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Zeng, Shangzhi

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Chairman:	Prof. Ng Joseph K Y Professor, Department of Computer Science, HKBU (Designated by Dean of Faculty of Science)
Internal Members:	Dr. Liu Hongyu Associate Professor, Department of Mathematics, HKBU (Designated by Head of Department of Mathematics)
	Prof. Yuan Xiaoming Professor, Department of Mathematics, HKBU
External Members:	Prof. Ye Jane Juan-Juan Professor Department of Mathematics and Statistics University of Victoria

Issued by Graduate School, HKBU

Algorithm-tailored Error Bound Conditions and the Linear Convergence Rate of ADMM

ZENG Shangzhi

A thesis submitted in partial fulfillment of the requirements
for the degree of
Master of Philosophy

Principal Supervisor:
Prof. YUAN Xiaoming (Hong Kong Baptist University)

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DECLARATION

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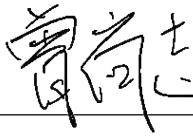
Signature: 

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Abstract

In the literature, error bound conditions have been widely used for studying the linear convergence rates of various first-order algorithms and the majority of literature focuses on how to sufficiently ensure these error bound conditions, usually posing more assumptions on the model under discussion. In this thesis, we focus on the alternating direction method of multipliers (ADMM), and show that the known error bound conditions for studying ADMM's linear convergence, can indeed be further weakened if the error bound is studied over the specific iterative sequence generated by ADMM. A so-called partial error bound condition, which is tailored for the specific ADMM's iterative scheme and weaker than known error bound conditions in the literature, is thus proposed to derive the linear convergence of ADMM. We further show that this partial error bound condition theoretically justifies the difference if the two primal variables are updated in different orders in implementing ADMM, which had been empirically observed in the literature yet no theory is known so far.

Keywords: Convex programming, alternating direction method of multipliers, calmness, partial error bound, linear convergence rate.

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Chapter 1

Introduction

Since the seminal work Glowinski and Marroco [1975]; Gabay and Mercier [1976]; Chan and Glowinski [1978], the Douglas-Rachford alternating direction method of multipliers (ADMM) has been widely used in various areas such as partial differential equations, image processing and statistical learning. In this thesis, we shall focus on error bound conditions to ensure the linear convergence rate of the alternating direction method of multipliers (ADMM). We specifically consider the following convex optimization problem with linear constraints and a separable objective function:

$$\begin{aligned} \min_{x\in\mathcal{X},y\in\mathcal{Y}} \quad & f(x)+g(y) \\ s.t. \quad & Ax+By=b, \end{aligned} \tag{1.1}$$

where $f : \boldsymbol{R}^{n_1} \to \boldsymbol{R}$ and $g : \boldsymbol{R}^{n_2} \to \boldsymbol{R}$ are both convex (not necessarily smooth) functions; $A \in \boldsymbol{R}^{m\times n_1}$ and $B \in \boldsymbol{R}^{m\times n_2}$ are given matrices; $\mathcal{X} \subset \boldsymbol{R}^{n_1}$ and $\mathcal{Y} \subset \boldsymbol{R}^{n_2}$ are convex sets; and $b \in \boldsymbol{R}^m$.

The main contents of this thesis is based on our recent paper Liu et al. [2017].

1.1 Alternating Direction Method of Multipliers (ADMM)

The iterative scheme of ADMM for (1.1) is:

$$\begin{cases} x^{k+1} = \arg\min\limits_{x\in\mathcal{X}} \ \{f(x) - (\lambda^k)^T(Ax + By^k - b) + \dfrac{\beta}{2}\|Ax + By^k - b\|^2\}, \\ y^{k+1} = \arg\min\limits_{y\in\mathcal{Y}} \ \{g(y) - (\lambda^k)^T(Ax^{k+1} + By - b) + \dfrac{\beta}{2}\|Ax^{k+1} + By - b\|^2\}, \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b), \end{cases} \tag{1.2}$$

where λ is the Lagrange multiplier and $\beta > 0$ is a penalty parameter. The subproblems arising in ADMM's iterations may be much easier than the original problem (1.1) and indeed they may have closed-form solutions when f and g are special enough. This feature makes the implementation of ADMM extremely easy for some applications arising in areas such as compressive sensing, image processing, statistical learning, sparse and low-rank optimization problems, etc., and it well explains the popularity of ADMM in various areas. We refer to Boyd et al. [2011]; Eckstein and Yao [2015]; Glowinski [2014] for some review papers on the ADMM.

