation, we have

$$\mathfrak{I}_2=\int B^\epsilon g_*(f^n)^2\langle v\rangle^{2l}dvdv_*d\sigma\leq A^\epsilon||g||_{L^1_\gamma}||f^n||^2_{L^2_{l+\gamma/2}}.\tag{3.22}$$

By coercivity estimate (3.6) and commutator estimate (3.8) of the Landau operator, we have

$$\begin{aligned}\mathfrak{I}_3 &= \langle Q_L(g,f^n\langle v\rangle^l),f^n\langle v\rangle^l\rangle+\{\langle Q_L(g,f^n)\langle v\rangle^l-Q_L(g,f^n\langle v\rangle^l),f^n\langle v\rangle^l\rangle\}\\ &\leq -C_1(g)||f^n||^2_{H^1_{l+\gamma/2}}+C_2(g)||f^n||^2_{L^2_{l+\gamma/2}}\\ &\quad+\Lambda C(l)||g||_{L^1_{\gamma+3}}||f^n||_{H^1_{l+\gamma/2}}||f^n||_{L^2_{l+\gamma/2}}.\end{aligned}\tag{3.23}$$

Now patching together the inequalities (3.21), (3.22) and (3.23), and using the basic inequality (1.10), we have

$$\frac{d}{dt}(\frac{1}{2}||f^n||^2_{L^2_l})+\frac{C_1}{2}\epsilon^{2-2s}||f^n||^2_{H^1_{l+\gamma/2}}\leq\frac{C_1}{8}\epsilon^{2-2s}||f^{n-1}||^2_{L^2_{l+\gamma/2}}+K_1||f^n||^2_{L^2_{l+\gamma/2}},$$

where C_1, K_1 are some positive constants depending on m, M, ϵ. For any $\lambda, s > 0$, one has

$$||f||_{L^2}^2 \leq \lambda ||f||_{H^s}^2 + \frac{4\pi}{3}\lambda^{-\frac{3}{2s}}||f||_{L^1}^2. \tag{3.24}$$

With the help of the above inequality, we have

$$\frac{d}{dt}||f^n||_{L_l^2}^2 + \frac{C_1}{2}\epsilon^{2-2s}||f^n||_{H_{l+\gamma/2}^1}^2 \leq \frac{C_1}{4}\epsilon^{2-2s}||f^{n-1}||_{L_{l+\gamma/2}^2}^2 + K_1||f^n||_{L_{l+\gamma/2}^1}^2,$$

for some new constant K_1. By the previous step, with the uniform upper bound of $||f^n||_{L_{l+\gamma/2}^1}$, we have

$$\frac{d}{dt}||f^n||_{L_l^2}^2 + \frac{C_1}{2}\epsilon^{2-2s}||f^n||_{L_{l+\gamma/2}^2}^2 \leq \frac{C_1}{4}\epsilon^{2-2s}||f^{n-1}||_{L_{l+\gamma/2}^2}^2 + K_1K_2, \tag{3.25}$$

where K_2 is some constant depending on $||f_0||_{L_{l+3\gamma/2}^1}$ and uniform upper bound of $||g||_{L_{l+3\gamma/2}^1}$. Now we use the same technique as in the previous step. Integrating both sides with respect to time, for any $t_n \in [0, t]$, we obtain

$$\begin{aligned} ||f^n(t_n)||_{L_l^2}^2 + \frac{C_1}{2}\epsilon^{2-2s}\int_0^{t_n}||f^n(r)||_{L_{l+\gamma/2}^2}^2 dr \;\; &\leq \;\; \frac{C_1}{4}\epsilon^{2-2s}\int_0^{t_n}||f^{n-1}(r)||_{L_{l+\gamma/2}^2}^2 dr \\ &\quad + ||f_0||_{L_l^2}^2 + K_1K_2t. \end{aligned}$$

Now denote $S^n(t_n) = \sum_{i=0}^n (\frac{1}{2})^i ||f^{n-i}(t_n)||_{L_l^2}^2$ and $x^n(t_n) = C_1\epsilon^{2-2s}\int_0^{t_n}||f^n(r)||_{L_{l+\gamma/2}^2}^2 dr$ for $n \geq 0$, by recursive derivation and noting that $x^0(t_n) \leq C_1\epsilon^{2-2s}t||f_0||_{L_{l+\gamma/2}^2}^2$, we obtain, for $n \geq 1$,

$$\begin{aligned} S^n(t_n) + \frac{1}{2}x^n(t_n) \;\; &\leq \;\; \sum_{i=0}^n (\frac{1}{2})^i(||f_0||_{L_l^2}^2 + K_1K_2t) + \frac{C_1}{2^{n+1}}\epsilon^{2-2s}t||f_0||_{L_{l+\gamma/2}^2}^2 \\ &\leq \;\; 2||f_0||_{L_l^2}^2 + 2K_1K_2t + \frac{C_1}{4}\epsilon^{2-2s}t||f_0||_{L_{l+\gamma/2}^2}^2. \end{aligned}$$

By tracking the definitions of constants K_1, K_2, we obtain

$$\sup_{0\leq s\leq t}\sup_n ||f^n(s)||_{L_l^2} \leq C(||f_0||_{L_{l+3\gamma/2}^1}, ||f_0||_{L_{l+\gamma/2}^2}, t, \sup_{0\leq s\leq t}||g(s)||_{L_{2l+\gamma+2}^2}). \tag{3.26}$$

Step 1.3: (Uniform Upper Bound in H_l^m with $m \geq 1$)

Fix an α with $|\alpha| \leq m$, one has

$$\begin{aligned} \partial_t\partial_v^\alpha f^n \;\; &= \;\; \sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}[Q^{\epsilon+}(\partial_v^{\alpha_1}g, \partial_v^{\alpha_2}f^{n-1}) - Q^{\epsilon-}(\partial_v^{\alpha_1}g, \partial_v^{\alpha_2}f^n) \\ &\quad + \epsilon^{2-2s}Q_L(\partial_v^{\alpha_1}g, \partial_v^{\alpha_2}f^n)]. \end{aligned}$$

Then we have

$$\begin{aligned}
\frac{d}{dt}(\frac{1}{2}||\partial_v^\alpha f^n||^2_{L^2_l}) &= \sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}[\langle Q^{\epsilon+}(\partial_v^{\alpha_1}g,\partial_v^{\alpha_2}f^{n-1}),\partial_v^\alpha f^n\langle v\rangle^{2l}\rangle \\
&\quad -\langle Q^{\epsilon-}(\partial_v^{\alpha_1}g,\partial_v^{\alpha_2}f^n),\partial_v^\alpha f^n\langle v\rangle^{2l}\rangle \\
&\quad +\epsilon^{2-2s}\langle Q_L(\partial_v^{\alpha_1}g,\partial_v^{\alpha_2}f^n),\partial_v^\alpha f^n\langle v\rangle^{2l}\rangle] \\
&\stackrel{\text{def}}{=} \sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}[\mathfrak{I}_1(\alpha_1,\alpha_2)-\mathfrak{I}_2(\alpha_1,\alpha_2)+\epsilon^{2-2s}\mathfrak{I}_3(\alpha_1,\alpha_2)].
\end{aligned}$$

As the same as (3.21), we have

$$|\mathfrak{I}_1(\alpha_1,\alpha_2)|\lesssim A^\epsilon||g||_{H^m_{2l+\gamma+2}}||f^{n-1}||_{H^m_{l+\gamma/2}}||f^n||_{H^m_{l+\gamma/2}}$$

As the same as (3.22), we have

$$|\mathfrak{I}_2(\alpha_1,\alpha_2)|\lesssim A^\epsilon||g||_{H^m_{\gamma+2}}||f^n||^2_{H^m_{l+\gamma/2}}.$$

When $|\alpha_2|\le|\alpha|-1\le m-1$, by upper bound estimate (3.3) and commutator estimate (3.8) of the Landau operator, we have

$$\begin{aligned}
|\mathfrak{I}_3(\alpha_1,\alpha_2)| &\le |\langle Q_L(\partial_v^{\alpha_1}g,\partial_v^{\alpha_2}f^n\langle v\rangle^l),\partial_v^\alpha f^n\langle v\rangle^l\rangle| \\
&\quad +|\{\langle Q_L(\partial_v^{\alpha_1}g,f^n)\langle v\rangle^l-Q_L(\partial_v^{\alpha_1}g,\partial_v^{\alpha_2}f^n\langle v\rangle^l),\partial_v^\alpha f^n\langle v\rangle^l\rangle\}| \\
&\lesssim ||g||_{H^m_{\gamma+4}}||f^n||_{H^m_{l+\gamma/2+2}}||f^n||_{H^{m+1}_{l+\gamma/2}}+||g||_{H^m_{\gamma+5}}||f^n||^2_{H^m_{l+\gamma/2}}.
\end{aligned}$$

When $\alpha_2=\alpha$, as the same as (3.23), we have

$$\begin{aligned}
\mathfrak{I}_3(0,\alpha) &\le -C_1(g)||\partial_v^\alpha f^n||^2_{H^1_{l+\gamma/2}}+C_2(g)||\partial_v^\alpha f^n||^2_{L^2_{l+\gamma/2}} \\
&\quad +\Lambda C(l)||g||_{H^m_{\gamma+5}}||f^n||_{H^{m+1}_{l+\gamma/2}}||f^n||_{H^m_{l+\gamma/2}}.
\end{aligned}$$

Now patching together the above estimates and taking sum over $|\alpha|\le m$, we have

$$\frac{1}{2}\frac{d}{dt}||f^n||^2_{H^m_l}+\frac{C_1}{2}\epsilon^{2-2s}||f^n||^2_{H^{m+1}_{l+\gamma/2}}\le\frac{C_1}{4}\epsilon^{2-2s}||f^{n-1}||^2_{H^m_{l+\gamma/2}}+K_1||f^n||^2_{H^m_{l+\gamma/2+2}},$$

where C_1,K_1 are some positive constants depending on uniform upper bound of $||g||_{H^m_{2l+\gamma+2}}$ and uniform lower bound of $||g||_{L^1}$. Thanks to interpolation theory and the basic inequality (1.10), for any $\eta>0$, there exists some constant C_η such that

$$||f^n||^2_{H^m_{l+\gamma/2+2}}\le\eta||f^n||^2_{H^{m+1}_{l+\gamma/2}}+C_\eta||f^n||^2_{L^1_{l+\gamma/2+2m+6}},$$

thus we have

$$\frac{d}{dt}||f^n||^2_{H^m_l} + \frac{C_1}{2}\epsilon^{2-2s}||f^n||^2_{H^m_{l+\gamma/2}} \leq \frac{C_1}{8}\epsilon^{2-2s}||f^{n-1}||^2_{H^m_{l+\gamma/2}} + K_1||f^n||^2_{L^1_{l+\gamma/2+2m+6}},$$

for some new constant K_1. By the previous step, with the uniform upper bound of $||f^n||_{L^1_{l+\gamma/2+2m+6}}$, we have

$$\frac{d}{dt}||f^n||^2_{L^2_l} + \frac{C_1}{2}\epsilon^{2-2s}||f^n||^2_{L^2_{l+\gamma/2}} \leq \frac{C_1}{4}\epsilon^{2-2s}||f^{n-1}||^2_{L^2_{l+\gamma/2}} + K_1K_2, \tag{3.27}$$

where K_2 is some constant depending on $||f_0||_{L^1_{l+3\gamma/2+2m+6}}$ and uniform upper bound of $||g||_{L^1_{l+3\gamma/2+2m+6}}$. Noticing that inequality (3.27) has exactly the same structure as inequality (3.25), we have

$$\begin{aligned}\sup_{0\leq s\leq t}\sup_n ||f^n(s)||_{H^m_l} \leq & C(||f_0||_{L^1_{l+3\gamma/2+2m+6}}, ||f_0||_{H^m_{l+\gamma/2}}, t, \\ & \sup_{0\leq s\leq t}||g(s)||_{H^m_{2l+\gamma+2}}, \sup_{0\leq s\leq t}||g(s)||_{L^1_{l+3\gamma/2+2m+6}}).\end{aligned} \tag{3.28}$$

Step 2: (Cauchy Sequence)

In this step, we prove that $\{f^n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in L^1_l for any $t\geq 0$. Set $h^n = f^n - f^{n-1}$ for $n\geq 1$. Then for $n\geq 2$, we have

$$\begin{cases}\partial_t h^n = Q^{\epsilon+}(g,h^{n-1}) - Q^{\epsilon-}(g,h^n) + \epsilon^{2-2s}Q_L(g,h^n), \\ h^n|_{t=0} = 0.\end{cases} \tag{3.29}$$

Because we are uncertain about the sign of h^n, we have to introduce the sign function $sgn(h^n)$. Similar as in (3.15), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3} Q^{\epsilon+}(g,h^{n-1})(v)sgn(h^n)\langle v\rangle^l dv \\ \leq & \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b^\epsilon(\cos\theta)|v-v_*|^\gamma g_*|h^{n-1}|\langle v'\rangle^l dvdv_*d\sigma \\ \leq & (1+\eta)^{\frac{\gamma}{2}}A^\epsilon||g||_{L^1}||h^{n-1}||_{L^1_{l+\gamma}} \\ & + (1+\frac{1}{\eta})^{\frac{\gamma}{2}}A^\epsilon||g||_{L^1_{l+\gamma}}||h^{n-1}||_{L^1} \\ & + C(l,\gamma,\eta)A^\epsilon||g||_{L^1_l}||h^{n-1}||_{L^1_l}.\end{aligned} \tag{3.30}$$

Similar as in (3.17)

$$\begin{aligned}\int_{\mathbb{R}^3} Q^{\epsilon-}(g,h^n)(v)sgn(h^n)\langle v\rangle^l dv &= \int_{\mathbb{R}^3} b^\epsilon|v-v_*|^\gamma g_*|h^n|\langle v\rangle^l dvdv_*d\sigma \\ &\geq \frac{A^\epsilon}{(1+\eta)^{\gamma/2}}||g||_{L^1}||h^n||_{L^1_{l+\gamma}} \\ &\quad -\eta^{-\gamma/2}A^\epsilon||g||_{L^1_\gamma}||h^n||_{L^1_l}.\end{aligned} \tag{3.31}$$

For the inner product $\langle Q_L(g,h^n), sgn(h^n)\langle v\rangle^l\rangle$, we can approximate Landau operator by Boltzmann operators. Let $b_\lambda(\cos\theta)=\lambda^{2s-2}b(\cos\theta)\mathbf{1}_{\theta\leq\lambda}$ for each $\lambda\leq\frac{\pi}{2}$, such that

$$\lim_{\lambda\downarrow 0}\int_{\mathbb{S}^2} b_\lambda(\cos\theta)\sin^2\theta d\sigma=\Lambda.$$

Let Q_λ be the Boltzmann operator associated to the kernel $b_\lambda(\cos\theta)|v-v_*|^\gamma$, then by lemma 7.1 in [52], there holds

$$\begin{aligned}&|\langle Q_L(g,h^n), sgn(h^n)\langle v\rangle^l\rangle_v-\langle Q_\lambda(g,h^n), sgn(h^n)\langle v\rangle^l\rangle_v| \\ \lesssim\ &\lambda\|g\|_{H^3_{l+\gamma+12}}\|h^n\|_{H^5_{l+\gamma+10}}.\end{aligned} \tag{3.32}$$

By the uniform estimate (3.28) and our assumption on g and f_0, we have

$$\begin{aligned}\sup_{0\leq t\leq T}\sup_n\|h^n(t)\|_{H^5_{l+\gamma+10}}\leq &C(||f_0||_{L^1_{l+5\gamma/2+16}},||f_0||_{H^5_{l+3\gamma/2+10}},t, \\ &\sup_{0\leq s\leq t}||g(s)||_{H^5_{2l+3\gamma+22}},\sup_{0\leq s\leq t}||g(s)||_{L^1_{l+5\gamma/2+16}}).\end{aligned} \tag{3.33}$$

Thanks to Proposition 3.1, for $l\geq 4$, we derive that

$$\begin{aligned}&\langle Q_\lambda(g,h^n), sgn(h^n)\langle v\rangle^l\rangle_v \\ =\ &\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b_\lambda|v-v_*|^\gamma g_*h^n(sgn(h^n(v'))\langle v'\rangle^l-sgn(h^n(v))\langle v\rangle^l)dvdv_*d\sigma \\ \leq\ &\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b_\lambda|v-v_*|^\gamma g_*|h^n|(\langle v'\rangle^l-\langle v\rangle^l)dvdv_*d\sigma \\ \leq\ &-\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b_\lambda|v-v_*|^\gamma g_*|h^n|\langle v\rangle^l(1-\cos^l\frac{\theta}{2})dvdv_*d\sigma \\ &+\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b_\lambda|v-v_*|^\gamma g_*|h^n|\langle v_*\rangle^l\sin^l\frac{\theta}{2}dvdv_*d\sigma \\ &+C(l)\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b_\lambda|v-v_*|^\gamma g_*|h^n|\langle v\rangle^{l-2}\langle v_*\rangle^{l-2}\sin^2\frac{\theta}{2}dvdv_*d\sigma.\end{aligned}$$

For λ small enough, we have

$$\frac{\Lambda}{2}\leq\int_{\mathbb{S}^2} b_\lambda(\cos\theta)\sin^2\theta d\sigma\leq 2\Lambda.$$

Thus we have

$$\int_{\mathbb{S}^2} b_\lambda(\cos\theta)(1-\cos^l\frac{\theta}{2})d\sigma \ge \int_{\mathbb{S}^2} b_\lambda(\cos\theta)\sin^2\frac{\theta}{2}d\sigma \ge \frac{\Lambda}{8},$$

and

$$\int_{\mathbb{S}^2} b_\lambda(\cos\theta)\sin^2\frac{\theta}{2}d\sigma \le \int_{\mathbb{S}^2} b_\lambda(\cos\theta)\frac{\sin^2\theta}{2}d\sigma \le \Lambda,$$

and finally

$$\int_{\mathbb{S}^2} b_\lambda(\cos\theta)\sin^l\frac{\theta}{2}d\sigma \le \frac{\lambda^2}{4}\int_{\mathbb{S}^2} b_\lambda(\cos\theta)\frac{\sin^2\theta}{2}d\sigma \le \frac{\lambda^2}{4}\Lambda.$$

With the help of the above three inequalities, we arrive at

$$\begin{aligned}\langle Q_\lambda(g,h^n), sgn(h^n)\langle v\rangle^l\rangle_v \;&\le\; -\frac{\Lambda}{16}||g||_{L^1}||h^n||_{L^1_{l+\gamma}} + \frac{\Lambda}{8}||g||_{L^1_\gamma}||h^n||_{L^1_l} \\ &\quad +C(l)\Lambda||g||_{L^1_l}||h^n||_{L^1_l} + \frac{\lambda^2}{4}\Lambda||g||_{L^1_{l+\gamma}}||h^n||_{L^1_\gamma}.\end{aligned}$$

Let λ tend to 0, by (3.32) and the uniform estimate (3.33), we have

$$\begin{aligned}\langle Q_L(g,h^n), sgn(h^n)\langle v\rangle^l\rangle_v \le -\frac{\Lambda}{16}||g||_{L^1}||h^n||_{L^1_{l+\gamma}} + \frac{\Lambda}{8}||g||_{L^1_\gamma}||h^n||_{L^1_l} \\ + C(l)\Lambda||g||_{L^1_l}||h^n||_{L^1_l}.\end{aligned} \tag{3.34}$$

Choose η small enough such that

$$(1+\eta)^{\frac{\gamma}{2}}A^\epsilon \le \frac{A^\epsilon}{(1+\eta)^{\gamma/2}} + \frac{1}{32}\epsilon^{2-2s}\Lambda,$$

and denote $a = \frac{A^\epsilon}{(1+\eta)^{\gamma/2}} + \frac{1}{32}\epsilon^{2-2s}\Lambda$, $\delta = \frac{1}{32}\epsilon^{2-2s}\Lambda$. Putting altogether (3.30), (3.31) and (3.34), we obtain

$$\begin{aligned}\frac{d}{dt}||h^n||_{L^1_l} \;&\le\; -(a+\delta)||g||_{L^1}||h^n||_{L^1_{l+\gamma}} + a||g||_{L^1}||h^{n-1}||_{L^1_{l+\gamma}} \\ &\quad +(1+\frac{1}{\eta})^{\frac{\gamma}{2}}A^\epsilon||g||_{L^1_{l+\gamma}}||h^{n-1}||_{L^1} + C(l,\gamma,\eta)A^\epsilon||g||_{L^1_l}||h^{n-1}||_{L^1_l} \\ &\quad +\eta^{-\gamma/2}A^\epsilon||g||_{L^1_\gamma}||h^n||_{L^1_l} + \frac{\Lambda}{8}\epsilon^{2-2s}||g||_{L^1_\gamma}||h^n||_{L^1_l} \\ &\quad +C(l)\epsilon^{2-2s}\Lambda||g||_{L^1_l}||h^n||_{L^1_l}.\end{aligned}$$