Under some mild conditions such as the nonemptyness of the solution set of the problem (1.1), the convergence of ADMM has been well studied in earlier literature, see e.g., Eckstein and Bertsekas [1992]; Eckstein et al. [1990]; Fortin and Glowinski [2000]; Gabay and Mercier [1976]; Glowinski and Le Tallec [1989]; Glowinski and Marroco [1975]; He and Yang [1998]; Lions and Mercier [1979]. Because of the applications recently found in various areas, research on the convergence analysis of the ADMM has regained attention from the community and more efforts have been put on the convergence rate analysis. In He and Yuan [2012, 2015]; Monteiro and Svaiter [2013], the worst-case $O(1/k)$ convergence rate measured by the iteration complexity has been established for the ADMM in both the ergodic and nonergodic senses, where k is the iteration counter. Such a convergence rate is of sublinear. Consequently, some results for the linear convergence rate of ADMM have been established either for special cases of the generic model (1.1) or for the scenarios where more assumptions are posed on the model (1.1). For example, it is shown in [Boley, 2013,

Theorem 6.4] that the local linear convergence rate of ADMM can be guaranteed for the special linear and quadratic cases of (1.1), if it is assumed that both the minimization subproblems in (1.2) have unique optimal solutions and additionally some strict complementarity conditions hold. Moreover, if f and/or g are/is strongly convex, one of them is differentiable and has a Lipschitz continuous gradient, and the generated iterative sequence is assumed to be bounded, together with some full rank conditions of the coefficient matrices, the global linear convergence rate of ADMM is proved in Deng and Yin [2016]. More results can be found in Nishihara et al. [2015] as well.

As in He and Yuan [2012], instead of the original ADMM scheme (1.2), our analysis is for the slightly generalized proximal version of the ADMM (PADMM for short)

$$\begin{cases} x^{k+1} = \arg\min\limits_{x\in\mathcal{X}} \ \{f(x) - (\lambda^k)^T(Ax + By^k - b) + \dfrac{\beta}{2}\|Ax + By^k - b\|^2 + \dfrac{1}{2}\|x - x^k\|_D^2\}, \\ y^{k+1} = \arg\min\limits_{y\in\mathcal{Y}} \ \{g(y) - (\lambda^k)^T(Ax^{k+1} + By - b) + \dfrac{\beta}{2}\|Ax^{k+1} + By - b\|^2\}, \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b), \end{cases} \tag{1.3}$$

where $D \in \boldsymbol{R}^{n_1\times n_1}$ is a symmetric and positive semi-definite matrix. Here, we slightly abuse the notation $\|x\|_D^2$ for the number x^TDx even though D may be only positive semi-definite. Throughout the penalty parameter β is fixed in our discussion. This scheme includes the original ADMM scheme (1.2) and the linearized ADMM (or, split inexact Uzawa method in Zhang et al. [2010]) as special cases with $D = 0$ and $D = (\sigma I_{n_1} - \beta A^TA)$ with $\sigma > \beta\|A^TA\|$, respectively. Note that the linearized version of ADMM has found many efficient applications, see Liu et al. [2013]; Wang and Yuan [2012]; Yang and Yuan [2013] to mention a few. Hence, we include this case into our discussion and consider the PADMM (1.3).

1.2 Error Bound Conditions for ADMM

Error bound conditions turn out to play an important role in studying the linear convergence rate of ADMM. To elucidate on error bound conditions, we first mention

the Karush-Kuhn-Tucker (KKT) system of the problem (1.1):

$$\begin{cases} 0 \in \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x), \\ 0 \in \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y), \\ 0 = Ax + By - b, \end{cases} \tag{1.4}$$

where "∂" denotes the subgradient of a convex function and $\mathcal{N}_{\mathcal{C}}(c) := \{\xi : \langle \xi, \zeta - c\rangle \leq 0,\ \forall \zeta \in \mathcal{C}\}$ denotes the normal cone at c to a given convex set $\mathcal{C}$. Let S^* be the solution set of the KKT system (1.4) and assume it to be nonempty. Furthermore, let $r : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m \to \boldsymbol{R}_+$ be a residual error function satisfying $r(x, y, \lambda) = 0$ iff $(x, y, \lambda) \in S^*$. We say that the KKT system (1.4) admits a local error bound around a given point $(x^*, y^*, \lambda^*) \in S^*$ with the residual error function $r(x, y, \lambda)$ if there exist a neighborhood

$$\mathcal{B}_\epsilon(x^*, y^*, \lambda^*) := \{(x, y, \lambda) : \|(x, y, \lambda) - (x^*, y^*, \lambda^*)\| \leq \epsilon\}$$

of the point (x^*, y^*, λ^*) and a constant $\kappa > 0$ such that

$$[\mathrm{EB}^{\mathrm{r}}] \qquad dist((x, y, \lambda), S^*) \leq \kappa \cdot r(x, y, \lambda) \quad \text{provided } \ (x, y, \lambda) \in \mathcal{B}_\epsilon(x^*, y^*, \lambda^*). \tag{1.5}$$

Throughout, we define $dist(c, \mathcal{C}) := \inf_{c' \in \mathcal{C}}\{\|c - c'\|\}$ for a given subset $\mathcal{C}$ and vector c in the same space, and $\|\cdot\|$ is the 2-norm without otherwise specified. If this estimate is valid for every $(x, y, \lambda) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m$, rather than merely $(x, y, \lambda) \in \mathcal{B}_\epsilon(x^*, y^*, \lambda^*)$, we say that the KKT system (1.4) admits a global error bound.

Error bound conditions of the KKT system (1.4) with various choices of the residual error function $r(x, y, \lambda)$ have inspired some works for studying the linear convergence rate of the ADMM. According to (1.4), it is natural to define an mapping $\phi : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m \rightrightarrows \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m$ as

$$\phi(x, y, \lambda) = \begin{pmatrix} \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x) \\ \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y) \\ Ax + By - b \end{pmatrix} \tag{1.6}$$

and then a residual error function as $r(x,y,\lambda) = dist(0,\phi(x,y,\lambda))$. We call ϕ defined in (1.6) the KKT mapping for obvious reasons and the KKT system (1.4) can be written as $0 \in \phi(x,y,\lambda)$. With $\phi(x,y,\lambda)$ given in (1.6), let us define $S : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m \rightrightarrows \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m$ as

$$S(p) := \{(x,y,\lambda) \mid p \in \phi(x,y,\lambda)\} \tag{1.7}$$

with $p = (p_1,p_2,p_3) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m$. Obviously, $S(0) = S^*$. Recall that we are interested in finding $0 \in \phi(x,y,\lambda)$, i.e., $(x,y,\lambda) \in S(0) = \{(x,y,\lambda) \mid 0 \in \phi(x,y,\lambda)\}$. Hence, p in (1.7) plays the role of a perturbation parameter and (1.7) can be regarded as a perturbed system of the KKT system (1.4). This is also the reason we purposively use the same letter S to define the mapping in (1.7) in addition to the notation S^* for the solution set of the KKT system (1.4).

Let us use the notation $w = (x,y,\lambda)$ for a more compact presentation. Now, using $dist(0,\phi(w))$ as the residual error function, the KKT system (1.4) is said to admit a local error bound around a feasible point $\bar{w} = (\bar{x},\bar{y},\bar{\lambda})$ if there exist a neighborhood $\mathcal{B}_\epsilon(\bar{w})$ of $\bar{w}$ and some constant $\kappa > 0$ such that

$$\text{[FEB]} \qquad dist(w,S(0)) \le \kappa \cdot dist(0,\phi(w)) \qquad \text{provided} \;\; w \in \mathcal{B}_\epsilon(\bar{w}). \tag{1.8}$$

Indeed, in terms of variational analysis, the existence of an error bound around the reference point $\bar{w}$ with the residual error function $r(w) = dist(0,\phi(w))$ is exactly the metric subregularity of the KKT mapping $\phi(w)$ at $(\bar{w},0)$. The set-valued mapping $\phi(w)$ is called metrically subregular around $(\bar{w},0)$ if there exists a neighbourhood $\mathcal{B}_\epsilon(\bar{w})$ of $\bar{w}$ and $\kappa > 0$ such that

$$dist\left(w,\phi^{-1}(0)\right) \le \kappa \cdot dist\left(0,\phi(w)\right) \;\; \text{provided} \quad w \in \mathcal{B}_\epsilon(\bar{w}).$$