For ease of notation, denote $K_1 = (1+\frac{1}{\eta})^{\frac{\gamma}{2}} A^\epsilon$, $K_2 = C(l,\gamma,\eta)A^\epsilon \sup_{0\le s\le t} ||g(s)||_{L^1_l}$ and $K_3 = (\eta^{-\gamma/2}A^\epsilon + \frac{\Lambda}{8}\epsilon^{2-2s} + C(l)\epsilon^{2-2s}\Lambda)\sup_{0\le s\le t}||g(s)||_{L^1_l}$. Then we have a much neater inequality on the interval $[0,t]$,

$$\begin{aligned} &\frac{d}{dt}||h^n||_{L^1_l} + (a+\delta)||g||_{L^1}||h^n||_{L^1_{l+\gamma}} \\ &\le a||g||_{L^1}||h^{n-1}||_{L^1_{l+\gamma}} + K_3||h^n||_{L^1_l} + (K_1||g||_{L^1_{l+\gamma}} + K_2)||h^{n-1}||_{L^1_l}. \end{aligned} \tag{3.35}$$

Using the same technique as in the previous step, define $y^n(t_n) = e^{-K_3 t_n}||h^n(t_n)||_{L^1_l}$ and $x^n(t_n) = \int_0^{t_n} e^{-K_3 s}||g(s)||_{L^1}||h^n(s)||_{L^1_{l+\gamma}} ds$, for $n \ge 1$ and $t_n \in [0,t]$. Then for $n \ge 2$, we derive that

$$y^n(t_n) + (a+\delta)x^n(t_n) \le ax^{n-1}(t_n) + \int_0^{t_n} (K_1||g(s)||_{L^1_{l+\gamma}} + K_2)y^{n-1}(s)ds,$$

where we have used the initial condition $h^n(0) = 0$. Now for $n \ge 1$ and $s \in [0,t]$, denoting $S^n(s) = \sum_{i=0}^{n-1}(\frac{a}{a+\delta})^i y^{n-i}(s)$, by recursive derivation, we obtain

$$\begin{aligned} S^n(t_n) + (a+\delta)x^n(t_n) \le\ & (\frac{a}{a+\delta})^{n-1}y^1(t_n) + (\frac{a}{a+\delta})^{n-2}ax^1(t_n) \\ & + \int_0^t (K_1||g(t_{n-1})||_{L^1_{l+\gamma}} + K_2)S^{n-1}(t_{n-1})dt_{n-1}. \end{aligned}$$

By estimate (3.19), we have

$$\begin{aligned} \sup_{0\le t_n\le t}\{y^1(t_n) + x^1(t_n)\} &\le C(||f_0||_{L^1_{l+\gamma}}, t, \sup_{0\le s\le t}||g(s)||_{L^1_l}, \int_0^t ||g(s)||_{L^1_{l+\gamma}} ds) \\ &\stackrel{\text{def}}{=} C(t). \end{aligned}$$

For ease of notation, for $n \ge 1$, let us define

$$b^n(t) = ((\frac{a}{a+\delta})^{n-1} + (\frac{a}{a+\delta})^{n-2}a)C(t).$$

Thus, by further recursive derivation, for any $t_n \in [0,t]$, we obtain

$$
\begin{aligned}
& S^n(t_n) + (a+\delta)x^n(t_n) \\
\leq\ & b^n(t) + \int_0^{t_n} (K_1||g(t_{n-1})||_{L^1_{l+\gamma}} + K_2) S^{n-1}(t_{n-1}) dt_{n-1} \\
\leq\ & \sum_{i=2}^{n} b^i(t) \frac{(\int_0^{t_n} (K_1||g(s)||_{L^1_{l+\gamma}} + K_2) ds)^{n-i}}{(n-i)!} \\
& + \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} S^1(t_1) \prod_{i=1}^{n-1} (K_1||g(t_i)||_{L^1_{l+\gamma}} + K_2) dt_{n-1} dt_{n-2} \cdots dt_1 \\
\leq\ & \sum_{i=1}^{n} b^i(t) \frac{(\int_0^{t_n} (K_1||g(s)||_{L^1_{l+\gamma}} + K_2) ds)^{n-i}}{(n-i)!},
\end{aligned}
$$

where we used the fact $S^1(t_1) \leq C(t) \leq (a+\delta+1)C(t) = b^1(t)$. Note that $b^n(t)$ is a geometric sequence and $b^n(t) = b^1(t)(\frac{a}{a+\delta})^{n-1}$ for any $n \geq 1$, thus we have

$$
\begin{aligned}
S^n(t_n) + (a+\delta)x^n(t_n) &\leq b^1(t) \sum_{i=1}^{n} (\frac{a}{a+\delta})^{i-1} \frac{(\int_0^{t_n} (K_1||g(s)||_{L^1_{l+\gamma}} + K_2) ds)^{n-i}}{(n-i)!} \\
&= b^1(t) (\frac{a}{a+\delta})^{n-1} \sum_{i=1}^{n} \frac{(\frac{a+\delta}{a} \int_0^{t_n} (K_1||g(s)||_{L^1_{l+\gamma}} + K_2) ds)^{n-i}}{(n-i)!} \\
&\leq b^1(t) (\frac{a}{a+\delta})^{n-1} \exp(\frac{a+\delta}{a} \int_0^{t_n} (K_1||g(s)||_{L^1_{l+\gamma}} + K_2) ds).
\end{aligned}
$$

By recalling the definitions of S^n and x^n, we arrive at

$$
\begin{aligned}
& \sup_{0\leq s\leq t} ||h^n(s)||_{L^1_l} + \int_0^t e^{K_3(t-s)} ||g(s)||_{L^1} ||h^n(s)||_{L^1_{l+\gamma}} ds \\
\leq\ & b^1(t)(\frac{a}{a+\delta})^{n-1} \exp(\frac{a+\delta}{a} \int_0^t (K_1||g(s)||_{L^1_{l+\gamma}} + K_2) ds + K_3 t).
\end{aligned}
$$

Since the series $\sum_n (\frac{a}{a+\delta})^{n-1}$ is finite, we conclude that $\{f^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty([0,t]; L^1_l) \cap L^1([0,t]; L^1_{l+\gamma})$. Due to the arbitrariness of $t \in [0,T]$, there is a function $f \in L^\infty([0,T]; L^1_l) \cap L^1([0,T]; L^1_{l+\gamma})$ such that

$$
\lim_{n\to\infty} \{ \sup_{0\leq s\leq T} ||f^n(s) - f(s)||_{L^1_l} + \int_0^T ||f^n(s) - f(s)||_{L^1_{l+\gamma}} ds \} = 0.
$$

It is obvious that f is the solution to (3.10). Thus the non-negativity of f is ensured by the non-negativity of f^n.

Step 3: (High Order Moments and Smoothness)

In this step, we prove the solution f constructed in the previous step actually lies in $L^\infty([0,T]; L^1_{l+5\gamma/2+16} \cap H^5_{l+\gamma+10}) \cap L^1([0,T]; L^1_{l+7\gamma/2+16})$. Let $q = l + 5\gamma/2 + 16$. By Proposition 3.1 and inequality (3.16), we first have

$$\begin{aligned}\int_{\mathbb{R}^3} Q^\epsilon(g,f)(v)\langle v\rangle^q dv &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b^\epsilon(\cos\theta)|v-v_*|^\gamma g_* f(\langle v'\rangle^q - \langle v\rangle^q) dv dv_* d\sigma \\ &\leq -\frac{C_\epsilon}{2}||g||_{L^1}||f||_{L^1_{q+\gamma}} + C_\epsilon ||g||_{L^1_\gamma}||f||_{L^1_q} \\ &\quad + A_2||g||_{L^1_{q+\gamma}}||f||_{L^1} + A_2 2^l ||g||_{L^1_q}||f||_{L^1_q}.\end{aligned}$$

Next, according to [26], one has

$$\langle Q_L(g,h), \langle v\rangle^q\rangle \leq -\Lambda q||g||_{L^1}||f||_{L^1_{l+\gamma}} + \Lambda(4q+2)q||g||_{L^1_4}||f||_{L^1_q}.$$

Therefore we have

$$\frac{d}{dt}||f||_{L^1_q} + \frac{C_\epsilon}{2}||g||_{L^1}||f||_{L^1_{q+\gamma}} \leq C(M,\Lambda,q)||g||_{L^1_q}||f||_{L^1_q} + A_2||f_0||_{L^1}||g||_{L^1_{q+\gamma}}.$$

By Gronwall's inequality, it is not difficult to derive

$$\sup_{0\leq s\leq T}||f||_{L^1_q} + \int_0^T ||f(t)||_{L^1_{q+\gamma}} dt \leq C(||f_0||_{L^1_q}, \sup_{0\leq t\leq T}||g||_{L^1_q}, \int_0^T ||g(t)||_{L^1_{q+\gamma}} dt).$$

Recalling the estimate (3.33) and the convergence of $\{f^n\}_{n\in\mathbb{N}}$ in $L^\infty([0,T]; L^1_l)$, we also have $f \in L^\infty([0,T]; H^5_{l+\gamma+10})$.

Step 4: (Uniqueness)

Suppose $f^1, f^2 \in L^\infty([0,T]; L^1_{l+5\gamma/2+16} \cap H^5_{\gamma+10+l}) \cap L^1([0,T]; L^1_{l+7\gamma/2+16})$ are two non-negative solutions of equation (3.10), let $h = f^1 - f^2$. Thus h is a solution to the following equation,

$$\begin{cases} \partial_t h = Q^\epsilon(g,h) + \epsilon^{2-2s} Q_L(g,h) \\ h|_{t=0} = 0. \end{cases} \tag{3.36}$$

Observe that the above equation is as the same as the equation (3.29) if $h^{n-1} = h^n$. With the same argument until inequality (3.35), we have

$$\frac{d}{dt}||h||_{L^1_l} + C_1||h^n||_{L^1_{l+\gamma}} \leq C_2||h||_{L^1_l},$$

where C_1 and C_2 are some positive constants depending on M and m. Then we have

$$||h(t)||_{L^1_l} \leq ||h(0)||_{L^1_l} e^{C_2 t},$$

which gives the uniqueness. □

3.2.2 First result on the well-posedness of approximate equation (1.7)

Based on the Picard iteration scheme, we derive that

Lemma 3.5. *Let $l \geq 4$ be a real number and N be a non-negative integer. Let w_H, w_L, w be functions defined by*

$$w_H(N,l) = \max\{w(N,l) + 3\gamma/2 + 4, 2l + 3 + \gamma/2\}, \tag{3.37}$$

$$\begin{aligned} w_L(N,l) = &\max\{q(2, w(N,l) + \gamma + 4), q(N, 2l+3), q(N+1, l+\gamma/2+2)\} \\ &+ \gamma, \end{aligned} \tag{3.38}$$

$$w(N,l) = \frac{(N+s+2)(2l+3) - (N+2)(l+\gamma/2)}{s}. \tag{3.39}$$

Suppose the non-negative datum $f_0 \in H^{(N+2)\vee 3}_{w_H(N,l)} \cap L^1_{w_L(N,l)}$ with $||f_0||_{L^1} > 0$, then our approximate equation (1.7) *admits a non-negative solution f in $L^\infty([0,T^*]; H^N_l \cap L^1_{w(N,l)})$ for some $T^* > 0$. Moreover, if $N \geq 2$ and $l \geq 8 + \gamma$, the solution is unique.*

Proof. Consider the sequence of functions $\{f^n\}_{n\in\mathbb{N}}$ defined by

$$\begin{cases} f^0(t) = f_0, \quad \text{for any } t \geq 0; \\ \partial_t f^n = Q^\epsilon(f^{n-1}, f^n) + \epsilon^{2-2s} Q_L(f^{n-1}, f^n), \quad n \geq 1, \\ f^n|_{t=0} = f_0. \end{cases} \tag{3.40}$$

We first mention that equation (3.40) conserves mass, that is, $||f^n(t)||_{L^1} = ||f_0||_{L^1}$ for any $n \geq 0$ and $t \geq 0$. By Lemma 3.4, $f^n \geq 0$ for any $n \in \mathbb{N}$.

Step 1: (Uniform L^1_l Upper Bound)

In this step we prove that $\{f^n\}_n$ has uniform upper bound in $L^\infty([0,T^*(l)]; L^1_l)$ with respect to n for some $T^*(l) > 0$ if $f_0 \in L^1_{l+\gamma}$. Thanks to Proposition 3.1, for any $l \geq 4$, we have

$$\begin{aligned}
\langle Q^\epsilon(f^{n-1}, f^n), \langle v\rangle^l\rangle &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b^\epsilon |v-v_*|^\gamma f_*^{n-1} f^n (\langle v'\rangle^l - \langle v\rangle^l) dv dv_* d\sigma \\
&\leq -\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b^\epsilon |v-v_*|^\gamma f_*^{n-1} f^n \langle v\rangle^l (1-\cos^l\frac{\theta}{2}) dv dv_* d\sigma \\
&\quad + \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b^\epsilon |v-v_*|^\gamma f_*^{n-1} f^n \langle v_*\rangle^l \sin^l\frac{\theta}{2} dv dv_* d\sigma \\
&\quad + C(l)\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b^\epsilon |v-v_*|^\gamma f_*^{n-1} f^n \langle v\rangle^{l-2}\langle v_*\rangle^{l-2} \sin^2\frac{\theta}{2} dv dv_* d\sigma.
\end{aligned}$$

In the following, denote $A_2^\epsilon = \int_{\mathbb{S}^2} b^\epsilon \sin^2\frac{\theta}{2} d\sigma \leq \frac{A_2}{2}$, then we have

$$\int_{\mathbb{S}^2} b^\epsilon(\cos\theta)(1-\cos^l\frac{\theta}{2}) d\sigma \geq \frac{3}{2}A_2^\epsilon,$$

and

$$\int_{\mathbb{S}^2} b^\epsilon(\cos\theta)\sin^l\frac{\theta}{2} d\sigma \leq \frac{1}{2}A_2^\epsilon,$$

where we used $1-\cos^l\frac{\theta}{2} \geq 1-\cos^4\frac{\theta}{2} \geq \frac{3}{2}\sin^2\frac{\theta}{2}$ and $\sin^l\frac{\theta}{2} \leq \frac{1}{2}\sin^2\frac{\theta}{2}$. Together with $|v-v_*|^\gamma \geq \frac{3}{4}\langle v\rangle^\gamma - c_1\langle v_*\rangle^\gamma$, $|v-v_*|^\gamma \leq 2(\langle v\rangle^\gamma + \langle v_*\rangle^\gamma)$, and $|v-v_*|^\gamma \leq \langle v\rangle^\gamma\langle v_*\rangle^\gamma$, we obtain

$$\begin{aligned}
\langle Q^\epsilon(f^{n-1}, f^n), \langle v\rangle^l\rangle \leq& -\frac{9}{8}A_2^\epsilon ||f^{n-1}||_{L^1}||f^n||_{L^1_{l+\gamma}} + \frac{3}{2}c_1 A_2^\epsilon ||f^{n-1}||_{L^1_\gamma}||f^n||_{L^1_l} \\
&+ A_2^\epsilon ||f^{n-1}||_{L^1_{l+\gamma}}||f^n||_{L^1} + A_2^\epsilon ||f^{n-1}||_{L^1_l}||f^n||_{L^1_\gamma} \\
&+ C(l)A_2^\epsilon ||f^{n-1}||_{L^1_l}||f^n||_{L^1_l}.
\end{aligned} \tag{3.41}$$

Recalling (3.18), we have

$$\langle Q_L(f^{n-1}, f^n), \langle v\rangle^l\rangle \leq -l\Lambda ||f^{n-1}||_{L^1}||f^n||_{L^1_{l+\gamma}} + (4l+2)l\Lambda ||f^{n-1}||_{L^1_4}||f^n||_{L^1_l}. \tag{3.42}$$

With (3.41) and (3.42) in hand, we have

$$\begin{aligned}
&\frac{d}{dt}||f^n||_{L^1_l} + \frac{9}{8}A_2^\epsilon ||f_0||_{L^1}||f^n||_{L^1_{l+\gamma}} \\
\leq& A_2^\epsilon ||f_0||_{L^1}||f^{n-1}||_{L^1_{l+\gamma}} + C(\epsilon, l, \Lambda)||f^{n-1}||_{L^1_l}||f^n||_{L^1_l},
\end{aligned} \tag{3.43}$$

where we denote $C(\epsilon, l, \Lambda) = A_2^\epsilon + \frac{3}{2}c_1 A_2^\epsilon + C(l)A_2^\epsilon + \epsilon^{2-2s}(4l+2)l\Lambda$. For simplicity, denote $m(l) = ||f_0||_{L^1_l}$. For any $n \in \mathbb{N}$ and $l \geq 0$, define

$$T^*(l) = \frac{\min\{\log\{\frac{11C(\epsilon,l,\Lambda)m^2(l)}{A_2^\epsilon m(0)m(l+\gamma)} + 1\}, \log(10/9)\}}{11C(\epsilon, l, \Lambda)m(l)},$$

and

$$C_{n,l} = \sup_{0 \leq t \leq T^*(l)} \sup_{0 \leq k \leq n} ||f^k(t)||_{L^1_l}.$$

We claim that for any $n \in \mathbb{N}$,

$$C_{n,l} \leq 11m(l). \tag{3.44}$$

We will prove (3.44) by induction. First, it is obvious $C_{0,l} \leq 11m(l)$. Next, fix a $n \geq 1$, suppose $C_{n-1,l} \leq 11m(l)$, then on the interval $[0, T^*(l)]$, for any $1 \leq k \leq n$, from (3.43), we have

$$\begin{aligned}
&\frac{d}{dt}||f^k||_{L^1_l} + \frac{9}{8}A_2^\epsilon m(0)||f^k||_{L^1_{l+\gamma}} \\
\leq\ & A_2^\epsilon m(0)||f^{k-1}||_{L^1_{l+\gamma}} + 11C(\epsilon, l, \Lambda)m(l)||f^k||_{L^1_l},
\end{aligned} \tag{3.45}$$

Thus for any $t \in [0, T^*(l)]$ and $1 \leq k \leq n$, we derive that

$$\begin{aligned}
&e^{-11C(\epsilon,l,\Lambda)m(l)t}||f^k(t)||_{L^1_l} + \frac{9}{8}A_2^\epsilon m(0)\int_0^t e^{-11C(\epsilon,l,\Lambda)m(l)s}||f^k(s)||_{L^1_{l+\gamma}}ds \\
\leq\ & A_2^\epsilon m(0)\int_0^t e^{-11C(\epsilon,l,\Lambda)m(l)s}||f^{k-1}(s)||_{L^1_{l+\gamma}}ds + m(l).
\end{aligned} \tag{3.46}$$

Multiplying the above inequality by $(\frac{8}{9})^{n-k}$ and taking sum over $1 \leq k \leq n$, we obtain

$$\begin{aligned}
&e^{-11C(\epsilon,l,\Lambda)m(l)t}\sum_{k=1}^n(\frac{8}{9})^{n-k}||f^k(t)||_{L^1_l} + \frac{9}{8}A_2^\epsilon m(0)\int_0^t e^{-11C(\epsilon,l,\Lambda)m(l)s}||f^n(s)||_{L^1_{l+\gamma}}ds \\
&\leq (\frac{8}{9})^{n-1}A_2^\epsilon m(0)m(l+\gamma)\frac{1-e^{-11C(\epsilon,l,\Lambda)m(l)t}}{11C(\epsilon, l, \Lambda)m(l)} + m(l)\sum_{k=1}^n(\frac{8}{9})^{n-k}.
\end{aligned}$$

Observing that $\sum_{k=1}^n(\frac{8}{9})^{n-k} \leq 9$, we arrive at

$$\begin{aligned}
&\sum_{k=1}^n(\frac{8}{9})^{n-k}||f^k(t)||_{L^1_l} + \frac{9}{8}A_2^\epsilon m(0)\int_0^t e^{11C(\epsilon,l,\Lambda)m(l)(t-s)}||f^n(s)||_{L^1_{l+\gamma}}ds \\
\leq\ & A_2^\epsilon m(0)m(l+\gamma)||f_0||_{L^1_{l+\gamma}}\frac{e^{11C(\epsilon,l,\Lambda)m(l)t}-1}{11C(\epsilon, l, \Lambda)m(l)} + 9m(l)e^{11C(\epsilon,l,\Lambda)m(l)t}.
\end{aligned}$$

Thus we have

$$
\begin{aligned}
\sup_{0\le t\le T^*(l)} ||f^n(t)||_{L^1_l} &\le A_2^\epsilon m(0)||f_0||_{L^1_{l+\gamma}} \frac{e^{11C(\epsilon,l,\Lambda)m(l)T^*(l)}-1}{11C(\epsilon,l,\Lambda)m(l)} \\
&\quad +9||f_0||_{L^1_l} e^{11C(\epsilon,l,\Lambda)m(l)T^*(l)} \\
&\le 11m(l),
\end{aligned}
$$

by the definition of $T^*(l)$. That is, $C_n \le 11m(l)$. Therefore the claim (3.44) is proved, which impiles

$$
\sup_{0\le t\le T^*(l)} \sup_{n\ge 0} ||f^n(t)||_{L^1_l} \le 11m(l). \tag{3.47}
$$

Step 2: (Uniform H^N_l Upper Bound)

In this step, we adopt the energy method to get a uniform upper bound of H^N_l norm of f^n with respect to n. Fix an α with $|\alpha|\le N$, one has

$$
\partial_t \partial_v^\alpha f^n = \sum_{\alpha_1+\alpha_2=\alpha} \binom{\alpha}{\alpha_1} [Q^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n) + \epsilon^{2-2s} Q_L(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n)].
$$

As before, we have

$$
\begin{aligned}
& \langle M^\epsilon(f^{n-1}, \partial_v^\alpha f^n), \partial_v^\alpha f^n \langle v\rangle^{2l}\rangle \\
= \; & \langle M^\epsilon(f^{n-1}, \partial_v^\alpha f^n\langle v\rangle^l), \partial_v^\alpha f^n \langle v\rangle^{l}\rangle \\
& +\{\langle M^\epsilon(f^{n-1}, \partial_v^\alpha f^n)\langle v\rangle^l - M^\epsilon(f^{n-1}, \partial_v^\alpha f^n\langle v\rangle^l), \partial_v^\alpha f^n \langle v\rangle^{l}\rangle\}.
\end{aligned}
$$

By coercivity estimate (3.5) and commutator estimates (3.7), (3.8), we have

$$
\begin{aligned}
& \langle M^\epsilon(f^{n-1}, \partial_v^\alpha f^n), \partial_v^\alpha f^n \langle v\rangle^{2l}\rangle + C_1(f_0)||\partial_v^\alpha f^n||^2_{\epsilon,l+\gamma/2} \\
\lesssim \; & C_2(f_0)||\partial_v^\alpha f^n||^2_{L^2_{l+\gamma/2}} + ||f^{n-1}||_{L^1_{2l+1}} ||\partial_v^\alpha f^n||_{\epsilon,l+\gamma/2} ||\partial_v^\alpha f^n||_{L^2_{l+\gamma/2}}.
\end{aligned} \tag{3.48}
$$

By upper bound estimate (3.1) and commutator estimates (3.7), (3.8), for $|\alpha_2|\le$

$N-1$, we have

$$
\begin{aligned}
&\langle M^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n), \partial_v^\alpha f^n \langle v\rangle^{2l}\rangle \\
=&\ \langle M^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n\langle v\rangle^l), \partial_v^\alpha f^n \langle v\rangle^{l}\rangle \\
&+\{\langle M^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n)\langle v\rangle^l, \partial_v^\alpha f^n \langle v\rangle^{l}\rangle \\
&-\langle M^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n\langle v\rangle^l), \partial_v^\alpha f^n \langle v\rangle^{l}\rangle\} \\
\lesssim&\ ||\partial_v^{\alpha_1} f^{n-1}||_{L^1_{\gamma+2}} ||\partial_v^{\alpha_2} f^n||_{H^s_{l+\gamma/2+2}} ||\partial_v^{\alpha} f^n||_{H^s_{l+\gamma/2}} \\
&+\epsilon^{2-2s}||\partial_v^{\alpha_1} f^{n-1}||_{L^1_{\gamma+2}} ||\partial_v^{\alpha_2} f^n||_{H^1_{l+\gamma/2+2}} ||\partial_v^{\alpha} f^n||_{H^1_{l+\gamma/2}} \\
&+||\partial_v^{\alpha_1} f^{n-1}||_{L^1_{2l+1}} ||\partial_v^{\alpha_2} f^n||_{H^s_{l+\gamma/2}} ||\partial_v^{\alpha} f^n||_{L^2_{l+\gamma/2}} \\
&+\epsilon^{2-2s}||\partial_v^{\alpha_1} f^{n-1}||_{L^1_{\gamma+3}} ||\partial_v^{\alpha_2} f^n||_{H^1_{l+\gamma/2}} ||\partial_v^{\alpha} f^n||_{L^2_{l+\gamma/2}}.
\end{aligned} \tag{3.49}
$$

When $N=0$, by (3.48), we have

$$
\begin{aligned}
&\langle M^\epsilon(f^{n-1}, f^n), f^n\langle v\rangle^{2l}\rangle + \frac{C_1(f_0)}{2}||f^n||^2_{\epsilon,l+\gamma/2} \\
\lesssim&\quad C_2(f_0)||f^n||^2_{L^2_{l+\gamma/2}} + \frac{1}{C_1(f_0)}||f^{n-1}||^2_{L^1_{2l+1}}||f^n||^2_{L^2_{l+\gamma/2}}.
\end{aligned}
$$

By (3.47), there holds

$$
\sup_{0\le t\le T^*(2l+1)} \sup_{n\ge 0} ||f^n(t)||_{L^1_{2l+1}} \le 11m(2l+1),
$$

so we have

$$
\frac{d}{dt}||f^n||^2_{L^2_l} + C_1(f_0)||f^n||^2_{\epsilon,l+\gamma/2} \lesssim C(||f_0||_{L^1_{2l+1}}, ||f_0||_{L\log L})||f^n||^2_{L^2_{l+\gamma/2}}.
$$

Thanks to the fact

$$
||f^n||^2_{L^2_{l+\gamma/2}} \le \eta||f^n||^2_{H^s_{l+\gamma/2}} + C(\eta)||f^n||^2_{L^1_{l+\gamma/2}},
$$

we have

$$
\frac{d}{dt}||f^n||^2_{L^2_l} + \frac{C_1(f_0)}{2}||f^n||^2_{\epsilon,l+\gamma/2} \le C(||f_0||_{L^1_{2l+1}}, ||f_0||_{L\log L}).
$$

By Gronwall's inequality, we obtain

$$
\sup_{0\le t\le T^*(2l+1)} ||f^n(t)||_{L^2_l} + \int_0^{T^*(2l+1)} ||f^n(s)||^2_{\epsilon,l+\gamma/2} ds \le C(||f_0||_{L^1_{2l+1}}, ||f_0||_{L^2_l}).
$$

With the help of uniform L^2_l norm and the above inequality, we can prove in a similar manner as in the second step in the proof of Theorem 1.1,

$$\sup_{0\le t\le T^*(\phi(s,l))} ||f^n(t)||_{H^s_l} \le C(||f_0||_{L^1_{\phi(s,l)}}, ||f_0||_{H^s_l}),$$

where $\phi(s,l) = \frac{(2l+4)(2+s)-2l}{s}$.

Now we turn to higher order regularity. Taking into account the fact $W^\epsilon(\xi) \le \langle \xi \rangle$, for the fixed ϵ, by (3.48) and (3.49), we have

$$\begin{aligned} \frac{d}{dt}(\frac{1}{2}||f^n||^2_{H^1_l}) + \frac{C_1(f_0)}{2}\epsilon^{2-2s}||f^n||^2_{H^2_{l+\gamma/2}} &\lesssim ||f^n||^2_{H^1_{l+\gamma/2}} + ||f^{n-1}||_{H^1_{2l+3}}||f^n||^2_{H^1_{l+\gamma/2}} \\ &\quad +||f^{n-1}||_{H^1_6}||f^n||_{H^1_{l+2+\gamma/2}}||f^n||_{H^2_{l+\gamma/2}}. \end{aligned}$$

Thanks interpolation theory and Young's inequality, one has

$$\begin{aligned} ||f^n||^2_{H^1_{l+\gamma/2}} &\le ||f^n||_{H^2_{l+\gamma/2}}||f^n||_{L^2_{l+\gamma/2}} \\ &\le \frac{C_1(f_0)}{8}\epsilon^{2-2s}||f^n||^2_{H^2_{l+\gamma/2}} + \frac{2}{C_1(f_0)\epsilon^{2-2s}}||f^n||^2_{L^2_{l+\gamma/2}}, \end{aligned}$$

$$\begin{aligned} ||f^{n-1}||_{H^1_{2l+3}}||f^n||^2_{H^1_{l+\gamma/2}} &\le ||f^{n-1}||^{1/2}_{H^2_{l+\gamma/2}}||f^{n-1}||^{1/2}_{L^2_{3l+6-\gamma/2}}||f^n||_{H^2_{l+\gamma/2}}||f^n||_{L^2_{l+\gamma/2}} \\ &\le \frac{C_1(f_0)}{8}\epsilon^{2-2s}||f^n||^2_{H^2_{l+\gamma/2}} \\ &\quad +\frac{2}{C_1(f_0)\epsilon^{2-2s}}||f^{n-1}||_{H^2_{l+\gamma/2}}||f^{n-1}||_{L^2_{3l+6-\gamma/2}}||f^n||^2_{L^2_{l+\gamma/2}} \\ &\le \frac{C_1(f_0)}{8}\epsilon^{2-2s}||f^n||^2_{H^2_{l+\gamma/2}} + \frac{C_1(f_0)}{32}\epsilon^{2-2s}||f^{n-1}||^2_{H^2_{l+\gamma/2}} \\ &\quad +\frac{32}{(C_1(f_0)\epsilon^{2-2s})^3}||f^{n-1}||^2_{L^2_{3l+6-\gamma/2}}||f^n||^4_{L^2_{l+\gamma/2}}, \end{aligned}$$

and finally

$$\begin{aligned} &||f^{n-1}||_{H^1_6}||f^n||_{H^1_{l+2+\gamma/2}}||f^n||_{H^2_{l+\gamma/2}} \\ \le\ &||f^{n-1}||^{\frac{1-s}{2-s}}_{H^2_6}||f^{n-1}||^{\frac{1}{2-s}}_{H^s_6}||f^n||^{3/2}_{H^2_{l+\gamma/2}}||f^n||^{1/2}_{L^2_{l+4+\gamma/2}} \\ \le\ &\frac{C_1(f_0)}{8}\epsilon^{2-2s}||f^n||^2_{H^2_{l+\gamma/2}} \\ &+\frac{1}{4}(\frac{32}{3C_1(f_0)}\epsilon^{2-2s})^3||f^{n-1}||^{\frac{4(1-s)}{2-s}}_{H^2_6}||f^{n-1}||^{\frac{4}{2-s}}_{H^s_6}||f^n||^2_{L^2_{l+4+\gamma/2}} \\ \le\ &\frac{C_1(f_0)}{8}\epsilon^{2-2s}||f^n||^2_{H^2_{l+\gamma/2}} + \frac{C_1(f_0)}{32}\epsilon^{2-2s}||f^{n-1}||^2_{H^2_6} \\ &+\frac{1}{p}(q\eta)^{-\frac{p}{q}}(\frac{1}{4}(\frac{32}{3C_1(f_0)}\epsilon^{2-2s})^3)^{\frac{2-s}{s}}||f^{n-1}||^{\frac{4}{s}}_{H^s_6}||f^n||^{\frac{2(2-s)}{s}}_{L^2_{l+4+\gamma/2}}, \end{aligned}$$

where we have used the Young's inequality (1.10) with $p = \frac{2-s}{s}, q = \frac{2-s}{s(1-s)}$ and $\eta = \frac{C_1(f_0)}{32}\epsilon^{2-2s}$. Thus we arrive at for any $n \geq 1$,

$$\frac{d}{dt}||f^n||^2_{H^1_l} + \frac{1}{4}C_1(f_0)\epsilon^{2-2s}||f^n||^2_{H^2_{l+\gamma/2}} \leq \frac{1}{8}C_1(f_0)\epsilon^{2-2s}||f^{n-1}||^2_{H^2_{l+\gamma/2}} + M,$$

where M is the uniform upper bound of $||f^n||_{H^s_6}, ||f^n||_{L^2_{l+5}}$ and $||f^n||_{L^1_{3l+6}}$ with respect to n on the time interval $[0, T^*]$. Here $T^* = T^*(\max\{\phi(s,6), 3l+6\})$. With the same technique as in dealing with (3.45), we obtain

$$\begin{aligned} & ||f^n(t)||^2_{H^1_l} + \frac{1}{4}C_1(f_0)\epsilon^{2-2s}\int_0^t ||f^n(r)||^2_{H^2_{l+\gamma/2}}dr \\ \leq \ & \frac{1}{8}C_1(f_0)\epsilon^{2-2s}||f_0||^2_{H^2_{l+\gamma/2}}t + 2(Mt + ||f_0||^2_{H^1_l}). \end{aligned}$$

The above inequality is true for any $n \geq 1$ and $t \in [0, T^*]$, so we have the desired result

$$\sup_n \sup_{0\leq t\leq T^*} ||f^n(t)||_{H^1_l} \leq C(||f_0||_{H^2_{l+\gamma/2}}, ||f_0||_{L^1_{\max\{\phi(s,6),3l+6\}}}, T^*).$$

Continuing the argument, there will be a function $q : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}$, such that

$$\sup_n \sup_{0\leq t\leq T^*(q(N,l))} ||f^n(t)||_{H^N_l} \leq C(||f_0||_{H^{N+1}_{l+\gamma/2}}, ||f_0||_{L^1_{q(N,l)+\gamma}}). \tag{3.50}$$

Step 3: (Cauchy Sequence)

Now we set to prove $\{f^n\}_n$ is a Cauchy sequence in the space $L^\infty([0, T^*]; L^1_l)$. Set $h^n = f^{n+1} - f^n$ for $n \geq 0$. Then for $n \geq 1$, h^n is the solution to the following equation

$$\begin{cases} \partial_t h^n = M^\epsilon(f^n, h^n) + M^\epsilon(h^{n-1}, f^n), \\ h^n|_{t=0} = 0. \end{cases} \tag{3.51}$$

As the same as (3.41), we have

$$\begin{aligned} \langle Q^\epsilon(f^n, h^n), sgn(h^n)\langle v\rangle^l\rangle &\leq \langle Q^\epsilon(f^n, |h^n|), \langle v\rangle^l\rangle \\ &\leq -\frac{9}{8}A_2^\epsilon||f^n||_{L^1}||h^n||_{L^1_{l+\gamma}} + \frac{3}{2}A_2^\epsilon||f^n||_{L^1_\gamma}||h^n||_{L^1_l} \\ &\quad + A_2^\epsilon||f^n||_{L^1_{l+\gamma}}||h^n||_{L^1} + A_2^\epsilon||f^n||_{L^1_l}||h^n||_{L^1_\gamma} \\ &\quad + C(l)A_2^\epsilon||f^n||_{L^1_l}||h^n||_{L^1_l}. \end{aligned} \tag{3.52}$$

As the same as (3.34), we have

$$\langle Q_L(f^n,h^n), sgn(h^n)\langle v\rangle^l\rangle_v \le -\frac{\Lambda}{16}||f^n||_{L^1}||h^n||_{L^1_{l+\gamma}} + \frac{\Lambda}{8}||f^n||_{L^1_\gamma}||h^n||_{L^1_l} + C(l)\Lambda||f^n||_{L^1_l}||h^n||_{L^1_l}. \tag{3.53}$$

Applying Proposition 3.1 again, we obtain

$$\begin{aligned} &\langle Q^\epsilon(h^{n-1},f^n), sgn(h^n)\langle v\rangle^l\rangle \\ \le &\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b^\epsilon|v-v_*|^\gamma|h^{n-1}_*|f^n(\langle v'\rangle^l+\langle v\rangle^l)dvdv_*d\sigma \\ \le &A_2^\epsilon(||f^n||_{L^1}||h^{n-1}||_{L^1_{l+\gamma}} + ||f^n||_{L^1_\gamma}||h^{n-1}||_{L^1_l}) \\ &+ C(l)A_2^\epsilon||f^n||_{L^1_l}||h^{n-1}||_{L^1_l} \\ &+ \ A^\epsilon||f^n||_{L^1_{l+\gamma}}||h^{n-1}||_{L^1_\gamma}. \end{aligned} \tag{3.54}$$

Recalling the Landau operator Q_L can be rewritten as:

$$Q_L(g,h) = \sum_{i,j=1}^{3}(a_{ij}*g)\partial_{ij}h - (c*g)h,$$

we have

$$\begin{aligned} \langle Q_L(h^{n-1},f^n), sgn(h^n)\langle v\rangle^l\rangle = &\sum_{i,j=1}^{3}\langle(a_{ij}*h^{n-1})\partial_{ij}f^n, sgn(h^n)\langle v\rangle^l\rangle \\ &- \langle(c*h^{n-1})f^n, sgn(h^n)\langle v\rangle^l\rangle \\ \le &\Lambda||h^{n-1}||_{L^1_{\gamma+2}}||f^n||_{H^2_{l+\gamma+4}} \\ &+ 2\Lambda(\gamma+3)||h^{n-1}||_{L^1_\gamma}||f^n||_{L^1_{l+\gamma}}. \end{aligned} \tag{3.55}$$

Putting together inequalities (3.52),(3.53),(3.54) and (3.55), we obtain

$$\begin{aligned} &\frac{d}{dt}||h^n||_{L^1_l} + \frac{9}{8}A_2^\epsilon m(0)||h^n||_{L^1_{l+\gamma}} \\ \le &A_2^\epsilon m(0)||h^{n-1}||_{L^1_{l+\gamma}} + K_1||h^n||_{L^1_l} + K_2||h^{n-1}||_{L^1_l}, \end{aligned} \tag{3.56}$$

where K_1 and K_2 are some constants depending at most on the uniform upper bound of $||f^n||_{H^2_{l+\gamma+4}}$, which is bounded by a constant depending on $||f_0||_{H^3_{l+3\gamma/2+4}}$ and $||f_0||_{L^1_{q(2,l+\gamma+4)+\gamma}}$. With a similar argument as in the previous lemma, for any $t\in[0,T^*(q(2,l+\gamma+4))]$, we can conclude

$$||h^n(t)||_{L^1_l} \le (\frac{8}{9})^n M(t)\exp(\frac{9K_2t}{8} + K_1t), \tag{3.57}$$

where $M(t) = \frac{9}{8}A_2^\epsilon m(0)\int_0^t e^{-K_1 s}||h^0(s)||_{L^1_{l+\gamma}}ds + 22m(l)$. Thus $\sum_n ||h^n(t)||_{L^1_l}$ is finite and $\{f^n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in L^1_l. Due to the arbitrariness of $t \in [0, T^*(q(2, l+\gamma+4))]$, there is a function $f \in L^\infty([0,T^*]; L^1_l)$ such that

$$\lim_{n\to\infty} \sup_{0\le t\le T^*} ||f^n(t) - f(t)||_{L^1_l} = 0.$$

In the following, we prove $\{f^n(t)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H^N_l. Fix an α with $|\alpha| \le N$, one has

$$\partial_t\partial_v^\alpha h^n = \sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}[M^\epsilon(\partial_v^{\alpha_1}f^n, \partial_v^{\alpha_2}h^n) + M^\epsilon(\partial_v^{\alpha_1}h^{n-1}, \partial_v^{\alpha_2}f^n)].$$

Then we have

$$\begin{aligned}\frac{d}{dt}(\frac{1}{2}||\partial_v^\alpha h^n||^2_{L^2_l}) &= \sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}[\langle M^\epsilon(\partial_v^{\alpha_1}f^n, \partial_v^{\alpha_2}h^n), \partial_v^\alpha h^n\langle v\rangle^{2l}\rangle \\ &\quad +\langle M^\epsilon(\partial_v^{\alpha_1}h^{n-1}, \partial_v^{\alpha_2}f^n), \partial_v^\alpha h^n\langle v\rangle^{2l}\rangle] \\ &\overset{\text{def}}{=} \sum_{\alpha_1+\alpha_2=\alpha}\binom{\alpha}{\alpha_1}[\mathfrak{I}_1(\alpha_1,\alpha_2) + \mathfrak{I}_2(\alpha_1,\alpha_2)].\end{aligned}$$