Equivalently, $\phi(w)$ is metrically subregular around $(\bar{w},0)$ if there exist a neighbourhood $\mathcal{B}_\epsilon(\bar{w})$ of $\bar{w}$ and $\kappa > 0$ such that

$$S(p) \cap \mathcal{B}_\epsilon(\bar{w}) \subset S(0) + \kappa \|p\| \cdot \mathcal{B}_1(0), \quad \forall p, \tag{1.9}$$

i.e., the set-valued mapping $S(p)$ is calm around $(0, \bar{w})$. We refer to Dontchev and Rockafellar [2014]; Rockafellar and Wets [2009] for more details of the concepts of metric subregularity and calmness and their relationship. Note that calmness was first introduced as the pseudo upper-Lipschitz continuity in Ye and Ye [1997]. Moreover, $S(p)$ in (1.7) considers the canonical perturbation p of $\phi(w)$. From now on, we call (1.8) a full error bound (FEB) condition since p fully perturbs ϕ in (1.7).

In the literature, other error bound conditions have been defined as well for studying the linear convergence rate of the ADMM and/or its variants. For instance, based on the so-called natural map (see [Facchinei and Pang, 2007, page 83]) in terms of the Moreau-Yosida proximal mapping, the following mapping is used in Han and Yuan [2013]:

$$R_1(w) = \begin{pmatrix} x - \mathrm{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda) \\ y - \mathrm{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda) \\ Ax + By - b \end{pmatrix}, \tag{1.10}$$

where δ is the indictor function of a convex set and Prox_h is the proximal mapping associated with the convex lower semi-continuous function h, i.e.,

$$\mathrm{Prox}_h(a) := \arg\min_{t \in \boldsymbol{R}^n} \left\{ h(t) + \frac{1}{2}\|t - a\|^2 \right\}.$$

The mapping defined in (1.10) is also called the proximal KKT mapping. Then, a residual error function is defined as $r(w) = dist(0, R_1(w))$ in Han and Yuan [2013]. Accordingly, the KKT system (1.4) is said to admit a local proximal error bound around $\bar{w}$ if there exist a neighborhood $\mathcal{B}_\epsilon(\bar{w})$ of $\bar{w}$ and some $\kappa > 0$ such that

$$[\text{Proximal EB} - \text{I}] \quad dist(w, S^*) \le \kappa \cdot \|R_1(w)\| \qquad \text{provided } w \in \mathcal{B}_\epsilon(\bar{w}). \tag{1.11}$$

Note that (1.11) is just the metric subregularity of $R_1(w)$ at $(\bar{w}, 0)$. Under the proximal error bound condition (1.11), the linear convergence rate of the ADMM (1.2) (and its variant with a relaxation factor) is obtained in Han and Yuan [2013] for the special case of the problem (1.1) where the objective function is quadratic. The conditions used in Boley [2013] such as the uniqueness of optimal solutions of the subproblems and the strict complementarity are not needed by the analysis in

Han and Yuan [2013].

Later, an alternative form of (1.10) is considered in Yang and Han [2016]:

$$R_2(w) = \begin{pmatrix} x - \mathrm{Proj}_{\mathcal{X}}(x - \partial f(x) + A^T\lambda) \\ y - \mathrm{Proj}_{\mathcal{Y}}(y - \partial g(y) + B^T\lambda) \\ Ax + By - b \end{pmatrix}, \tag{1.12}$$

where $\mathrm{Proj}_{\mathcal{C}}(\zeta) := \arg\min_{\xi\in\mathcal{C}} \{\|\xi - \zeta\|\}$ is the canonical projection operator onto a given convex set $\mathcal{C}$. Accordingly, the residual error function is defined as $r(w) = dist(0, R_2(w))$ and the KKT system (1.4) is said to admit a local error bound around $\bar{w}$ if there exist a neighborhood $\mathcal{B}_\epsilon(\bar{w})$ of $\bar{w}$ and some $\kappa > 0$ such that

$$[\text{Proximal EB} - \text{II}] \quad dist(w, S^*) \le \kappa \cdot dist(0, R_2(w)) \quad \text{provided} \quad w \in \mathcal{B}_\epsilon(\bar{w}). \tag{1.13}$$