As the same as (3.48), on the time interval $[0, T^*(2l+1)]$, we have

$$\begin{aligned}&\mathfrak{I}_1(0,\alpha) + C_1(f_0)||\partial_v^\alpha h^n||^2_{\epsilon,l+\gamma/2} \\ \lesssim\ & C_2(f_0)||\partial_v^\alpha h^n||^2_{L^2_{l+\gamma/2}} + ||f^n||_{L^1_{2l+1}}||\partial_v^\alpha h^n||_{\epsilon,l+\gamma/2}||\partial_v^\alpha h^n||_{L^2_{l+\gamma/2}},\end{aligned}$$

and thus

$$\mathfrak{I}_1(0,\alpha) + \frac{C_1(f_0)}{2}||\partial_v^\alpha h^n||^2_{\epsilon,l+\gamma/2} \lesssim C(||f_0||_{L^1_{2l+1}}, ||f_0||_{L^2})||\partial_v^\alpha h^n||_{L^2_{l+\gamma/2}}.$$

As the same as (3.49), for $|\alpha_2| \le |\alpha| - 1 \le N-1$ and any $\eta > 0$, on the time interval $[0, T^*(q(N, 2l+3))]$, we have

$$\begin{aligned}\mathfrak{I}_1(\alpha_1,\alpha_2) &\lesssim ||\partial_v^{\alpha_1}f^n||_{L^1_{\gamma+2}}||\partial_v^{\alpha_2}h^n||_{H^s_{l+\gamma/2+2}}||\partial_v^\alpha h^n||_{H^s_{l+\gamma/2}} \\ &\quad +\epsilon^{2-2s}||\partial_v^{\alpha_1}f^n||_{L^1_{\gamma+2}}||\partial_v^{\alpha_2}h^n||_{H^1_{l+\gamma/2+2}}||\partial_v^\alpha h^n||_{H^1_{l+\gamma/2}} \\ &\quad +||\partial_v^{\alpha_1}f^n||_{L^1_{2l+1}}||\partial_v^{\alpha_2}h^n||_{H^s_{l+\gamma/2}}||\partial_v^\alpha h^n||_{L^2_{l+\gamma/2}} \\ &\quad +\epsilon^{2-2s}||\partial_v^{\alpha_1}f^n||_{L^1_{\gamma+3}}||\partial_v^{\alpha_2}h^n||_{H^1_{l+\gamma/2}}||\partial_v^\alpha h^n||_{L^2_{l+\gamma/2}} \\ &\lesssim ||f^n||_{H^N_{2l+3}}||h^n||_{H^N_{l+\gamma/2+2}}||\partial_v^\alpha h^n||_{\epsilon,l+\gamma/2},\end{aligned}$$

which implies, for any $\eta > 0$,

$$\mathfrak{I}_1(\alpha_1,\alpha_2) - \eta||\partial_v^\alpha h^n||^2_{\epsilon,l+\gamma/2} \lesssim C(\eta, ||f_0||_{H^{N+1}_{2l+3+\gamma/2}}, ||f_0||_{L^1_{q(N,2l+3)+\gamma}})||h^n||^2_{H^N_{l+\gamma/2+2}}.$$

Similarly, on the time interval $[0, T^*(q(N+1, l+\gamma/2+2))]$, we have

$$\mathfrak{I}_2(\alpha_1,\alpha_2) \lesssim ||h^{n-1}||_{H^N_{2l+3}}||f^n||_{H^{N+1}_{l+\gamma/2+2}}||\partial_v^\alpha h^n||_{\epsilon,l+\gamma/2},$$

and so for any $\eta > 0$,

$$\mathfrak{I}_2(\alpha_1,\alpha_2) - \eta||\partial_v^\alpha h^n||^2_{\epsilon,l+\gamma/2} \lesssim C(\eta, ||f_0||_{H^{N+2}_{l+\gamma+2}}, ||f_0||_{L^1_{q(N+1,l+\gamma/2+2)+\gamma}})||h^{n-1}||^2_{H^N_{2l+3}}.$$

Taking a suitable η, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\frac{1}{2}||\partial_v^\alpha h^n||^2_{L^2_l}) + \frac{C_1(f_0)}{4}||\partial_v^\alpha h^n||^2_{\epsilon,l+\gamma/2}\\
\lesssim\ & C(||f_0||_{H^{N+1}_{2l+3+\gamma/2}}, ||f_0||_{L^1_{q(N,2l+3)+\gamma}})||h^n||^2_{H^N_{l+\gamma/2+2}}\\
&+C(||f_0||_{H^{N+2}_{l+\gamma+2}}, ||f_0||_{L^1_{q(N+1,l+\gamma/2+2)+\gamma}})||h^{n-1}||^2_{H^N_{2l+3}}.
\end{aligned}$$

Now taking sum over $|\alpha| \le N$, we arrive at

$$\begin{aligned}
&\frac{d}{dt}||h^n||^2_{H^N_l} + \frac{C_1(f_0)}{2}||h^n||^2_{H^{N+s}_{l+\gamma/2}}\\
\lesssim\ & C(||f_0||_{H^{N+1}_{2l+3+\gamma/2}}, ||f_0||_{L^1_{q(N,2l+3)+\gamma}})||h^n||^2_{H^N_{l+\gamma/2+2}}\\
&+C(||f_0||_{H^{N+2}_{l+\gamma+2}}, ||f_0||_{L^1_{q(N+1,l+\gamma/2+2)+\gamma}})||h^{n-1}||^2_{H^N_{2l+3}}.
\end{aligned}$$

By interpolation theory, one has

$$||h^n||^2_{H^N_{l+\gamma/2+2}} \le \eta||h^n||^2_{H^{N+s}_{l+\gamma/2}} + c(\eta)||h^n||^2_{L^1_{w_1(N,l,s,\gamma)}},$$

and

$$||h^{n-1}||^2_{H^N_{2l+3}} \le \lambda||h^{n-1}||^2_{H^{N+s}_{l+\gamma/2}} + c(\lambda)||h^{n-1}||^2_{L^1_{w_2(N,l,s,\gamma)}},$$

where $w_1(N,l,s,\gamma) = l+\gamma/2+\frac{2(N+s+2)}{s}$ and $w_2(N,l,s,\gamma) = \frac{(N+s+2)(2l+3)-(N+2)(l+\gamma/2)}{s}$. It is easy to check $w_1 \le w_2$. Choosing suitable η and λ, we have

$$\begin{aligned}
&\frac{d}{dt}||h^n||^2_{H^N_l} + \frac{C_1(f_0)}{4}||h^n||^2_{H^{N+s}_{l+\gamma/2}}\\
\le\ & (\frac{8}{9})\frac{C_1(f_0)}{4}||h^{n-1}||^2_{H^{N+s}_{l+\gamma/2}}\\
&+C(||f_0||_{H^{N+1}_{2l+3+\gamma/2}}, ||f_0||_{L^1_{q(N,2l+3)+\gamma}})||h^n||^2_{L^1_{w_1(N,l,s,\gamma)}}\\
&+C(||f_0||_{H^{N+2}_{l+\gamma+2}}, ||f_0||_{L^1_{q(N+1,l+\gamma/2+2)+\gamma}})||h^{n-1}||^2_{L^1_{w_2(N,l,s,\gamma)}}.
\end{aligned}$$

Thanks to (3.57), on the time interval $[0, T^*(q(2, w_2 + \gamma + 4))]$, there holds

$$||h^n(t)||_{L^1_{w_2}} \leq (\frac{8}{9})^n M(T^*) \exp(\frac{9K_2T^*}{8} + K_1T^*) \stackrel{\text{def}}{=} (\frac{8}{9})^n M,$$

where $T^* = T^*(q(w_2 + \gamma + 4, 2))$ and $M(T^*) = \frac{9}{8} A_2^\epsilon m(0) \int_0^{T^*} e^{-K_1 s} ||h^0(s)||_{L^1_{w_2+\gamma}} ds + 22m(w_2)$. For ease of notation, let $K_3 = C(\eta, ||f_0||_{H^{N+1}_{2l+3+\gamma/2}}, ||f_0||_{L^1_{q(N,2l+3)+\gamma}})$ and $K_4 = C(\lambda, ||f_0||_{H^{N+2}_{l+\gamma+2}}, ||f_0||_{L^1_{q(N+1,l+\gamma/2+2)+\gamma}})$. Then we have

$$\begin{aligned} & \frac{d}{dt} ||h^n||^2_{H^N_l} + \frac{C_1(f_0)}{4} ||h^n||^2_{H^{N+s}_{l+\gamma/2}} \\ \leq \quad & (\frac{8}{9}) \frac{C_1(f_0)}{4} ||h^{n-1}||^2_{H^{N+s}_{l+\gamma/2}} + M(K_3 + \frac{9}{8} K_4)(\frac{8}{9})^n. \end{aligned}$$

Integrating both sides with respect to time over $[0, t]$ for any $t \in [0, T^*]$, we have

$$\begin{aligned} & ||h^n(t)||^2_{H^N_l} + \frac{C_1(f_0)}{4} \int_0^t ||h^n(r)||^2_{H^{N+s}_{l+\gamma/2}} dr \\ \leq \quad & (\frac{8}{9}) \frac{C_1(f_0)}{4} \int_0^t ||h^{n-1}(r)||^2_{H^{N+s}_{l+\gamma/2}} dr + MT^*(K_3 + \frac{9}{8} K_4)(\frac{8}{9})^n \\ \\ \leq \quad & (\frac{8}{9})^n \frac{C_1(f_0)}{4} \int_0^t ||h^0(r)||^2_{H^{N+s}_{l+\gamma/2}} dr + MT^*(K_3 + \frac{9}{8} K_4)(\frac{8}{9})^n n. \end{aligned}$$

Thus $\sum_n ||h^n(t)||_{H^N_l}$ is finite and $\{f^n(t)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in H^N_l. So there is a function $f \in L^\infty([0, T^*]; H^N_l)$ such that

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T^*} ||f^n(t) - f(t)||_{H^N_l} = 0.$$

The condition on f_0 can be summarized by the definitions of K_3, K_4 and the previous step as

$$f_0 \in H^{(N+2) \vee 3}_{w_H(N,l)} \cap L^1_{w_L(N,l)}.$$

Under this condition, actually $\{f^n(t)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $H^N_l \cap L^1_{w(N,l)}$.

It is obvious that f is the solution to (1.7). Because f^n is non-negative for each $n \geq 0$, the limit function f is also non-negative.

Step 4: (Uniqueness)

Suppose $f, g \in L^\infty([0,T]; H^2_{l+\gamma+4})$ are two non-negative solutions to (1.7). Set $F = f - g$ and $G = f + g$. Then F is a solution to the following equation,

$$\begin{cases} \partial_t F = M^\epsilon(G, F) + M^\epsilon(F, G) \\ F|_{t=0} = 0. \end{cases} \tag{3.58}$$

Note that the above equation is as the same as equation (3.51) if $h^n = h^{n-1}$. Thus following the same argument until inequality (3.56), we have

$$\frac{d}{dt}||F||_{L^1_l} + \frac{1}{8}A^\epsilon_2||G||_{L^1}||F||_{L^1_{l+\gamma}} \leq K||F||_{L^1_l}$$

where K is some constant depending on the uniform upper bound of $||G||_{H^2_{l+\gamma+4}}$. Note that the previous estimate holds true for $l \geq 4$. Therefore, our approximate equation (1.7) has at most one solution in the space $L^\infty([0,T]; H^N_l)$ if $N \geq 2$ and $l \geq 8 + \gamma$.

□

3.2.3 Improvement of the well-posedness result of approximate equation (1.7)

In this subsection, by using the symmetric property of the collision operators, we will prove the propagation of L^1_l and H^N_l norms of the solution f to (1.7) and then extend the lifespan T^* in Lemma 3.5 to be global. Thanks to Lemma 3.5, we may assume that solution f^ϵ to our approximate equation is non-negative and suitably smooth. It means that in this subsection we only need to give the *a priori* estimates to the solution.

In order to prove the propagation of L^1_l of the solution f^ϵ, we first give two propositions. The first proposition is related to the Boltzmann operator, while the second deals with the Landau operator.

Proposition 3.2. *Let $p \geq 3$ and $k_p = [\frac{p+1}{2}]$. Suppose*

$$\Theta(v, v_*) \stackrel{def}{=} \int_{\mathbb{S}^2} b(\cos\theta)|v - v_*|^\gamma(\langle v'\rangle^{2p} + \langle v'_*\rangle^{2p} - \langle v\rangle^{2p} - \langle v_*\rangle^{2p})d\sigma, \tag{3.59}$$

then one has

$$
\begin{aligned}
\Theta(v,v_*) &\leq -\frac{1}{4}A_2(\langle v\rangle^{2p+\gamma}+\langle v_*\rangle^{2p+\gamma})+\frac{1}{2}A_2(\langle v\rangle^{2p}\langle v_*\rangle^{\gamma}+\langle v\rangle^{\gamma}\langle v_*\rangle^{2p}) \\
&\quad +A_2\sum_{k=1}^{k_p}\binom{p}{k}\{\langle v\rangle^{2k+\gamma}\langle v_*\rangle^{2(p-k)}+\langle v\rangle^{2(p-k)+\gamma}\langle v_*\rangle^{2k}+ \\
&\quad +\langle v\rangle^{2(p-k)}\langle v_*\rangle^{2k+\gamma}+\langle v\rangle^{2k}\langle v_*\rangle^{2(p-k)+\gamma}\} \\
&\quad +2p(p-1)A_2\sum_{k=0}^{k_p-1}\binom{p-2}{k}\{\langle v\rangle^{2(k+1)+\gamma}\langle v_*\rangle^{2(p-k-1)} \\
&\quad +\langle v\rangle^{2(p-k-1)+\gamma}\langle v_*\rangle^{2(k+1)}+\langle v\rangle^{2(p-k-1)}\langle v_*\rangle^{2(k+1)+\gamma} \\
&\quad +\langle v\rangle^{2(k+1)}\langle v_*\rangle^{2(p-k-1)+\gamma}\} \\
&\leq -\frac{1}{4}A_2(\langle v\rangle^{2p+\gamma}+\langle v_*\rangle^{2p+\gamma})+2^{2p+1}A_2\langle v\rangle^{2p}\langle v_*\rangle^{2p}.
\end{aligned}
$$

Proof. One may refer to lemma 3.6 in [63] for the proof. □

Remark 3.1. Lemma 3.6 in [63] only deals with the case $p \geq 3$, however, the conclusion is also valid in the case $2 \leq p < 3$ but with a different and smaller coefficient coming out instead of the constant $\frac{1}{4}$ before the highest order $2p+\gamma$.

Proposition 3.3. *Let $p > 2$ and f be a non-negative function, then*

$$
\langle Q_L(f,f),\langle v\rangle^p\rangle \leq -\Lambda p||f||_{L^1}||f||_{L^1_{p+\gamma}}+\Lambda p(4p+2)||f||_{L^1_2}||f||_{L^1_p}. \tag{3.60}
$$

Proof. One may refer to [26] for the proof. □

Now we are ready to prove the propagation of moments and smoothness.

Proof of Theorem 1.1: The proof will be divided into four steps.

Step 1: Propagation of the moments.

We consider the $2l$ moment. Assume $l \geq 3$, for the case $2 \leq l < 3$, the proof is similar thanks to Remark 3.1. By the definition of M^ϵ, we have

$$
\begin{aligned}
\frac{d}{dt}||f^\epsilon||_{L^1_{2l}} &= \langle Q^\epsilon(f^\epsilon,f^\epsilon),\langle v\rangle^{2l}\rangle+\epsilon^{2-2s}\langle Q_L(f^\epsilon,f^\epsilon),\langle v\rangle^{2l}\rangle \\
&\stackrel{\text{def}}{=} \mathfrak{I}_1+\mathfrak{I}_2.
\end{aligned}
$$

The term $\mathfrak{I}_1$ can be written as:

$$\begin{aligned}
\mathfrak{I}_1 &= \int_{\mathbb{R}^6\times\mathbb{S}^2} b^\epsilon(\cos\theta)|v-v_*|^\gamma f_*^\epsilon f^\epsilon(\langle v'\rangle^{2l}-\langle v\rangle^{2l})d\sigma dv_* dv \\
&= \frac{1}{2}\int_{\mathbb{R}^6\times\mathbb{S}^2} b^\epsilon(\cos\theta)|v-v_*|^\gamma f_*^\epsilon f^\epsilon(\langle v'\rangle^{2l}+\langle v_*'\rangle^{2l}-\langle v\rangle^{2l}-\langle v_*\rangle^{2l})d\sigma dv_* dv
\end{aligned}$$

Let $A_2^\epsilon = \int_{\mathbb{S}^2} b^\epsilon(\cos\theta)\sin^2\theta d\sigma$, then by Proposition 3.2, we have

$$\begin{aligned}
\mathfrak{I}_1 \le\ & -\frac{A_2^\epsilon}{4}||f^\epsilon||_{L^1_0}||f^\epsilon||_{L^1_{2l+\gamma}} + \frac{A_2^\epsilon}{2}||f^\epsilon||_{L^1_2}||f^\epsilon||_{L^1_{2l}} \\
&+A_2^\epsilon\sum_{k=1}^{k_l}\binom{l}{k}\{||f^\epsilon||_{L^1_{2k+2}}||f^\epsilon||_{L^1_{2(l-k)}}+||f^\epsilon||_{L^1_{2k}}||f^\epsilon||_{L^1_{2(l-k)+2}}\} \\
&+2l(l-1)A_2^\epsilon\sum_{k=0}^{k_l-1}\binom{l-2}{k}\{||f^\epsilon||_{L^1_{2(k+1)+2}}||f^\epsilon||_{L^1_{2(l-k-1)}} \\
&+||f^\epsilon||_{L^1_{2(l-k)}}||f^\epsilon||_{L^1_{2(k+1)}}\},
\end{aligned}$$

where we have used the assumption $\gamma \le 2$. By interpolation, for any $2 \le p, q \le 2l$ with $p+q = 2l+2$, we have

$$||f^\epsilon||_{L^1_p}||f^\epsilon||_{L^1_q} \le ||f^\epsilon||_{L^1_2}||f^\epsilon||_{L^1_{2l}}.$$

Using the fact $2\sum_{k=1}^{k_l}\binom{l}{k} \le 2^l$, we can conclude:

$$\mathfrak{I}_1 \le -\frac{A_2^\epsilon}{4}||f^\epsilon||_{L^1_0}||f^\epsilon||_{L^1_{2l+\gamma}} + 2^{2l+1}A_2^\epsilon||f^\epsilon||_{L^1_2}||f^\epsilon||_{L^1_{2l}}. \tag{3.61}$$

For the term $\mathfrak{I}_2$, we apply Proposition 3.3 with $p = 2l$ and obtain

$$\mathfrak{I}_2 \le -2l\Lambda\epsilon^{2-2s}||f||_{L^1}||f||_{L^1_{2l+\gamma}} + 2l(8l+2)\Lambda\epsilon^{2-2s}||f||_{L^1_2}||f||_{L^1_{2l}}.$$

Let $0 < \epsilon_* < \frac{\sqrt{2}}{2}$ be the point such that $A_2^{\epsilon_*} = \frac{A_2}{2}$, then for any $0 < \epsilon \le \epsilon_*$, we have

$$\frac{d}{dt}||f^\epsilon||_{L^1_{2l}} \le -\frac{A_2}{8}||f^\epsilon||_{L^1}||f^\epsilon||_{L^1_{2l+\gamma}} + (2^{2l+1}A_2 + 2l(8l+2)\Lambda)||f^\epsilon||_{L^1_2}||f^\epsilon||_{L^1_{2l}}.$$

For any $\eta > 0$, there exists a constant $K_1(\eta, l)$ such that

$$\langle v\rangle^{2l} \le \eta\langle v\rangle^{2l+\gamma} + K_1(\eta, l).$$

Thus we have $||f^\epsilon||_{L^1_{2l}} \leq \eta||f^\epsilon||_{L^1_{2l+\gamma}} + K_1(\eta, l)||f^\epsilon||_{L^1}$. With the preservation of mass and energy, by denoting $K_2(l) = 2^{2l+1}A_2 + 2l(8l+2)\Lambda$ and taking $\eta(f_0) = \frac{A||f_0||_{L^1}}{16K_2(l)||f_0||_{L^1_2}}$, we have

$$\frac{d}{dt}||f^\epsilon||_{L^1_{2l}} \leq -\frac{A_2}{16}||f_0||_{L^1}||f^\epsilon||_{L^1_{2l+\gamma}} + K_2(l)K_1(\eta(f_0), l)||f_0||_{L^1_2}||f_0||_{L^1}.$$

Let $a = K_2(l)K_1(\eta(f_0), l)||f_0||_{L^1_2}||f_0||_{L^1}$ and $b = -\frac{A_2}{16}||f_0||_{L^1}$, by Gronwall's inequality (1.11), we have the following:

$$||f^\epsilon(t)||_{L^1_{2l}} \leq ||f_0||_{L^1_{2l}} + \frac{a}{|b|} \stackrel{\text{def}}{=} ||f_0||_{L^1_{2l}} + K(f_0, l).$$

The constant $K(f_0, l)$ depends only on l, $||f_0||_{L^1}$ and $||f_0||_{L^1_2}$.