Under the error bound condition (1.13), which also reads as the metric subregularity of $R_2(w)$ at $(\bar{w}, 0)$, the linear convergence rate of the ADMM (1.2) and its linearized variant is established in Yang and Han [2016] for the special case of (1.1) where ∂f and ∂g are both polyhedral multifunctions. Recall that a set-valued mapping is called polyhedral multifunction if its graph is the union of finitely many convex polyhedra. Note that the projection operator onto a closed convex set $\mathcal{C}$ can be regarded as the proximal operator associated with the indicator function over $\mathcal{C}$. We also call (1.13) a local proximal error bound of the KKT system (1.4).

Existing error bound conditions, including (1.8), (1.11) and (1.13) used in the mentioned literature, are all proposed on the basis of the KKT system (1.4). Generally, they are assumed only dependently on the model (1.1) per se, while irrelevant to any specific algorithm under discussion. We thus call them generic error bound conditions. Obviously, they are somehow too "sufficient" for studying the convergence rate of a specific algorithm. Indeed, most of the efforts, e.g., Boley [2013]; Han and Yuan [2013]; Han et al. [to appear]; Yang and Yuan [2013] in the literature, have been put on how to sufficiently ensure these error bound conditions, usually by posing more assumptions or special structures on the model (1.1), so that the linear convergence rate of ADMM can be guaranteed. In other words, the structures and features

of an specific algorithm are ignored when its linear convergence rate is studied via error bound conditions; and thus directly using these generic error bound conditions indeed shrinks the range that validates the linear convergence rate of ADMM.

1.3 Contributions

We are going to show how some weakened error bound properties quickly yield linear convergence guarantees. For this purpose, we first clarify that to ease the analysis of the desired convergence rate of PADMM, the normally-used KKT mapping $\phi : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^{m} \rightrightarrows \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^{m}$,

$$\phi(x, y, \lambda) = \begin{pmatrix} \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x) \\ \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y) \\ Ax + By - b, \end{pmatrix}$$

which directly follows the KKT system should be used to defined the residual error function $r = dist(0, \phi)$. In fact, according to the recently proposed perturbational characterization framework in Wang et al. [2017], for the purpose of a simplified analysis, the PADMM algorithmic optimality residual suggests that $dist(0, \phi)$ should be the best choice of the residual function. However, in contrast, among the existing literature regarding this topic, for instance, Han and Yuan [2013]; Han et al. [to appear]; Yang and Han [2016], variants of proximal KKT mappings were used as the surrogates to characterize error bounds. The proximal KKT mapping is defined by the so-called natural map (see [Facchinei and Pang, 2007, page 83]) in terms of Moreau-Yosida proximal mapping. For instance, in Han and Yuan [2013], the authors considered the following proximal form KKT mapping

$$R_1(x, y, \lambda) = \begin{pmatrix} x - \mathrm{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda) \\ y - \mathrm{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda) \\ Ax + By - b \end{pmatrix},$$

and therefore defined the residual function $r = dist(0, R_1)$, where $Prox_h$ is the proximal mapping associated with convex lower semi-continuous function h.

The natural map has some advantages (see Facchinei and Pang [2007]) and the usefulness of the proximal map-based error bound can be seen from the existing literatures. However, according to the clarification in Wang et al. [2017], the residual error function should be defined algorithm-tailored, and the proximal form mapping does not match PADMM very appropriately.

In the language of variational analysis, the existence of an error bound with residual function $r = dist(0, \phi)$ is exactly "metric subregularity" of the KKT mapping ϕ. The metric subregularity is equivalent to the calmness at the origin of the inverse of the KKT mapping. In the recent paper Drusvyatskiy and Lewis [2016], the authors investigate unconstrained separable convex optimization problem and illustrate that subregularity of the gradient-like mapping is equivalent to subregularity of its subdifferetial (see also Wang et al. [2017] from another perspective). Therefore, they employ the quadratic growth condition as the characterization of error bound condition which succeeds to yield a linear rate convergence for the prox-gradient method. In this thesis, for the constrained problem, we will show that the error bound defined in terms of the KKT mapping ϕ is equivalent to the one of the proximal KKT mapping R_1. This observation thereby allows people to call on extensive literature relating the metric regularity/subregularity of KKT mapping ϕ, see e.g., Gfrerer [2013]; Gfrerer and Klatte [2016]; Gfrerer and Mordukhovich [2017]; Gfrerer and Ye [2017]; Henrion et al. [2002].

When analyzing the PADMM/original ADMM algorithmic linear rate convergence, we encounter a surprise. Instead of assuming a full error bound admitted by the KKT system around the reference point, we only need to estimate the distance from each generated point (x^k, y^k, λ^k) to the KKT solution set S. Interesting therefore is the observation that, at each (x^k, y^k, λ^k) generated by the PADMM, the second part of the KKT optimality condition

$$0 \in \partial g(y^k) - B^T\lambda^k + \mathcal{N}_{\mathcal{Y}}(y^k)$$

constantly holds valid. In the language of perturbation analysis, focusing on the

sequence (x^k, y^k, λ^k), no perturbation occurs in the part

$$0 \in \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y)$$

of the KKT system $0 \in \phi(x, y, \lambda)$. Inspired by this observation, we will show that the weaker partial error bound is sufficient to ensure the desired linear rate convergence. One example is presented to demonstrate the advantages of using the partial error bound.

The new application of (partial) error bound thereby allows us to call on extensive literature concerning the metric subregularity of KKT mapping ϕ. Given the generality of these techniques, we expect that the approach we describe here, rooted in understanding linear convergence through natural (partial) KKT mapping, should motivate broad investigation on calmness of multifunction to be employed. In fact, the study on the calmness condition has enjoyed a prosperous time since the recent paper Gfrerer [2011] of significance. We refer the reader to Gfrerer and Mordukhovich [2015]; Gfrerer and Outrata [2016]; Gfrerer and Ye [2017] for several very recent advance on this subject.

An important byproduct of our analysis, worthy of independent interest, relates to the fact that our partial error bound theory may help interpret the convergence rate change affected by updating order. Particularly, it is known that the updating order of primal variables has nothing to do with the algorithmic convergence. However, the updating order should interfere with the position where perturbation occurs and hence the expression of partial error bound. That is, if we update y in front of x in the PADMM, then at each (x^k, y^k, λ^k) generated by the PADMM, the first part of the KKT optimality condition

$$0 \in \partial f(x^k) - A^T\lambda^k + \mathcal{N}_{\mathcal{X}}(x^k)$$

constantly holds valid. Therefore, it is possible for people to choose an appropriate updating order such that the associated partial error bound is somehow easier to meet. By doing so, one may attain a convergence rate guarantee in theory.

1.4 Outline of the Thesis

In Chapter 2, we summarize some necessary preliminaries concerning global convergence of ADMM.

In Chapter 3, we propose an error bound condition tailored for the specific iterative scheme (1.3) and prove that it suffices to ensure the linear convergence of the PADMM (1.3). We also show that the generic FEB (1.8) is sufficient to ensure this error bound condition.

In Chapter 4, we clarify the equivalence between the FEB (1.8) and the proximal EB-I (1.11) and proximal EB-II (1.13). Because of the equivalence, theoretically we can choose anyone from (1.8), (1.11) and (1.13). We further explain in this chapter why we choose the FEB (1.8) to conduct the convergence analysis for the PADMM (1.3) from a perturbation analysis perspective.

In Chapter 5, with the purpose of studying PADMM-tailored error bound conditions, we find that a more meticulous analysis for the sequence generated by (1.3) immediately gives us an insight and helps us further weaken the mentioned error bound conditions but still ensure the linear convergence rate of the PADMM (1.3). More specifically, for the sequence $\{w^k\}$ generated by (1.3), the second part of the KKT system (1.4), i.e., $0 \in \partial g(y^k) - B^T\lambda^k + \mathcal{N}_{\mathcal{Y}}(y^k)$, always holds for all iterates. In language of perturbation analysis, the sequence $\{w^k\}$ generated by the PADMM (1.3) introduces no perturbation to the part $0 \in \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y)$ in (1.7). This interesting observation suggests that there is no need to fully satisfy a general error bound condition that is derived based on the KKT system (1.4) and a partial error bound condition without consideration of the perturbation to the part $\partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y)$ is sufficient for studying the linear convergence rate of the PADMM (1.3). In particular, an example is constructed to illustrate that the partial error bound condition is indeed weaker than the known full counterparts.