Step 2: Propagation of L^2_l norm.

By the definition of M^ϵ, we have

$$\begin{aligned} \frac{d}{dt}(\frac{1}{2}||f^\epsilon||^2_{L^2_l}) &= \langle M^\epsilon(f^\epsilon, f^\epsilon\langle v\rangle^l), f^\epsilon\langle v\rangle^l\rangle \\ &\quad + \{\langle M^\epsilon(f^\epsilon, f^\epsilon)\langle v\rangle^l - M^\epsilon(f^\epsilon, f^\epsilon\langle v\rangle^l), f^\epsilon\langle v\rangle^l\rangle\} \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2. \end{aligned}$$

Applying coercivity estimate (3.5) with $g = f^\epsilon, f = f^\epsilon\langle v\rangle^l$, we have

$$\mathfrak{I}_1 \leq -C_1(f_0)||f^\epsilon||^2_{\epsilon, l+\gamma/2} + C_2(f_0)||f^\epsilon||^2_{L^2_{l+\gamma/2}}. \tag{3.62}$$

Applying commutator estimate (3.9) with $g = f^\epsilon, h = f^\epsilon, f = f^\epsilon\langle v\rangle^l, N_2 = l + \gamma/2, N_3 = \gamma/2$ and $N_1 = 2l+5$, we have

$$\mathfrak{I}_2 \lesssim ||f^\epsilon||_{L^1_{2l+5}}(||f^\epsilon||_{H^s_{l+\gamma/2}} + \epsilon^{2-2s}||f^\epsilon||_{H^1_{l+\gamma/2}})||f^\epsilon||_{L^2_{l+\gamma/2}}.$$

Thanks to the facts $||\cdot||^2_{H^s_{l+\gamma/2}} \leq ||\cdot||^2_{\epsilon, l+\gamma/2}$ and $\epsilon^{2-2s}||\cdot||^2_{H^1_{l+\gamma/2}} \leq ||\cdot||^2_{\epsilon, l+\gamma/2}$, we have

$$\mathfrak{I}_2 - \frac{C_1(f_0)}{2}||f^\epsilon||^2_{\epsilon, l+\gamma/2} \lesssim \frac{1}{C_1(f_0)}||f^\epsilon||^2_{L^1_{2l+5}}||f^\epsilon||^2_{L^2_{l+\gamma/2}}. \tag{3.63}$$

Now patching together (3.62),and (3.63), we get

$$\begin{aligned} &\frac{d}{dt}(\frac{1}{2}||f^\epsilon||^2_{L^2_l}) + \frac{C_1(f_0)}{2}||f^\epsilon||^2_{\epsilon, l+\gamma/2} \\ \lesssim\ & (C_2(f_0) + \frac{1}{C_1(f_0)}||f^\epsilon||^2_{L^1_{2l+5}})||f^\epsilon||^2_{L^2_{l+\gamma/2}} \\ \lesssim\ & C_3(||f_0||_{L^1_{2l+5}}, ||f_0||_{L\log L})||f^\epsilon||^2_{L^2_{l+\gamma/2}}, \end{aligned}$$

where the existence of $C_3(f_0,l) = C_3(||f_0||_{L^1_{2l+5}}, ||f_0||_{L\log L}, l)$ is ensured by the previous step. By applying (3.24) with $\lambda = \frac{C_1(f_0)}{4C_3(f_0,l)}$, we have

$$\frac{d}{dt}(\frac{1}{2}||f^\epsilon||^2_{L^2_l}) + \frac{C_1(f_0)}{4}||f^\epsilon||^2_{\epsilon,l+\gamma/2} \lesssim C_3(f_0,l)(\frac{C_1(f_0)}{4C_3(f_0,l)})^{-\frac{3}{2s}}||f^\epsilon||^2_{L^1_{l+\gamma/2}}.$$

Thanks to Gronwall's inequality, there is a constant $C(||f_0||_{L^1_{2l+5}}, ||f_0||_{L^2_l})$ such that for any $t \geq 0$,

$$||f^\epsilon(t)||^2_{L^2_l} + \int_t^{t+1} ||f^\epsilon(r)||^2_{\epsilon,l+\gamma/2} dr \leq C(||f_0||_{L^1_{2l+5}}, ||f_0||_{L^2_l}). \tag{3.64}$$

Inequality (1.17) is obtained in the case of $N = 0$.

Step 3: Propagation of H^s_l norm.

We first introduce some notations for the fractional derivative. We set

$$(\triangle_s f)(v) = \frac{(\tau_h f)(v) - f(v)}{|h|^{\frac{3}{2}+s}},$$

with $(\tau_h f)(v) = f(v+h)$ and $0 < s < 1$. Then there holds

$$\begin{aligned} \triangle_s(fg) &= \triangle_s f g + \tau_h f \triangle_s g \\ &= f\triangle_s g + \triangle_s f \tau_h g. \end{aligned}$$

By the definition of fractional Sobolev space, one has:

$$\|g\|^2_{H^s} \sim \int_{|h|\leq\frac{1}{2}} \|\triangle_s g\|^2_{L^2} dh + \|g\|^2_{L^2}. \tag{3.65}$$

Moreover, we also have, for $|h| \leq \frac{1}{2}$ and $m \in \mathbb{R}$,

$$\|g\langle v\rangle^k \triangle_s \langle v\rangle^l\|_{H^m} \lesssim |h|^{-(\frac{1}{2}+s)}\|g\langle v\rangle^{l+k}\|_{H^m}, \tag{3.66}$$

$$\|\tau_h g\|_{H^m_l} \sim \|g\|_{H^m_l}, \tag{3.67}$$

and

$$\begin{aligned} &\|g\|^2_{H^m_l} + \int_{|h|\leq\frac{1}{2}} \|\langle v\rangle^l \triangle_s g\|^2_{H^m} dh \\ \sim\ &\|g\|^2_{H^m_l} + \int_{|h|\leq\frac{1}{2}} \|\triangle_s(g\langle v\rangle^l)\|^2_{H^m} dh \sim \|g\|^2_{H^{m+s}_l}. \end{aligned} \tag{3.68}$$

One may check the proof of (3.66), (3.67) and (3.68) in the appendix of [53].

Let $g^\epsilon = f^\epsilon \langle v\rangle^l$. It is not difficult to check that $\triangle_s g^\epsilon$ satisfies the following equation:

$$\begin{aligned}
&\partial_t(\triangle_s g^\epsilon) - M^\epsilon(f^\epsilon, \triangle_s g^\epsilon)\\
= \;& M^\epsilon(\triangle_s f^\epsilon, \tau_h g^\epsilon) + [M^\epsilon(\triangle_s f^\epsilon, f^\epsilon)\langle v\rangle^l - M^\epsilon(\triangle_s f^\epsilon, f^\epsilon\langle v\rangle^l)]\\
&+[M^\epsilon(\tau_h f^\epsilon, \triangle_s f^\epsilon)\langle v\rangle^l - M^\epsilon(\tau_h f^\epsilon, \triangle_s f^\epsilon\langle v\rangle^l)]]\\
&+[M^\epsilon(\tau_h f^\epsilon, \tau_h f^\epsilon)\triangle_s\langle v\rangle^l - M^\epsilon(\tau_h f^\epsilon, \tau_h f^\epsilon\triangle_s\langle v\rangle^l)]\\
\stackrel{\text{def}}{=} \;& \sum_{i=1}^4 F_i.
\end{aligned}$$

By the upper bound estimate (3.1), noting $\gamma \le 2$, we have

$$\langle F_1, \triangle_s g^\epsilon\rangle \lesssim ||\triangle_s f^\epsilon||_{L^1_4}(||g^\epsilon||_{H^s_3}||\triangle_s g^\epsilon||_{H^s_{\gamma/2}} + \epsilon^{2-2s}||g^\epsilon||_{H^1_3}||\triangle_s g^\epsilon||_{H^1_{\gamma/2}}),$$

which implies, for any $\eta_1 > 0$,

$$\begin{aligned}
&\langle F_1, \triangle_s g^\epsilon\rangle - \eta_1||\triangle_s g^\epsilon||^2_{\epsilon,\gamma/2}\\
\lesssim\;& \frac{1}{2\eta_1}(||\triangle_s f^\epsilon||^2_{L^1_4}||g^\epsilon||^2_{H^s_3} + \epsilon^{2-2s}||\triangle_s f^\epsilon||^2_{L^1_4}||g^\epsilon||^2_{H^1_3}).
\end{aligned}\tag{3.69}$$

By the commutator estimates (3.7) and (3.8), we have

$$\begin{aligned}
\langle F_2, \triangle_s g^\epsilon\rangle \lesssim\;& ||\triangle_s f^\epsilon||_{L^1_{2l+1}}||f^\epsilon||_{H^s_{l+\gamma/2}}||\triangle_s g^\epsilon||_{L^2_{\gamma/2}}\\
&+\epsilon^{2-2s}||\triangle_s f^\epsilon||_{L^1_{\gamma+3}}||f^\epsilon||_{H^1_{l+\gamma/2}}||\triangle_s g^\epsilon||_{L^2_{\gamma/2}},
\end{aligned}$$

which implies, for any $\eta_1 > 0$,

$$\begin{aligned}
&\langle F_2, \triangle_s g^\epsilon\rangle - \eta_1||\triangle_s g^\epsilon||^2_{\epsilon,\gamma/2}\\
\lesssim\;& \frac{1}{2\eta_1}(||\triangle_s f^\epsilon||^2_{L^1_{2l+1}}||f^\epsilon||^2_{H^s_{l+1}} + \epsilon^{2-2s}||\triangle_s f^\epsilon||^2_{L^1_5}||f^\epsilon||^2_{H^1_{l+1}}).
\end{aligned}\tag{3.70}$$

Similarly, we have

$$\begin{aligned}
\langle F_3, \triangle_s g^\epsilon\rangle \lesssim\;& ||f^\epsilon||_{L^1_{2l+1}}||\triangle_s f^\epsilon||_{H^s_{l+\gamma/2}}||\triangle_s g^\epsilon||_{L^2_{\gamma/2}}\\
&+\epsilon^{2-2s}||f^\epsilon||_{L^1_{\gamma+3}}||\triangle_s f^\epsilon||_{H^1_{l+\gamma/2}}||\triangle_s g^\epsilon||_{L^2_{\gamma/2}},
\end{aligned}$$

which implies, for any $\eta_2 > 0$,

$$\begin{aligned}&\langle F_3, \triangle_s g^\epsilon\rangle - \eta_2 ||\triangle_s f^\epsilon||^2_{\epsilon, l+\gamma/2}\\ \lesssim\ & \frac{1}{2\eta_2}(||f^\epsilon||^2_{L^1_{2l+1}}||\triangle_s g^\epsilon||^2_{L^2_1} + \epsilon^{2-2s}||f^\epsilon||^2_{L^1_5}||\triangle_s g^\epsilon||^2_{L^2_1}).\end{aligned} \tag{3.71}$$

Also by the upper bound estimate (3.1), we have

$$\begin{aligned}\langle F_4, \triangle_s g^\epsilon\rangle \ \ \lesssim\ \ & ||f^\epsilon||_{L^1_{l+5}}||f^\epsilon||_{H^s_{l+3}}||(\triangle_s g^\epsilon)(\triangle_s\langle v\rangle^l)||_{H^s_{-l+\gamma/2}}\\ &+\epsilon^{2-2s}||f^\epsilon||_{L^1_{l+5}}||f^\epsilon||_{H^1_{l+3}}||(\triangle_s g^\epsilon)(\triangle_s\langle v\rangle^l)||_{H^1_{-l+\gamma/2}}\\ &+||f^\epsilon||_{L^1_4}||(\tau_h f^\epsilon)(\triangle_s\langle v\rangle^l)||_{H^s_3}||\triangle_s g^\epsilon||_{H^s_{\gamma/2}}\\ &+\epsilon^{2-2s}||f^\epsilon||_{L^1_4}||(\tau_h f^\epsilon)(\triangle_s\langle v\rangle^l)||_{H^1_3}||\triangle_s g^\epsilon||_{H^1_{\gamma/2}},\end{aligned}$$

which implies, for any $\eta_1 > 0$,

$$\begin{aligned}&\langle F_4, \triangle_s g^\epsilon\rangle - \eta_1 ||\triangle_s g^\epsilon||^2_{\epsilon,\gamma/2}\\ \lesssim\ & \frac{1}{\eta_1}|h|^{-(1+2s)}(||f^\epsilon||^2_{L^1_{l+5}}||f^\epsilon||^2_{H^s_{l+3}} + \epsilon^{2-2s}||f^\epsilon||^2_{L^1_{l+5}}||f^\epsilon||^2_{H^1_{l+3}}).\end{aligned} \tag{3.72}$$

By the coercivity estimate (3.5), we have

$$\langle M^\epsilon(f^\epsilon, \triangle_s g^\epsilon), \triangle_s g^\epsilon\rangle \le -C_1(f_0)||\triangle_s g^\epsilon||^2_{\epsilon,\gamma/2} + C_2(f_0)||\triangle_s g^\epsilon||^2_{L^2_{\gamma/2}}. \tag{3.73}$$

Thanks to

$$\frac{d}{dt}(\frac{1}{2}||\triangle_s g^\epsilon||^2_{L^2}) \ \ = \ \ \langle\partial_t\triangle_s g^\epsilon, \triangle_s g^\epsilon\rangle = \langle M^\epsilon(f^\epsilon, \triangle_s g^\epsilon), \triangle_s g^\epsilon\rangle + \sum_{i=1}^4\langle F_i, \triangle_s g^\epsilon\rangle,$$

patching together all the above estimates, taking $\eta_1 = \frac{C_1(f_0)}{6}$ in (3.69),(3.70),(3.72), we arrive at, for $|h| \le \frac{1}{2}$,

$$
\begin{aligned}
&\frac{d}{dt}\Big(\frac{1}{2}||\triangle_s g^\epsilon||_{L^2}^2\Big) + \frac{C_1(f_0)}{2}||\triangle_s g^\epsilon||_{\epsilon,\gamma/2}^2 - \eta_2||\triangle_s f^\epsilon||_{\epsilon,l+\gamma/2}^2 \\
\lesssim\ & +C_2(f_0)||\triangle_s g^\epsilon||_{L^2_{\gamma/2}}^2 + \frac{1}{2\eta_1}(||\triangle_s f^\epsilon||_{L^1_4}^2||g^\epsilon||_{H^s_3}^2 + \epsilon^{2-2s}||\triangle_s f^\epsilon||_{L^1_4}^2||g^\epsilon||_{H^1_3}^2) \\
& +\frac{1}{2\eta_1}(||\triangle_s f^\epsilon||_{L^1_{2l+1}}^2||f^\epsilon||_{H^s_{l+1}}^2 + \epsilon^{2-2s}||\triangle_s f^\epsilon||_{L^1_5}^2||f^\epsilon||_{H^1_{l+1}}^2) \\
& +\frac{1}{2\eta_2}(||f^\epsilon||_{L^1_{2l+1}}^2||\triangle_s g^\epsilon||_{L^2_1}^2 + \epsilon^{2-2s}||f^\epsilon||_{L^1_5}^2||\triangle_s g^\epsilon||_{L^2_1}^2) \\
& +\frac{1}{\eta_1}|h|^{-(1+2s)}(||f^\epsilon||_{L^1_{l+5}}^2||f^\epsilon||_{H^s_{l+3}}^2 + \epsilon^{2-2s}||f^\epsilon||_{L^1_{l+5}}^2||f^\epsilon||_{H^1_{l+3}}^2) \\
\lesssim\ & +C_2(f_0)||\triangle_s g^\epsilon||_{L^2_{\gamma/2}}^2 + \frac{1}{\eta_1}||\triangle_s f^\epsilon||_{L^2_6}^2||g^\epsilon||_{\epsilon,3}^2 \\
& +\frac{1}{\eta_1}(||\triangle_s f^\epsilon||_{L^2_{2l+3}}^2||f^\epsilon||_{H^s_{l+1}}^2 + \epsilon^{2-2s}||\triangle_s f^\epsilon||_{L^2_7}^2||f^\epsilon||_{H^1_{l+1}}^2) \\
& +\frac{1}{\eta_2}||f^\epsilon||_{L^1_{2l+5}}^2||\triangle_s g^\epsilon||_{L^2_1}^2 + \frac{1}{\eta_1}|h|^{-(1+2s)}||f^\epsilon||_{L^1_{l+5}}^2||f^\epsilon||_{\epsilon,l+3}^2.
\end{aligned}
$$

where we have used the fact $||\langle\cdot\rangle^{-2}||_{L^2} \le \sqrt{2}\pi$. Integrating both sides from 0 to t with respect to time, we obtain

$$
\begin{aligned}
&||\triangle_s g^\epsilon(t)||_{L^2}^2 + C_1(f_0)\int_0^t ||\triangle_s g^\epsilon(r)||_{\epsilon,\gamma/2}^2 dr - 2\eta_2\int_0^t ||\triangle_s f^\epsilon(r)||_{\epsilon,l+\gamma/2}^2 dr \\
\lesssim\ & ||\triangle_s g^\epsilon(0)||_{L^2}^2 + C_2(f_0)\int_0^t ||\triangle_s g^\epsilon(r)||_{L^2_{\gamma/2}}^2 dr \\
& + \frac{1}{\eta_1}\int_0^t ||\triangle_s f^\epsilon(r)||_{L^2_6}^2||g^\epsilon(r)||_{\epsilon,3}^2 dr \\
& + \frac{1}{\eta_1}\int_0^t (||\triangle_s f^\epsilon(r)||_{L^2_{2l+3}}^2||f^\epsilon(r)||_{H^s_{l+1}}^2 + \epsilon^{2-2s}||\triangle_s f^\epsilon(r)||_{L^2_7}^2||f^\epsilon(r)||_{H^1_{l+1}}^2)dr \\
& + \frac{1}{\eta_2}\int_0^t ||f^\epsilon(r)||_{L^1_{2l+5}}^2||\triangle_s g^\epsilon(r)||_{L^2_1}^2 dr \\
& + \frac{1}{\eta_1}|h|^{-(1+2s)}\int_0^t ||f^\epsilon(r)||_{L^1_{l+5}}^2||f^\epsilon(r)||_{\epsilon,l+3}^2 dr.
\end{aligned}
\tag{3.74}
$$

Integrating both sides with respect to h over the ball $B(0,\frac{1}{2})$, noting that $\int_{|h|\le\frac{1}{2}} |h|^{-(1+2s)}dh$ is finite, thanks to the facts (3.65) and (3.68), taking a small enough

η_2, we derive that

$$
\begin{aligned}
&||g^\epsilon(t)||^2_{H^s} + \frac{C_1(f_0)}{2}\int_0^t \int_{|h|\le\frac{1}{2}} ||\triangle_s g^\epsilon(r)||^2_{\epsilon,\gamma/2} dhdr \\
\lesssim\ & ||g^\epsilon(0)||^2_{H^s} + ||g^\epsilon(t)||^2_{L^2} + C_2(f_0)\int_0^t ||g^\epsilon(r)||^2_{H^s_1} dr \\
&+ \frac{1}{\eta_1}\int_0^t ||f^\epsilon(r)||^2_{H^s_6} ||g^\epsilon(r)||^2_{\epsilon,3} dr \\
&+ \frac{1}{\eta_1}\int_0^t (||f^\epsilon(r)||^2_{H^s_{2l+3}} ||f^\epsilon(r)||^2_{H^s_{l+1}} + \epsilon^{2-2s}||f^\epsilon(r)||^2_{H^s_7}||f^\epsilon(r)||^2_{H^1_{l+1}}) dr \\
&+ \frac{1}{\eta_2}\int_0^t ||f^\epsilon(r)||^2_{L^1_{2l+5}} ||g^\epsilon(r)||^2_{H^s_1} dr + \frac{1}{\eta_1}\int_0^t ||f^\epsilon(r)||^2_{L^1_{l+5}} ||f^\epsilon(r)||^2_{\epsilon,l+3} dr.
\end{aligned}
\tag{3.75}
$$