In Chapter 6, we interpret the observation that the updating order may affect convergence rate by the PADMM-tailored partial error bound condition. It has been empirically observed that the convergence speed may be different if we swap the order of x and y in the ADMM (1.2) despite that there is no difference from the theoretical convergence-proof point of view. So far it seems that no rigorous theory

is known for explaining this difference. We shall show by an example that swapping the order of x and y in (1.2) does make difference in satisfying the partial error bound condition tailored for the ADMM (1.2). This theoretical justification gives hints to users to decide a more appropriate order of updating the primal variables for a specific application of the problem (1.1) so that the associated partial error bound can be meet more easily and hence the linear convergence rate of ADMM can be yielded.

In Chapter 7, we mention some conclusions and possible future works.

Chapter 2

Preliminaries

In this Chapter, we state assumptions under which our further analysis will be conducted, recall the variational inequality characterization of the problem (1.1) and provide some known or obvious convergence results of the PADMM (1.3).

2.1 Basic Assumptions

To characterize the solution set of the problem (1.1) by the first-order optimality conditions, we need certain constraint qualification such as the strong conical hull intersection property (Strong CHIP for short) for the sets $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{F}$ defined by

$$\mathcal{F} := \{(x, y) \mid Ax + By = b\}. \tag{2.1}$$

In particular, for any (x, y) feasible for the problem (1.1), there holds

$$\mathcal{N}_{\mathcal{F}\cap\mathcal{X}\times\mathcal{Y}}(x, y) := \mathcal{N}_{\mathcal{F}}(x, y) + \mathcal{N}_{\mathcal{X}}(x) \times \mathcal{N}_{\mathcal{Y}}(y).$$

The strong CHIP plays a similar role as the Abadie constraint qualification, which is regarded as not restrictive. Throughout, to avoid triviality, the following nonemptyness assumption is assumed.

Assumption 2.1. *The optimal solution set of problem (1.1) is nonempty.*

Under Assumption 2.1 and strong CHIP, $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ is an optimal solution

point of the problem (1.1) if and only if there exists a Lagrange multiplier $\lambda^* \in \boldsymbol{R}^m$ such that (x^*, y^*, λ^*) solves the KKT system (1.4).

2.2 Variational inequality characterization of (1.1)

As analyzed in He and Yuan [2012], the problem (1.1) can be characterized by the variational inequality: finding $w^* = (x^*, y^*, \lambda^*) \in \Omega := \mathcal{X} \times \mathcal{Y} \times \boldsymbol{R}^m$ such that

$$\mathrm{VI}(\Omega, F, \theta): \qquad \theta(u) - \theta(u^*) + (w - w^*)^T F(w^*) \geq 0, \quad \forall\, w \in \Omega, \tag{2.2}$$

where

$$u = (x, y), \quad \theta(u) = f(x) + g(y) \ \text{ and } \ F(w) = \begin{pmatrix} -A^T\lambda \\ -B^T\lambda \\ Ax + By - b \end{pmatrix}. \tag{2.3}$$

Note that the mapping $F(w)$ defined in (2.3) is monotone as it is affine with a skew-symmetric matrix. Since S^* is assumed to be nonempty, the solution set of $\mathrm{VI}(\Omega, F, \theta)$, denoted by Ω^*, is also nonempty.

2.3 Convergence of (1.3)

Our main purpose of this thesis is discussing error bound conditions that can ensure the linear convergence rate of the PADMM (1.3) under the by-default assumption that the convergence of (1.3) is given. As a prerequisite of the analysis to be delineated, the convergence of (1.3) can be easily given by various results in the literature. In this section, we briefly mention the convergence of (1.3) and give a particular sufficient condition to ensure it.