Using the fact $||f^\epsilon||^2_{H^s_{2l+3}} ||f^\epsilon||^2_{H^s_{l+1}} \le ||f^\epsilon||^2_{H^s_l} ||f^\epsilon||^2_{H^s_{2l+4}}$, substituting into the uniform upper bound of $||f^\epsilon||_{L^1_{2l+5}}$ and $||f^\epsilon||_{L^2_l}$ derived in previous two steps, we have

$$
\begin{aligned}
&||f^\epsilon(t)||^2_{H^s_l} + \frac{C_1(f_0)}{2}\int_0^t \int_{|h|\le\frac{1}{2}} ||\triangle_s g^\epsilon(r)||^2_{\epsilon,\gamma/2} dhdr \\
\lesssim\ & ||f_0||^2_{H^s_l} + C(||f_0||_{L^2_l}, ||f_0||_{L^1_{2l+5}}) \\
&+ C(||f_0||_{L^1_{2l+5}}, ||f_0||_{L\log L})\int_0^t ||f^\epsilon(r)||^2_{\epsilon,l+3} dr \\
&+ C(||f_0||_{L^1_1}, ||f_0||_{L\log L})\int_0^t ||f^\epsilon(r)||^2_{H^s_l} ||f^\epsilon(r)||^2_{\epsilon,2l+4} dr.
\end{aligned}
\tag{3.76}
$$

Actually, inequality (3.76) holds true on any bounded interval. Therefore, for any $t_1 < t_2$ with $t_2 - t_1 \le 2$, we have

$$
\begin{aligned}
&||f^\epsilon(t_2)||^2_{H^s_l} + \frac{C_1(f_0)}{2}\int_{t_1}^{t_2} \int_{|h|\le\frac{1}{2}} ||\triangle_s g^\epsilon(r)||^2_{\epsilon,\gamma/2} dhdr \\
\lesssim\ & ||f^\epsilon(t_1)||^2_{H^s_l} + C(||f_0||_{L^2_l}, ||f_0||_{L^1_{2l+5}}) \\
&+ C(||f_0||_{L^1_{2l+5}}, ||f_0||_{L\log L})\int_{t_1}^{t_2} ||f^\epsilon(r)||^2_{\epsilon,l+3} dr \\
&+ C(||f_0||_{L^1_1}, ||f_0||_{L\log L})\int_{t_1}^{t_2} ||f^\epsilon(r)||^2_{H^s_l} ||f^\epsilon(r)||^2_{\epsilon,2l+4} dr.
\end{aligned}
\tag{3.77}
$$

By Gronwall's inequality (1.12) and uniform upper bound (3.64) for integral of $||f^\epsilon||^2_{\epsilon,l}$ on any bounded interval, we arrive at

$$
||f^\epsilon(t_2)||^2_{H^s_l} \lesssim C(||f_0||_{L^1_{4l+13}}, ||f_0||_{L^2_{2l+4}})\{||f^\epsilon(t_1)||^2_{H^s_l} + C(||f_0||_{L^1_{2l+11}}, ||f_0||_{L^2_{l+3}})\}. \tag{3.78}
$$

Also from (3.64), we conclude that, in any unit interval $[t, t+1]$, there exists at least one point t_* such that

$$||f^\epsilon(t_*)||^2_{H^s_l} \lesssim C(||f_0||_{L^1_{2l+5}}, ||f_0||_{L^2_l}). \tag{3.79}$$

Combining (3.78) and (3.79), we have

$$||f^\epsilon(t)||^2_{H^s_l} \lesssim C(||f_0||_{H^s_l}, ||f_0||_{L^1_{4l+13}}, ||f_0||_{L^2_{2l+4}}). \tag{3.80}$$

Together with (3.77), we finally arrive at

$$\begin{aligned} &||f^\epsilon(t)||^2_{H^s_l} + \frac{C_1(f_0)}{2}\int_t^{t+1}\int_{|h|\le\frac{1}{2}} ||\triangle_s g^\epsilon(r)||^2_{\epsilon,\gamma/2} dhdr \\ \lesssim\, & C(||f_0||_{H^s_l}, ||f_0||_{L^1_{4l+13}}, ||f_0||_{L^2_{2l+4}}). \end{aligned} \tag{3.81}$$

By interpolation theory, there holds

$$||f_0||_{L^2_{2l+4}} \lesssim ||f_0||_{H^s_l} + ||f_0||_{H^{-2}_{\phi(s,l)}} \lesssim ||f_0||_{H^s_l} + ||f_0||_{L^1_{\phi(s,l)}},$$

where $\phi(s,l) = \frac{(2l+4)(2+s)-2l}{s} \ge 4l+13$. Therefore we have

$$||f^\epsilon(t)||^2_{H^s_l} \lesssim C(||f_0||_{H^s_l}, ||f_0||_{L^1_{\phi(s,l)}}). \tag{3.82}$$

Step 4: Propagation of H^N_l norm when $N \ge 1$.

We prove the propagation by induction on N. Let $m \ge 1$ be an integer. Suppose inequality (1.17) holds true for all $N \le m-1$, we now prove that it is also valid for $N = m$.

Set $g^\epsilon = \partial^\alpha_v f^\epsilon \langle v\rangle^l$ with $|\alpha| \le m$, then g^ϵ solves

$$\begin{aligned} \partial_t g^\epsilon &= M^\epsilon(f^\epsilon, g^\epsilon) + [M^\epsilon(f^\epsilon, \partial^\alpha_v f^\epsilon)\langle v\rangle^l - M^\epsilon(f^\epsilon, \partial^\alpha_v f^\epsilon\langle v\rangle^l)] \\ &\quad + \sum_{|\alpha_1|\ge 1, \alpha_1+\alpha_2=\alpha} \binom{\alpha}{\alpha_1} M^\epsilon(\partial^{\alpha_1}_v f^\epsilon, \partial^{\alpha_2}_v f^\epsilon)\langle v\rangle^l. \end{aligned} \tag{3.83}$$

By the coercivity estimate (3.5), we have

$$\langle M^\epsilon(f^\epsilon, g^\epsilon), g^\epsilon\rangle \le -C_1(f_0)||g^\epsilon||^2_{\epsilon,\gamma/2} + C_2(f_0)||g^\epsilon||^2_{L^2_{\gamma/2}}. \tag{3.84}$$

By the commutator estimate (3.9), we have

$$|\langle M^\epsilon(f^\epsilon,\partial_v^\alpha f^\epsilon)\langle v\rangle^l - M^\epsilon(f^\epsilon,\partial_v^\alpha f^\epsilon\langle v\rangle^l),g^\epsilon\rangle| \lesssim ||f^\epsilon||_{L^1_{2l+5}}||g^\epsilon||_{\epsilon,\gamma/2}||g^\epsilon||_{L^2_{\gamma/2}},$$

which implies, for any $\eta_1>0$,

$$\begin{aligned}&|\langle M^\epsilon(f^\epsilon,\partial_v^\alpha f^\epsilon)\langle v\rangle^l - M^\epsilon(f^\epsilon,\partial_v^\alpha f^\epsilon\langle v\rangle^l),g^\epsilon\rangle| - \eta_1||g^\epsilon||^2_{\epsilon,\gamma/2}\\ \lesssim\ &\frac{1}{\eta_1}||f^\epsilon||^2_{L^1_{2l+5}}||g^\epsilon||^2_{L^2_{\gamma/2}}.\end{aligned}\tag{3.85}$$

For the remaining terms in the right hand of (3.83) with $|\alpha_1|\geq 1$, we split each of them into two terms:

$$\begin{aligned}M^\epsilon(\partial_v^{\alpha_1}f^\epsilon,\partial_v^{\alpha_2}f^\epsilon)\langle v\rangle^l &= \{M^\epsilon(\partial_v^{\alpha_1}f^\epsilon,\partial_v^{\alpha_2}f^\epsilon)\langle v\rangle^l - M^\epsilon(\partial_v^{\alpha_1}f^\epsilon,\partial_v^{\alpha_2}f^\epsilon\langle v\rangle^l)\}\\ &\quad +M^\epsilon(\partial_v^{\alpha_1}f^\epsilon,\partial_v^{\alpha_2}f^\epsilon\langle v\rangle^l)\\ &\overset{\text{def}}{=} \mathfrak{I}_1+\mathfrak{I}_2.\end{aligned}$$

By the commutator estimate (3.9), for the case $|\alpha_1|=|\alpha|\leq m$, we have

$$|\langle \mathfrak{I}_1,g^\epsilon\rangle| \lesssim ||\partial_v^{\alpha_1}f^\epsilon||_{L^1_{2l+5}}||f^\epsilon||_{\epsilon,l+\gamma/2}||g^\epsilon||_{L^2_{\gamma/2}} \lesssim ||f^\epsilon||_{H^m_{2l+7}}||f^\epsilon||_{\epsilon,l+\gamma/2}||g^\epsilon||_{L^2_{\gamma/2}},$$

which implies, for any $\eta_2>0$,

$$|\langle \mathfrak{I}_1,g^\epsilon\rangle| - \eta_2||g^\epsilon||^2_{\epsilon,\gamma/2} \lesssim \frac{1}{\eta_2}||f^\epsilon||^2_{H^m_{2l+7}}||f^\epsilon||^2_{\epsilon,l+\gamma/2}\tag{3.86}$$

For the case $1\leq|\alpha_1|\leq|\alpha|-1\leq m-1$, we have

$$\begin{aligned}|\langle \mathfrak{I}_1,g^\epsilon\rangle| &\lesssim ||\partial_v^{\alpha_1}f^\epsilon||_{L^1_{2l+5}}||\partial_v^{\alpha_2}f^\epsilon||_{\epsilon,l+\gamma/2}||g^\epsilon||_{L^2_{\gamma/2}}\\ &\lesssim (||f^\epsilon||_{H^1_{2l+7}}||f^\epsilon||_{H^m_{l+\gamma/2}}\mathbf{1}_{m\geq 2}+||f^\epsilon||_{H^{m-1}_{2l+7}}||f^\epsilon||_{H^{m-1}_{l+\gamma/2}})||g^\epsilon||_{L^2_{\gamma/2}},\end{aligned}$$

which implies, for any $\eta_2>0$,

$$\begin{aligned}&|\langle \mathfrak{I}_1,g^\epsilon\rangle| - \eta_2||g^\epsilon||^2_{\epsilon,\gamma/2}\\ \lesssim\ &\frac{1}{\eta_2}(||f^\epsilon||^2_{H^1_{2l+7}}||f^\epsilon||^2_{H^m_{l+\gamma/2}}\mathbf{1}_{m\geq 2}+||f^\epsilon||^2_{H^{m-1}_{2l+7}}||f^\epsilon||^2_{H^{m-1}_{l+\gamma/2}}).\end{aligned}\tag{3.87}$$

By the upper bound estimate (3.1), for the case $|\alpha_1|=|\alpha|\leq m$, we have,

$$|\langle \mathfrak{I}_2,g^\epsilon\rangle| \lesssim ||\partial_v^{\alpha_1}f^\epsilon||_{L^1_4}(||f^\epsilon||_{H^s_{l+3}}||g^\epsilon||_{H^s_{\gamma/2}}+\epsilon^{2-2s}||f^\epsilon||_{H^1_{l+3}}||g^\epsilon||_{H^1_{\gamma/2}}),$$

which implies, for any $\eta_3 > 0$,

$$|\langle \mathfrak{I}_2, g^\epsilon\rangle| - \eta_3||g^\epsilon||^2_{\epsilon,\gamma/2} \lesssim \frac{1}{\eta_3}||f^\epsilon||^2_{H^m_6}||f^\epsilon||^2_{\epsilon,l+3}. \tag{3.88}$$

While for the case $1 \leq |\alpha_1| \leq |\alpha| - 1 \leq m-1$, we similarly have, for any $\eta_3 > 0$,

$$|\langle \mathfrak{I}_2, g^\epsilon\rangle| - \eta_3||g^\epsilon||^2_{\epsilon,\gamma/2} \lesssim \frac{1}{\eta_3}(||f^\epsilon||^2_{H^1_6}||f^\epsilon||^2_{H^m_{l+3}}\mathbf{1}_{m\geq 2} + ||f^\epsilon||^2_{H^{m-1}_6}||f^\epsilon||^2_{H^{m-1}_{l+3}}). \tag{3.89}$$

Now choosing suitable η_1 in (3.85), η_2 in (3.86) and (3.87), and η_3 in (3.88) and (3.89), we have

$$\begin{aligned}
&\frac{d}{dt}||f^\epsilon||^2_{H^m_l} + \frac{C_1(f_0)}{2}||f^\epsilon||^2_{\epsilon,m,l+\gamma/2} \\
\lesssim\ & C(||f_0||_{L^1_{2l+5}}, ||f_0||_{L^2})||f^\epsilon||^2_{H^m_{l+\gamma/2}} + C(C_1(f_0))\{||f^\epsilon||^2_{H^m_{2l+7}}||f^\epsilon||^2_{\epsilon,l+3} \\
&+ ||f^\epsilon||^2_{H^1_{2l+7}}||f^\epsilon||^2_{H^m_{l+3}}\mathbf{1}_{m\geq 2} + ||f^\epsilon||^2_{H^{m-1}_{2l+7}}||f^\epsilon||^2_{H^{m-1}_{l+3}}\}.
\end{aligned} \tag{3.90}$$

When $m = 1$, inequality (3.90) reduces to

$$\begin{aligned}
&\frac{d}{dt}||f^\epsilon||^2_{H^1_l} + \frac{C_1(f_0)}{2}||f^\epsilon||^2_{\epsilon,1,l+\gamma/2} \\
\lesssim\ & C(||f_0||_{L^1_{2l+5}}, ||f_0||_{L^2})||f^\epsilon||^2_{H^1_{l+\gamma/2}} \\
&+C(C_1(f_0))\{||f^\epsilon||^2_{H^1_{2l+7}}||f^\epsilon||^2_{\epsilon,l+3} + ||f^\epsilon||^2_{L^2_{2l+7}}||f^\epsilon||^2_{L^2_{l+3}}\}.
\end{aligned}$$

Remembering that

$$||f^\epsilon||^2_{\epsilon,l+3} = ||f^\epsilon||^2_{H^s_{l+3}} + \epsilon^{2-2s}||f^\epsilon||^2_{H^1_{l+3}},$$

by interpolation theory and the basic inequality (1.10), for any $\eta > 0$, we have

$$\begin{aligned}
||f^\epsilon||^2_{H^1_{2l+7}} &\leq ||f^\epsilon||^{2(1-s)}_{H^{1+s}_{l+\gamma/2}}||f^\epsilon||^{2s}_{H^s_{\psi(l)}} \\
&\leq \eta||f^\epsilon||^2_{H^{1+s}_{l+\gamma/2}} + s(\frac{\eta}{1-s})^{-\frac{1-s}{s}}||f^\epsilon||^2_{H^s_{x(l)}},
\end{aligned}$$

where $x(l) = \frac{2l+7}{s} - \frac{1-s}{s}(l + \frac{\gamma}{2})$, and

$$\begin{aligned}
||f^\epsilon||^2_{H^1_{2l+7}}||f^\epsilon||^2_{H^1_{l+3}} &\leq ||f^\epsilon||^4_{H^1_{2l+7}} \\
&\leq ||f^\epsilon||^{\frac{4-4s}{2-s}}_{H^2_{l+\gamma/2}}||f^\epsilon||^{\frac{4}{2-s}}_{H^s_{\psi'(l)}} \\
&\leq \eta||f^\epsilon||^2_{H^2_{l+\gamma/2}} + \frac{s}{2-s}(\frac{2\eta - s\eta}{2-2s})^{-\frac{2-2s}{s}}||f^\epsilon||^{4/s}_{H^s_{\tilde{x}(l)}},
\end{aligned}$$

where $\tilde{x}(l) = (2l+7)(2-s) - (1-s)(l+\gamma/2) \le x(l)$. Taking a small enough $\eta > 0$, we finally have

$$\frac{d}{dt}||f^\epsilon||^2_{H^1_l} + \frac{C_1(f_0)}{4}||f^\epsilon||^2_{\epsilon,1,l+\gamma/2} \lesssim C(||f^\epsilon||_{H^s_{x(l)}}, ||f_0||_{L^1_{2l+5}}).$$

Then by Gronwall's inequality and the uniform upper bound (3.82) of H^s norm, we arrive at

$$||f^\epsilon(t)||^2_{H^1_l} + \frac{C_1(f_0)}{4}\int_t^{t+1} ||f^\epsilon(r)||^2_{\epsilon,1,l+\gamma/2} dr \lesssim C(||f_0||_{L^1_{\phi(s,x(l))}}, ||f_0||_{H^s_{x(l)}}, ||f_0||_{H^1_l}).$$

Once again by interpolation theory, there holds

$$||f_0||_{H^s_{x(l)}} \lesssim ||f_0||_{H^1_l} + ||f_0||_{L^1_{y(l)}},$$

where $y(l) = \frac{3x(l)-(s+2)l}{1-s}$. By setting $\phi(1,l) = \max\{\phi(s,x(l)), y(l)\}$, we have

$$||f^\epsilon(t)||^2_{H^1_l} + \frac{C_1(f_0)}{4}\int_t^{t+1} ||f^\epsilon(r)||^2_{\epsilon,1,l+\gamma/2} dr \lesssim C(||f_0||_{L^1_{\phi(1,l)}}, ||f_0||_{H^1_l}).$$

When $m \ge 2$, $||f^\epsilon||^2_{H^1_{2l+7}}$ has uniform bound by assumption. According to the interpolation inequality and the basic inequality (1.10), one has

$$||f^\epsilon||^2_{H^m_{2l+7}} \le \eta||f^\epsilon||^2_{H^{m+s}_{l+\gamma/2}} + (\frac{1+s}{s}\eta)^{-\frac{1}{s}}||f^\epsilon||^2_{H^{m-1}_{z(l)}}, \tag{3.91}$$

where $z(l) = 2l+7+\frac{l+7}{s}$. With the fact $||f^\epsilon||_{\epsilon,l+3} \lesssim ||f^\epsilon||_{H^1_{l+3}}$, we finally arrive at

$$\frac{d}{dt}||f^\epsilon||^2_{H^m_l} + \frac{C_1(f_0)}{4}||f^\epsilon||^2_{\epsilon,m,l+\gamma/2} \lesssim C(||f_0||_{L^1_{2l+5}}, ||f^\epsilon||_{H^{m-1}_{z(l)}}).$$

Then by Gronwall's inequality and the assumed uniform bound of H^{m-1} norm,

$$\begin{aligned} &||f^\epsilon(t)||^2_{H^m_l} + \frac{C_1(f_0)}{4}\int_t^{t+1} ||f^\epsilon(r)||^2_{\epsilon,m,l+\gamma/2} dr \\ \lesssim\ & C(||f_0||_{L^1_{\phi(m-1,z(l))}}, ||f_0||_{H^{m-1}_{z(l)}}, ||f_0||_{H^m_l}). \end{aligned}$$

By interpolation theory, there holds

$$||f_0||_{H^{m-1}_{z(l)}} \lesssim ||f_0||_{H^m_l} + ||f_0||_{L^1_{u(m,l)}},$$

where $u(m,l) = (m+2)z(l) - (m+1)l$. Now by setting $\phi(m,l) = \max\{u(m,l), \phi(m-1, z(l))\}$, we arrive at

$$||f^\epsilon(t)||^2_{H^m_l} + \frac{C_1(f_0)}{4}\int_t^{t+1} ||f^\epsilon(r)||^2_{\epsilon,m,l+\gamma/2} dr \lesssim C(||f_0||_{L^1_{\phi(m,l)}}, ||f_0||_{H^m_l}). \tag{3.92}$$

The proof of Theorem 1.1 is complete now.

Remark 3.2. Since $L^1 \subset H^{-m}$ if $m > 3/2$, one can obtain lower weight requirement in the space L^1. We use H^{-2} as one interpolation space just for a neat expression. For the same reason, we replace γ by 2.

Chapter 4

Error Estimates of the Approximation

In this chapter, we prove the last two theorems stated in section 1.2 of the introduction, which show that our approximation error is of order $3-2s$.

4.1 Moment error

In this section, we give a proof to Theorem 1.2. As one can see in the proof below, the key point is to control the weight on the error function F_R^ϵ defined below in 4.1. Since we already have the propagation property of solutions f and f^ϵ accordingly to Theorem 1.1, we may transfer weight from F_R^ϵ to them as much as we need. Moreover, we could exploit smoothness of solutions f and f^ϵ thanks to the assumed smoothness of the initial datum.

Proof of Theorem 1.2: For each $0<\epsilon\le\sqrt{2}/2$, we define F_R^ϵ and Q_ϵ respectively as follows:

$$F_R^\epsilon = \frac{f^\epsilon - f}{\epsilon^{3-2s}}, \tag{4.1}$$

$$Q_\epsilon = Q - Q^\epsilon. \tag{4.2}$$

Take the difference between equations (1.1) and (1.7), and divide both sides by ϵ^{3-2s}, we have

$$\partial_t F_R^\epsilon = \Upsilon(f^\epsilon) + Q(f^\epsilon, F_R^\epsilon) + Q(F_R^\epsilon, f) \tag{4.3}$$

where

$$\Upsilon(f^\epsilon) = \frac{1}{\epsilon}[Q_L(f^\epsilon, f^\epsilon) - \epsilon^{2s-2} Q_\epsilon(f^\epsilon, f^\epsilon)]. \tag{4.4}$$

We now show that L^1_{2l} norm of F^ϵ_R is bounded by some norm of the initial datum f_0 and time t. According to (4.3), we have

$$\begin{aligned} \frac{d}{dt}||F^\epsilon_R||_{L^1_{2l}} &= \langle \Upsilon(f^\epsilon) + Q(f^\epsilon, F^\epsilon_R) + Q(F^\epsilon_R, f), sgn(F^\epsilon_R)\langle v\rangle^{2l}\rangle \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3. \end{aligned} \tag{4.5}$$

According to lemma 7.1 in [52], we have

$$\mathfrak{I}_1 \leq C(7.1)||f^\epsilon||^2_{H^5_{2l+\gamma+12}}. \tag{4.6}$$

Now we deal with $\mathfrak{I}_2$. Note that

$$\begin{aligned} \mathfrak{I}_2 &= \int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma f^\epsilon_* F^\epsilon_R(sgn(F^\epsilon_R(v'))\langle v'\rangle^{2l} - sgn(F^\epsilon_R(v))\langle v\rangle^{2l}) d\sigma dv_* dv \\ &\leq \int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma f^\epsilon_* |F^\epsilon_R|(\langle v'\rangle^{2l} - \langle v\rangle^{2l}) d\sigma dv_* dv \\ &\stackrel{\text{def}}{=} \mathfrak{I}_{2,1} + \mathfrak{I}_{2,2}, \end{aligned}$$

where

$$\mathfrak{I}_{2,1} = \int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma f^\epsilon_* |F^\epsilon_R|(\langle v'\rangle^{2l} + \langle v'_*\rangle^{2l} - \langle v\rangle^{2l} - \langle v_*\rangle^{2l}) d\sigma dv_* dv,$$

and

$$\mathfrak{I}_{2,2} = -\int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)|v-v_*|^\gamma f^\epsilon_* |F^\epsilon_R|(\langle v'_*\rangle^{2l} - \langle v_*\rangle^{2l}) d\sigma dv_* dv.$$

According to Proposition 3.2, we have

$$\mathfrak{I}_{2,1} \leq -\frac{1}{4}A_2(||f^\epsilon||_{L^1}||F^\epsilon_R||_{L^1_{2l+\gamma}} + ||f^\epsilon||_{L^1_{2l+\gamma}}||F^\epsilon_R||_{L^1}) + 2^{2l+1}A_2||f^\epsilon||_{L^1_{2l}}||F^\epsilon_R||_{L^1_{2l}}. \tag{4.7}$$

Now we turn to $\mathfrak{I}_{2,2}$. By Taylor expansion,

$$\begin{aligned} \langle v'_*\rangle^{2l} - \langle v_*\rangle^{2l} &= (v'_* - v_*)\cdot(\nabla\langle\cdot\rangle^{2l})(v_*) \\ &\quad + \int_0^1 \frac{1-\kappa}{2}(v'_* - v_*)\otimes(v'_* - v_*) : (\nabla^2\langle\cdot\rangle^{2l})(v(\kappa)) d\kappa, \end{aligned}$$

where $v(\kappa) = v_* + \kappa(v'_* - v_*) = v_* - \kappa(v' - v)$. By symmetry,

$$\int_{\mathbb{S}^2} b(\cos\theta)(v'_* - v_*)d\sigma = (v - v_*)\int_{\mathbb{S}^2} b(\cos\theta)\sin^2\frac{\theta}{2}d\sigma. \tag{4.8}$$

Observe that the matrix $\nabla^2\langle\cdot\rangle^{2l}$ is positive definite, we are only left with

$$\begin{aligned}\mathfrak{I}_{2,2} &\le 2l\int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)\sin^2\frac{\theta}{2}|v - v_*|^{\gamma+1}\langle v_*\rangle^{2l-1}f^\epsilon_*|F^\epsilon_R|d\sigma dv_* dv\\ &\le A_2 l||f^\epsilon||_{L^1_{2l+\gamma}}||F^\epsilon_R||_{L^1_{\gamma+1}}.\end{aligned} \tag{4.9}$$

Split $\mathfrak{I}_3$ into two parts:

$$\begin{aligned}\mathfrak{I}_3 &= \int_{\mathbb{R}^6\times\mathbb{S}^2} BF^\epsilon_R(v_*)f(sgn(F^\epsilon_R(v'))\langle v'\rangle^{2l} - sgn(F^\epsilon_R(v))\langle v\rangle^{2l})d\sigma dv_* dv\\ &= \int_{\mathbb{R}^6\times\mathbb{S}^2} B\mathbf{1}_{\theta\le|v-v_*|^{-\alpha}}F^\epsilon_R(v_*)f(sgn(F^\epsilon_R(v'))\langle v'\rangle^{2l} - sgn(F^\epsilon_R(v))\langle v\rangle^{2l})d\sigma dv_* dv\\ &\quad + \int_{\mathbb{R}^6\times\mathbb{S}^2} B\mathbf{1}_{\theta\ge|v-v_*|^{-\alpha}}F^\epsilon_R(v_*)f(sgn(F^\epsilon_R(v'))\langle v'\rangle^{2l} - sgn(F^\epsilon_R(v))\langle v\rangle^{2l})d\sigma dv_* dv\\ &\overset{\text{def}}{=} \mathfrak{I}_{3,1} + \mathfrak{I}_{3,2},\end{aligned}$$

where $\alpha = \frac{\gamma+2}{2-2s}$. Such a choice of α is to ensure that $\mathfrak{I}_{3,1,2}$ defined below is controlled by $||F^\epsilon_R||_{L^1_{2l}}$.

For $\mathfrak{I}_{3,1}$, we have

$$\begin{aligned}\mathfrak{I}_{3,1} &= \int_{\mathbb{R}^6\times\mathbb{S}^2} B\mathbf{1}_{\theta\le|v-v_*|^{-\alpha}}(F^\epsilon_R)_*(sgn(F^\epsilon_R(v'))f'\langle v'\rangle^{2l} - sgn(F^\epsilon_R(v))f\langle v\rangle^{2l})d\sigma dv_* dv\\ &\quad + \int_{\mathbb{R}^6\times\mathbb{S}^2} B\mathbf{1}_{\theta\le|v-v_*|^{-\alpha}}(F^\epsilon_R)_*(f - f')sgn(F^\epsilon_R(v'))\langle v'\rangle^{2l}d\sigma dv_* dv\\ &\overset{\text{def}}{=} \mathfrak{I}_{3,1,1} + \mathfrak{I}_{3,1,2}.\end{aligned}$$

By cancellation lemma,

$$|\mathfrak{I}_{3,1,1}| \le C(cancel)||f||_{L^1_{2l+\gamma}}||F^\epsilon_R||_{L^1_\gamma}, \tag{4.10}$$

where $C(cancel) = 2^{\frac{5+\gamma}{2}}A_2$. Now we turn to the term $\mathfrak{I}_{3,1,2}$. By Taylor expansion:

$$f(v) - f(v') = (v - v')\cdot\nabla_v f(v') + \int_0^1 \frac{1-\kappa}{2}(v - v')\otimes(v - v') : \nabla^2_v f(v(\kappa))d\kappa,$$

where $v(\kappa) = v' + \kappa(v - v')$. For fixed v_*, it is easy to check

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} b(\cos\theta)\mathbf{1}_{\theta\le|v-v_*|^{-\alpha}}|v - v_*|^\gamma(v - v')\cdot\nabla_v f(v')sgn(F^\epsilon_R(v'))\langle v'\rangle^{2l}d\sigma dv = 0.$$

Thus we are only left with

$$|\mathfrak{I}_{3,1,2}| \leq \int_0^1 \int_{\mathbb{R}^6\times\mathbb{S}^2} \frac{1-\kappa}{2} b(\cos\theta)\sin^2\frac{\theta}{2}\mathbf{1}_{\theta\leq|v-v_*|^{-\alpha}}|v-v_*|^{\gamma+2} \times F_R^\epsilon(v_*)|\nabla_v^2 f(v(\kappa))|\langle v'\rangle^{2l} d\kappa d\sigma dv_* dv. \tag{4.11}$$

Set $u = v' + \kappa(v-v')$, then we have

$$\begin{aligned}\langle v'\rangle^2 &= 1+|v'|^2 = 1+|v'+\kappa(v-v')-\kappa(v-v')|^2\\ &\leq 1+2|u|^2+2\kappa^2|v-v'|^2 \leq 2\langle u\rangle^2 + 2\kappa^2|u-v_*|^2,\end{aligned}$$

and

$$\langle v'\rangle^{2l} \leq 2^{2l-1}\langle u\rangle^{2l} + 2^{2l-1}\kappa^{2l}|u-v_*|^{2l} \leq 2^{2l}\langle u\rangle^{2l}\langle v_*\rangle^{2l}.$$

In the change of variable: $v\to u$, the Jacobian matrix is

$$\frac{du}{dv} = \frac{1+k}{2}\left(I + \frac{1-k}{1+k}\frac{v-v_*}{|v-v_*|}\otimes\sigma\right),$$

with its Jacobian

$$\left|\frac{du}{dv}\right| = \frac{(1+k)^3}{8}\left(1+\frac{1-k}{1+k}\frac{v-v_*}{|v-v_*|}\cdot\sigma\right) \geq \frac{1}{8}.$$

Thanks to $|u-v_*| \leq |v-v_*| \leq \sqrt{2}|u-v_*|$, we obtain

$$\begin{aligned}|\mathfrak{I}_{3,1,2}| &\leq 2^{2l+3}\pi K\int_{\mathbb{R}^6}\int_0^{|u-v_*|^{-\alpha}\wedge\pi/2} \theta^{1-2s}|u-v_*|^{\gamma+2}\\ &\quad\times F_R^\epsilon(v_*)|\nabla_v^2 f(u)|\langle u\rangle^{2l}\langle v_*\rangle^{2l} d\theta dv_* du\\ &\leq 2^{2l+2}\frac{\pi K}{l-s}||\nabla_v^2 f||_{L^1_{2l}}||F_R^\epsilon||_{L^1_{2l}}\\ &\leq 2^{2l+\frac{5}{2}}\frac{\pi^2 K}{l-s}||f||_{H^2_{2l+2}}||F_R^\epsilon||_{L^1_{2l}},\end{aligned} \tag{4.12}$$

where we have used the fact $||\langle\cdot\rangle^{-2}||_{L^2} \leq \sqrt{2}\pi$.

Now we turn to $\mathfrak{I}_{3,2}$. Note that

$$\begin{aligned}\mathfrak{I}_{3,2} &\leq \int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)\mathbf{1}_{\theta\geq|v-v_*|^{-\alpha}}|v-v_*|^\gamma|F_R^\epsilon(v_*)|f(\langle v'\rangle^{2l}+\langle v\rangle^{2l})d\sigma dv_* dv\\ &= \int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)\mathbf{1}_{\theta\geq|v-v_*|^{-\alpha}}|v-v_*|^\gamma|F_R^\epsilon(v_*)|f(\langle v'\rangle^{2l}-\langle v\rangle^{2l})d\sigma dv_* dv\\ &\quad +2\int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)\mathbf{1}_{\theta\geq|v-v_*|^{-\alpha}}|v-v_*|^\gamma|F_R^\epsilon(v_*)|f\langle v\rangle^{2l}d\sigma dv_* dv\\ &\stackrel{\text{def}}{=} \mathfrak{I}_{3,2,1}+\mathfrak{I}_{3,2,2}.\end{aligned}$$

First look at the term $\mathfrak{I}_{3,2,1}$. Recall that $j=\frac{u-(u\cdot n)n}{|u-(u\cdot n)n|}$ in Proposition 3.1, then we have $j\cdot n=0$, and thus

$$\int_{\mathbb{S}^2} b(\cos\theta)\mathbf{1}_{\theta\geq|v-v_*|^{-\alpha}}(E(\theta))^{p-1}h(j\cdot\omega)\sin\theta d\sigma=0.$$

Applying Proposition 3.1 and the above equality, we obtain

$$\begin{aligned}
\mathfrak{I}_{3,2,1} \quad &\leq \quad \int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)\sin^{2l}\frac{\theta}{2}|v-v_*|^\gamma|F_R^\epsilon(v_*)|f\langle v_*\rangle^{2l}d\sigma dv_* dv\\
&\quad\quad +c_l\int_{\mathbb{R}^6\times\mathbb{S}^2} b(\cos\theta)\sin^2\theta|v-v_*|^\gamma|F_R^\epsilon(v_*)|f\langle v_*\rangle^{2l-2}\langle v\rangle^{2l-2}d\sigma dv_* dv\\
&\overset{\text{def}}{=} \quad \mathfrak{I}_{3,2,1,1}+\mathfrak{I}_{3,2,1,2},
\end{aligned}$$

where $c_l=2^{l-3}(l(l-1)+4)$. Thanks to the following fact:

$$\begin{aligned}
\int_{\mathbb{S}^2} b(\cos\theta)\sin^{2l}\frac{\theta}{2}d\sigma \quad &\leq \quad 2\pi K\int_0^{\pi/2}\theta^{-1-2s}\sin^{2l}\frac{\theta}{2}d\theta\\
&= \quad 2^{1-2s}\pi K\int_0^{\pi/4}\eta^{-1-2s}\sin^{2l}\eta d\eta\\
&\leq \quad \frac{2^{-2s}\pi K}{l-s}(\frac{\pi}{4})^{2l-2s},
\end{aligned}$$

and $|v-v_*|^\gamma\leq 2(\langle v\rangle^\gamma+\langle v_*\rangle^\gamma)$, we have

$$\mathfrak{I}_{3,2,1,1}\leq\frac{2^{1-2s}\pi K}{l-s}(\frac{\pi}{4})^{2l-2s}(||f||_{L^1}||F_R^\epsilon||_{L^1_{2l+\gamma}}+||f||_{L^1_\gamma}||F_R^\epsilon||_{L^1_{2l}}). \tag{4.13}$$

Due to $|v-v_*|^\gamma\leq\langle v\rangle^2\langle v_*\rangle^2$, we obtain

$$\mathfrak{I}_{3,2,1,2}\leq c_l A_2||f||_{L^1_{2l}}||F_R^\epsilon||_{L^1_{2l}}. \tag{4.14}$$

As for the term $\mathfrak{I}_{3,2,2}$, we have

$$\begin{aligned}
|\mathfrak{I}_{3,2,2}| &\leq 4\pi K\int_{\mathbb{R}^6}\int_{|v-v_*|^{-\alpha}\wedge\pi/2}^{\pi/2}\theta^{-1-2s}|v-v_*|^\gamma|F_R^\epsilon(v_*)|f\langle v\rangle^{2l}d\theta dv_* dv\\
&\leq\frac{2\pi K}{s}||f||_{L^1_{4l}}||F_R^\epsilon||_{L^1_{2l}},
\end{aligned}\tag{4.15}$$

provided $2\alpha s+\gamma\leq 2l$.

Putting together the above inequalities (4.6),(4.7),(4.9),(4.10), and (4.12)-(4.15), for those l such that $\frac{2^{1-2s}\pi K}{l-s}(\frac{\pi}{4})^{2l-2s} \leq \frac{A_2}{8}$, we have the following desired result:

$$\begin{aligned}
\frac{d}{dt}||F_R^\epsilon||_{L^1_{2l}} \leq & -\frac{A_2}{8}||f_0||_{L^1}||F_R^\epsilon||_{L^1_{2l+\gamma}} + C(7.1)||f^\epsilon||^2_{H^5_{2l+\gamma+12}} \\
& +\{2^{2l+1}A_2||f^\epsilon||_{L^1_{2l}} + A_2 l||f^\epsilon||_{L^1_{2l+\gamma}} + C(cancel)||f||_{L^1_{2l+\gamma}} \\
& +2^{2l+\frac{1}{2}}\frac{\pi^2 K}{1-s}||f||_{H^2_{2l+2}} + \frac{A_2}{8}||f||_{L^1_\gamma} \\
& +c_l A_2||f||_{L^1_{2l}} + \frac{2\pi K}{s}||f||_{L^1_{4l}}\}||F_R^\epsilon||_{L^1_{2l}},
\end{aligned}$$

where we have used the mass conservation property: $||f_t||_{L^1} = ||f_t^\epsilon||_{L^1} = ||f_0||_{L^1}$. The propagation of L and H norms of f and f^ϵ allows us to conclude:

$$\begin{aligned}
\frac{d}{dt}||F_R^\epsilon||_{L^1_{2l}} \leq & -\frac{A_2}{8}||f_0||_{L^1}||F_R^\epsilon||_{L^1_{2l+\gamma}} + C(||f_0||_{L^1_{\phi(5,2l+\gamma+12)}}, ||f_0||_{H^5_{2l+\gamma+12}}) \\
& + C(||f_0||_{L^1_{\max\{41,\phi(2,2l+2)\}}}, ||f_0||_{H^2_{2l+2}})||F_R^\epsilon||_{L^1_{2l}}.
\end{aligned} \tag{4.16}$$

Applying Gronwall's inequality (1.11) with $a = C(||f_0||_{L^1_{\phi(5,2l+\gamma+12)}}, ||f_0||_{H^5_{2l+\gamma+12}})$ and $b = C(||f_0||_{L^1_{\max\{41,\phi(2,2l+2)\}}}, ||f_0||_{H^2_{2l+2}})$, we have

$$||F_R^\epsilon(t)||_{L^1_{2l}} \leq \frac{a}{b}(e^{bt}-1) \overset{\text{def}}{=} C(f_0,t).$$

The proof of Theorem 1.2 is complete now.

4.2 Smoothness error

Once we have the estimate of moment error, we may use it to prove smoothness error. The proof is divided into two steps. In the first step, we prove that $||F_R^\epsilon(t)||_{L^2_l}$ is bounded by some norm of the initial datum and time t. In the second step, we prove by induction that high order weighted Sobolev norm of $F_R^\epsilon(t)$ is also bounded by some norm of the initial datum and time t accordingly. We now prove Theorem 1.3 in the rest of this chapter.

Proof of Theorem 1.3:

Step 1: (Case $N = 0$)

Taking the difference between equations (1.1) and (1.7), and dividing both sides by ϵ^{3-2s}, we have

$$\partial_t F_R^\epsilon = \Upsilon(f) + M^\epsilon(f^\epsilon, F_R^\epsilon) + M^\epsilon(F_R^\epsilon, f).$$

Then we have

$$\begin{aligned} \frac{d}{dt}(\frac{1}{2}||F_R^\epsilon||_{L_l^2}^2) &= \langle \Upsilon(f) + M^\epsilon(f^\epsilon, F_R^\epsilon) + M^\epsilon(F_R^\epsilon, f), F_R^\epsilon \langle v\rangle^{2l}\rangle \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3. \end{aligned}$$

According to lemma 7.1 in [52], we have

$$\mathfrak{I}_1 \lesssim ||f||_{H^5_{l+\gamma+10}}^2 ||F_R^\epsilon||_{L_l^2},$$

which implies, for any $\eta > 0$,

$$\mathfrak{I}_1 - \eta||F_R^\epsilon||_{L^2}^2 \lesssim \frac{1}{\eta}||f||_{H^5_{l+\gamma+10}}^4. \tag{4.17}$$

Split $\mathfrak{I}_2$ into two terms

$$\begin{aligned} \mathfrak{I}_2 &= \langle M^\epsilon(f^\epsilon, F_R^\epsilon\langle v\rangle^l), F_R^\epsilon\langle v\rangle^l\rangle + \{\langle M^\epsilon(f^\epsilon, F_R^\epsilon)\langle v\rangle^l - M^\epsilon(f^\epsilon, F_R^\epsilon\langle v\rangle^l), F_R^\epsilon\langle v\rangle^l\rangle\} \\ &\stackrel{\text{def}}{=} \mathfrak{I}_{2,1} + \mathfrak{I}_{2,2}. \end{aligned}$$

By coercivity estimate (3.5), we have

$$\mathfrak{I}_{2,1} \leq -C_1(f_0)||F_R^\epsilon||_{\epsilon,l+\gamma/2}^2 + C_2(f_0)||F_R^\epsilon||_{L^2_{l+\gamma/2}}^2. \tag{4.18}$$

By commutator estimate (3.9) with $N_2 = l + \gamma/2, N_3 = \gamma/2$, we have,

$$\mathfrak{I}_{2,2} \lesssim ||f^\epsilon||_{L^1_{2l+5}}||F_R^\epsilon||_{\epsilon,l+\gamma/2}||F_R^\epsilon||_{L^2_{l+\gamma/2}},$$

which implies, for any $\eta > 0$,

$$\mathfrak{I}_{2,2} - \eta||F_R^\epsilon||_{\epsilon,l+\gamma/2}^2 \lesssim \frac{1}{\eta}||f^\epsilon||_{L^1_{2l+5}}^2||F_R^\epsilon||_{L^2_{l+\gamma/2}}^2. \tag{4.19}$$

Split $\mathfrak{I}_3$ into two terms

$$\begin{aligned} \mathfrak{I}_3 &= \langle M^\epsilon(F_R^\epsilon, f\langle v\rangle^l), F_R^\epsilon\langle v\rangle^l\rangle_v + \{\langle M^\epsilon(F_R^\epsilon, f)\langle v\rangle^l - M^\epsilon(F_R^\epsilon, f\langle v\rangle^l), F_R^\epsilon\langle v\rangle^l\rangle_v\} \\ &\stackrel{\text{def}}{=} \mathfrak{I}_{3,1} + \mathfrak{I}_{3,2}. \end{aligned}$$

Applying upper bound estimate (3.1) with $w_1 = \gamma/2 + 2, w_2 = \gamma/2$, we have

$$\mathfrak{I}_{3,1} \lesssim ||F_R^\epsilon||_{L^1_{\gamma+2}} ||f||_{H^1_{l+3}} ||F_R^\epsilon||_{\epsilon,l+\gamma/2},$$

which implies, for any $\eta > 0$,

$$\mathfrak{I}_{3,1} - \eta ||F_R^\epsilon||^2_{\epsilon,l+\gamma/2} \lesssim \frac{1}{\eta} ||F_R^\epsilon||^2_{L^1_{\gamma+2}} ||f||^2_{H^1_{l+3}}. \tag{4.20}$$

By commutator estimate (3.9), we have

$$\mathfrak{I}_{3,2} \lesssim ||F_R^\epsilon||_{L^1_{2l+5}} ||f||_{H^1_{l+\gamma/2}} ||F_R^\epsilon||_{L^2_{l+\gamma/2}},$$

which implies, for any $\eta > 0$,

$$\mathfrak{I}_{3,2} - \eta ||F_R^\epsilon||^2_{L^2_{l+\gamma/2}} \lesssim \frac{1}{\eta} ||F_R^\epsilon||^2_{L^1_{2l+5}} ||f||^2_{H^1_{l+\gamma/2}}. \tag{4.21}$$

Now setting $\eta = \frac{C_1(f_0)}{8}$ in (4.17),(4.19),(4.20),(4.21), and together with (4.18), we have

$$\begin{aligned}
&\frac{d}{dt} ||F_R^\epsilon||^2_{L^2_l} + C_1(f_0) ||F_R^\epsilon||^2_{\epsilon,l+\gamma/2} \\
\lesssim& \{C_2(f_0) + \frac{1}{\eta} ||f^\epsilon||^2_{L^1_{2l+5}}\} ||F_R^\epsilon||^2_{L^2_{l+\gamma/2}} + \frac{1}{\eta} ||f||^4_{H^5_{l+\gamma+10}} \\
&+ \frac{1}{\eta} ||F_R^\epsilon||^2_{L^1_{\gamma+2}} ||f||^2_{H^1_{l+3}} + \frac{1}{\eta} ||F_R^\epsilon||^2_{L^1_{2l+5}} ||f||^2_{H^1_{l+\gamma/2}}.
\end{aligned} \tag{4.22}$$

Now choosing $\lambda = \frac{C_1(f_0)}{2} (C_2(f_0) + \frac{1}{\eta} ||f^\epsilon||^2_{L^1_{2l+5}})^{-1}$ in (3.24), we have

$$\begin{aligned}
&\frac{d}{dt} ||F_R^\epsilon||^2_{L^2_l} + \frac{C_1(f_0)}{2} ||F_R^\epsilon||^2_{\epsilon,l+\gamma/2} \\
\lesssim& \{C_2(f_0) + \frac{1}{\eta} ||f^\epsilon||^2_{L^1_{2l+5}}\} \lambda^{-\frac{3}{2s}} ||F_R^\epsilon||^2_{L^1_{l+\gamma/2}} + \frac{1}{\eta} ||f||^4_{H^5_{l+\gamma+10}} \\
&+ \frac{1}{\eta} ||F_R^\epsilon||^2_{L^1_{\gamma+2}} ||f||^2_{H^1_{l+3}} + \frac{1}{\eta} ||F_R^\epsilon||^2_{L^1_{2l+5}} ||f||^2_{H^1_{l+\gamma/2}}.
\end{aligned} \tag{4.23}$$

According to Theorem 1.2, we have

$$||F_R^\epsilon(t)||_{L^1_{2l+5}} \leq C(||f_0||_{L^1_{\phi(5,2l+\gamma+17)}}, ||f_0||_{H^5_{2l+\gamma+17}}, t).$$

The other terms of the right hand side of (4.23) are also bounded by some lower order or lower weight norm of initial datum f_0, thus we arrive at

$$||F_R^\epsilon(t)||_{L^2_l} \leq C(||f_0||_{L^1_{\phi(5,2l+\gamma+17)}}, ||f_0||_{H^5_{2l+\gamma+17}}, t).$$

We remark that the dependence on t is also at most exponential.

Step 2: (Case $N \geq 1$)

Fix an integer $m \geq 1$, and suppose inequality (1.21) holds true for all $N \leq m-1$. We now prove that it is also valid for $N = m$.

Let $g^\epsilon_{\alpha,l} = \langle v\rangle^l \partial^\alpha_v F^\epsilon_R$ with $|\alpha| \leq m$, then $g^\epsilon_{\alpha,l}$ solves

$$\partial_t g^\epsilon_{\alpha,l} = \sum_{\alpha_1 \leq \alpha} \binom{\alpha}{\alpha_1} [M^\epsilon(\partial^{\alpha_1}_v f^\epsilon, \partial^{\alpha_2}_v F^\epsilon_R) + M^\epsilon(\partial^{\alpha_1}_v F^\epsilon_R, \partial^{\alpha_2}_v f) + \Upsilon(\partial^{\alpha_1}_v f, \partial^{\alpha_2}_v f)]\langle v\rangle^l.$$

Therefore we have

$$\begin{aligned} \frac{d}{dt}(\frac{1}{2}||g^\epsilon_{\alpha,l}||^2_{L^2}) &= \langle \partial_t g^\epsilon_{\alpha,l}, g^\epsilon_{\alpha,l}\rangle \\ &= \sum_{\alpha_1+\alpha_2=\alpha} \binom{\alpha}{\alpha_1} \{\langle M^\epsilon(\partial^{\alpha_1}_v f^\epsilon, \partial^{\alpha_2}_v F^\epsilon_R)\langle v\rangle^l, g^\epsilon_{\alpha,l}\rangle \\ &\quad + \langle M^\epsilon(\partial^{\alpha_1}_v F^\epsilon_R, \partial^{\alpha_2}_v f)\langle v\rangle^l, g^\epsilon_{\alpha,l}\rangle \\ &\quad + \langle \Upsilon(\partial^{\alpha_1}_v f, \partial^{\alpha_2}_v f)\langle v\rangle^l, g^\epsilon_{\alpha,l}\rangle\} \\ &\overset{\text{def}}{=} \sum_{\alpha_1+\alpha_2=\alpha} \binom{\alpha}{\alpha_1} \{\mathfrak{I}_1(\alpha_1,\alpha_2) + \mathfrak{I}_2(\alpha_1,\alpha_2) + \mathfrak{I}_3(\alpha_1,\alpha_2)\}. \end{aligned} \tag{4.24}$$

Again by lemma 7.1 in the Appendix of [52], we have

$$\mathfrak{I}_3(\alpha_1,\alpha_2) = \langle \Upsilon(\partial^{\alpha_1}_v f, \partial^{\alpha_2}_v f)\langle v\rangle^l, g^\epsilon_{\alpha,l}\rangle \lesssim ||f||^2_{H^{m+5}_{l+\gamma+10}} ||g^\epsilon_{\alpha,l}||_{L^2},$$

which implies, for any $\eta > 0$,

$$\mathfrak{I}_3(\alpha_1,\alpha_2) - \eta||g^\epsilon_{\alpha,l}||^2_{L^2} \lesssim \frac{1}{\eta}||f||^4_{H^{m+5}_{l+\gamma+10}}. \tag{4.25}$$

Splitting $\mathfrak{I}_1(\alpha_1,\alpha_2)$ into two terms, we have

$$\begin{aligned} \mathfrak{I}_1(\alpha_1,\alpha_2) &= \langle M^\epsilon(\partial^{\alpha_1}_v f^\epsilon, \langle v\rangle^l \partial^{\alpha_2}_v F^\epsilon_R), g^\epsilon_{\alpha,l}\rangle \\ &\quad + \{\langle M^\epsilon(\partial^{\alpha_1}_v f^\epsilon, \partial^{\alpha_2}_v F^\epsilon_R)\langle v\rangle^l, g^\epsilon_{\alpha,l}\rangle - \langle M^\epsilon(\partial^{\alpha_1}_v f^\epsilon, \langle v\rangle^l \partial^{\alpha_2}_v F^\epsilon_R), g^\epsilon_{\alpha,l}\rangle\} \\ &\overset{\text{def}}{=} \mathfrak{I}_{1,1}(\alpha_1,\alpha_2) + \mathfrak{I}_{1,2}(\alpha_1,\alpha_2). \end{aligned}$$

By coercivity estimate (3.5), we have

$$\mathfrak{I}_{1,1}(0,\alpha) \leq -C_1(f_0)||g^\epsilon_{\alpha,l}||^2_{\epsilon,\gamma/2} + C_2(f_0)||g^\epsilon_{\alpha,l}||^2_{L^2_{\gamma/2}}. \tag{4.26}$$

For $1 \le |\alpha_1| \le |\alpha| \le m$, by upper bound estimate (3.1) with $w_1 = \gamma/2+2, w_2 = \gamma/2$, we have

$$\begin{aligned} \mathfrak{I}_{1,1}(\alpha_1,\alpha_2) &\lesssim ||\partial_v^{\alpha_1} f^\epsilon||_{L^1_4} ||\partial_v^{\alpha_2} F_R^\epsilon||_{H^1_{l+\gamma/2+2}} ||g^\epsilon_{\alpha,l}||_{\epsilon,\gamma/2} \\ &\lesssim ||f^\epsilon||_{H^m_6} ||F_R^\epsilon||_{H^m_{l+\gamma/2+2}} ||g^\epsilon_{\alpha,l}||_{\epsilon,\gamma/2}, \end{aligned}$$

which implies, for any $\eta_1 > 0$,

$$\mathfrak{I}_{1,1}(\alpha_1,\alpha_2) - \eta_1 ||g^\epsilon_{\alpha,l}||^2_{\epsilon,\gamma/2} \lesssim \frac{1}{\eta_1} ||f^\epsilon||^2_{H^m_6} ||F_R^\epsilon||^2_{H^m_{l+\gamma/2+2}}. \tag{4.27}$$

By commutator estimate (3.9) with $N_2 = l+\gamma/2, N_3 = \gamma/2$, we have

$$\mathfrak{I}_{1,2}(\alpha_1,\alpha_2) \lesssim ||f^\epsilon||_{H^m_{2l+7}} ||\partial_v^{\alpha_2} F_R^\epsilon||_{\epsilon,l+\gamma/2} ||g^\epsilon_{\alpha,l}||_{L^2_{\gamma/2}},$$

which implies, for any $\eta_2 > 0$,

$$\mathfrak{I}_{1,2}(\alpha_1,\alpha_2) - \eta_2 ||\partial_v^{\alpha_2} F_R^\epsilon||^2_{\epsilon,l+\gamma/2} \lesssim \frac{1}{\eta_2} ||f^\epsilon||^2_{H^m_{2l+7}} ||g^\epsilon_{\alpha,l}||^2_{L^2_{\gamma/2}}. \tag{4.28}$$

Splitting $\mathfrak{I}_2(\alpha_1,\alpha_2)$ into two terms, we have

$$\begin{aligned} \mathfrak{I}_2(\alpha_1,\alpha_2) &= \langle M^\epsilon(\partial_v^{\alpha_1} F_R^\epsilon, \langle v\rangle^l \partial_v^{\alpha_2} f), g^\epsilon_{\alpha,l}\rangle \\ &\quad + \{\langle M^\epsilon(\partial_v^{\alpha_1} F_R^\epsilon, \partial_v^{\alpha_2} f)\langle v\rangle^l, g^\epsilon_{\alpha,l}\rangle - \langle M^\epsilon(\partial_v^{\alpha_1} F_R^\epsilon, \langle v\rangle^l \partial_v^{\alpha_2} f), g^\epsilon_{\alpha,l}\rangle\} \\ &\overset{\text{def}}{=} \mathfrak{I}_{2,1}(\alpha_1,\alpha_2) + \mathfrak{I}_{2,2}(\alpha_1,\alpha_2). \end{aligned}$$

Applying upper bound estimate (3.1) with $w_1 = \gamma/2+2, w_2 = \gamma/2$, we may have

$$\begin{aligned} \mathfrak{I}_{2,1}(\alpha_1,\alpha_2) &\lesssim ||\partial_v^{\alpha_1} F_R^\epsilon||_{L^1_4} ||\partial_v^{\alpha_2} f||_{H^1_{l+\gamma/2+2}} ||g^\epsilon_{\alpha,l}||_{\epsilon,\gamma/2} \\ &\lesssim ||F_R^\epsilon||_{H^m_6} ||f||_{H^{m+1}_{l+\gamma/2+2}} ||g^\epsilon_{\alpha,l}||_{\epsilon,\gamma/2}, \end{aligned}$$

which implies, for any $\eta_1 > 0$,

$$\mathfrak{I}_{2,1}(\alpha_1,\alpha_2) - \eta_1 ||g^\epsilon_{\alpha,l}||^2_{\epsilon,\gamma/2} \lesssim \frac{1}{\eta_1} ||f||^2_{H^{m+1}_{l+\gamma/2+2}} ||F_R^\epsilon||^2_{H^m_6}. \tag{4.29}$$

By commutator estimate (3.9) with $N_2 = l+\gamma/2, N_3 = \gamma/2$, we have

$$\mathfrak{I}_{2,2}(\alpha_1,\alpha_2) \lesssim ||F_R^\epsilon||_{H^m_{2l+7}} ||f||_{H^{m+1}_{l+\gamma/2}} ||g^\epsilon_{\alpha,l}||_{L^2_{\gamma/2}},$$

which implies, for any $\eta_1 > 0$,

$$\mathfrak{I}_{2,2}(\alpha_1, \alpha_2) - \eta_1 ||g^\epsilon_{\alpha,l}||^2_{L^2_{\gamma/2}} \lesssim \frac{1}{\eta_1} ||f||^2_{H^{m+1}_{l+\gamma/2}} ||F^\epsilon_R||^2_{H^m_{2l+7}}. \tag{4.30}$$

Patching all together (4.25),(4.26),(4.27),(4.28),(4.29),(4.30), and taking a small enough η_1, we arrive at

$$\begin{aligned}
&\frac{d}{dt}(\frac{1}{2}||g^\epsilon_{\alpha,l}||^2_{L^2}) + \frac{3C_1(f_0)}{4}||g^\epsilon_{\alpha,l}||^2_{\epsilon,\gamma/2} - \eta_2 ||F^\epsilon_R||^2_{\epsilon,m,l+\gamma/2} \\
\lesssim\ & ||f||^4_{H^{m+5}_{l+\gamma+10}} + (C_2(f_0) + \frac{1}{\eta_2}||f^\epsilon||^2_{H^m_{2l+7}})||g^\epsilon_{\alpha,l}||^2_{L^2_{\gamma/2}} \\
&+ \frac{1}{\eta_1}(||f^\epsilon||^2_{H^m_6} + ||f||^2_{H^{m+1}_{l+\gamma/2+2}} + ||f||^2_{H^{m+1}_{l+\gamma/2}})||F^\epsilon_R||^2_{H^m_{2l+7}}.
\end{aligned} \tag{4.31}$$

Let $a(m) = \sum_{r=0}^m \binom{r+2}{r}$. Summing over $|\alpha| \le m$, by taking $\eta_2 = \frac{C_1(f_0)}{4}\frac{1}{a(m)}$, we have

$$\begin{aligned}
&\frac{d}{dt}(\frac{1}{2}||F^\epsilon_R||^2_{H^m_l}) + \frac{C_1(f_0)}{2}||F^\epsilon_R||^2_{\epsilon,m,l+\gamma/2} \\
\lesssim\ & ||f||^4_{H^{m+5}_{l+\gamma+10}} + (C_2(f_0) + \frac{1}{\eta_2}||f^\epsilon||^2_{H^m_{2l+7}})||F^\epsilon_R||^2_{H^m_{l+\gamma/2}} \\
&+ \frac{1}{\eta_1}(||f^\epsilon||^2_{H^m_6} + ||f||^2_{H^{m+1}_{l+\gamma/2+2}} + ||f||^2_{H^{m+1}_{l+\gamma/2}})||F^\epsilon_R||^2_{H^m_{2l+7}}.
\end{aligned} \tag{4.32}$$

Thanks to (3.91), we may conclude

$$\frac{d}{dt}||F^\epsilon_R||^2_{H^m_l} + \frac{C_1(f_0)}{2}||F^\epsilon_R||^2_{\epsilon,m,l+\gamma/2} \lesssim C(||f||_{H^{m+5}_{l+\gamma+10}}, ||F^\epsilon_R||_{H^{m-1}_{z(l)}}, ||f_0||_{LlogL}, ||f_0||_{L^1_1}).$$

By Theorem 1.1 and Remark 1.1, for any $t \ge 0$, we have

$$||f(t)||_{H^{m+5}_{l+\gamma+10}} \lesssim C(||f_0||_{L^1_{\phi(m+5,l+\gamma+10)}}, ||f_0||_{H^{m+5}_{l+\gamma+10}}).$$

By assumption, there holds

$$||F^\epsilon_R(t)||_{H^{m-1}_{z(l)}} \lesssim C(||f_0||_{L^1_{\varphi(m-1,z(l))}}, ||f_0||_{H^{m+4}_{\psi(m-1,z(l))}}, t).$$

On the other hand, by interpolation, we have

$$||f_0||_{H^{m+4}_{\psi(m-1,z(l))}} \lesssim ||f_0||_{H^{m+5}_{l+\gamma+10}} + ||f_0||_{L^1_{\rho(m,l)}},$$

where $\rho(m,l) = (m+7)\psi(m-1,z(l)) - (m+6)(l+\gamma+10)$. By defining $\varphi(m,l) = \max\{\varphi(m-1,z(l)), \rho(m,l)\}$, we have

$$||F^\epsilon_R(t)||^2_{H^m_l} \le C(||f_0||_{L^1_{\varphi(m,l)}}, ||f_0||_{H^{m+5}_{l+\gamma+10}}, t).$$

The proof of Theorem 1.3 is complete now.

Chapter 5

Conclusion and Future Work

In this chapter, we make a summary and show some future work.

5.1 Summary of the thesis

In the thesis, we prove a dynamic programming principle for the stochastic control problem under expectation constraint. The principle is demonstrated to be powerful by deriving the DPP for several specific problems. The main tool is measurable selection theorem, which is used to construct an admissible ϵ-optimal control.

We also propose a new model to approximate the Boltzmann equation, which is showed to be more accurate than the angular cut-off model. The result relies mainly on grazing collision limit and coercivity estimates.

5.2 Future research

Following the results in the thesis, we intend to do some projects in the future, which are unfolded into four parts.

First, the control problem in the thesis involves only linear expectation. We will consider non-linear expectation which can be induced by backward stochastic differential equation.

Second, we may consider the stability of the control problem under expectation constraint. By that, we mean the problem whether one can use piecewise constant controls to achieve ϵ-optimal results.

Third, the case we consider in the thesis is homogenous Boltzmann with hard potential. We will consider soft potential in the future. At least in the case of moderately soft potential $\gamma + 2s > 0$, this is possible since the global solution exists and propagation properties still hold.

Fourth, as illustrated in subsection 1.2.4, we will try to construct a Kac's like program to simulate the solution of our approximate equation, which is a numerical method to solve the original homogeneous Boltzmann equation. This project is very promising in the case of hard potentials. But for soft potential, even if we make the theoretical error analysis of our approximate solution, there is still a lot of uncertain work to derive a numerical method.

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CURRICULUM VITAE

Academic qualifications of the thesis author, Mr. ZHOU Yulong:

- Received the degree of Bachelor of Science from Sichuan University, June 2011.
- Received the degree of Master of Economics from Xiamen University, June 2014.

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