With the given model (1.1) and the iterative scheme of the PADMM (1.3), let us define the matrix H and its submatrix H_0 as follows to simplify the notation in our

analysis:

$$H = \begin{pmatrix} D & 0 & 0 \\ 0 & \beta B^T B & 0 \\ 0 & 0 & \frac{1}{\beta} I \end{pmatrix} \quad \text{and} \quad H_0 = \begin{pmatrix} \beta B^T B & 0 \\ 0 & \frac{1}{\beta} I \end{pmatrix}. \tag{2.4}$$

Moreover, let us make the following assumption.

Assumption 2.2. *One of the following conditions satisfies:*

(1) $D \succeq 0$*, and both* A *and* B *are full column rank; or*

(2) $D \succ 0$*, and* B *is full column rank.*

Obviously, $H \succeq 0$ and $H_0 \succ 0$ for either of the cases in Assumption 2.2. In particular, $H \succ 0$ if Case (2) of Assumption 2.2 holds. Hereafter, we also slightly abuse the notation $\|w\|_H$ for $\sqrt{w^T H w}$ even though H might only be positive semi-definite. Moreover, there exists a constant $L_H > 0$ such that

$$\|w\|_H \leq L_H \|w\|, \qquad \forall w \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m.$$

To derive the convergence of (1.3), first notice that the iterative scheme (1.3) can be written as

$$\begin{cases} 0 \in \partial f(x^{k+1}) - A^T \lambda^{k+1} + \beta A^T B(y^k - y^{k+1}) + D(x^{k+1} - x^k) + \mathcal{N}_{\mathcal{X}}(x^{k+1}), \\ 0 \in \partial g(y^{k+1}) - B^T \lambda^{k+1} + \mathcal{N}_{\mathcal{Y}}(y^{k+1}), \\ 0 = Ax^{k+1} + By^{k+1} - b + \dfrac{1}{\beta}(\lambda^{k+1} - \lambda^k). \end{cases} \tag{2.5}$$

We recall some inequalities established in the literature (see., e.g., Fang et al. [2015]; Han and Yuan [2013]; He and Yuan [2015]; Yang and Han [2016]) for deriving the convergence of the ADMM (1.2), the PADMM (1.3), and their variants. Some of the proofs are omitted.

Lemma 2.3. *Let* $\{w^k = (x^k, y^k, \lambda^k)\}$ *be the sequence generated by the PADMM (1.3), then we have*

$$\theta(w) - \theta(w^{k+1}) + \left(w - w^{k+1}\right)^T \left\{F(w) + \eta(y^k, y^{k+1}) + H(w^{k+1} - w^k)\right\} \geq 0, \ \ \forall\, w \in \Omega, \tag{2.6}$$

where

$$\eta(y^k, y^{k+1}) := \beta \begin{pmatrix} A^T \\ B^T \\ 0 \end{pmatrix} B(y^k - y^{k+1}).$$

The next proposition gives some important inequalities for the sequence $\{w^k\}$ generated by the PADMM (1.3).

Proposition 2.4. *Let $\{w^k = (x^k, y^k, \lambda^k)\}$ be the sequence generated by the PADMM (1.3). For any point $w^* = (x^*, y^*, \lambda^*)$ in S^*, we have*

$$\|w^{k+1} - w^*\|_H^2 \leq \|w^k - w^*\|_H^2 - \|w^{k+1} - w^k\|_H^2, \tag{2.7}$$

and consequently it holds that

$$\sum_{k=0}^{\infty} \|w^{k+1} - w^k\|_H^2 \leq \infty. \tag{2.8}$$

Then, we show that Assumptions 2.1 and 2.2, and strong CHIP are sufficient to ensure the convergence of the PADMM (1.3).

Theorem 2.5. *Let $\{w^k\}$ be the sequence generated by the PADMM (1.3). If Assumptions 2.1 and 2.2, and strong CHIP are all satisfied, then $\{w^k\}$ converges to a solution point $w^* \in S^*$.*

Proof. We first consider Case (1) of Assumptions 2.2. For this case, $H \succeq 0$ but both A and B are full column rank. It follows from (2.7) that the sequence $\{v^k = (y^k, \lambda^k)\}$ is bounded. Moreover, (2.8) in Proposition 2.4 implies that $\|w^{k+1} - w^k\|_H \to 0$ and hence the boundedness of the sequences $\{\frac{1}{\beta}(\lambda^k - \lambda^{k+1})\}$ and $\{B(y^k - y^{k+1})\}$, by the definition of H in (2.4). We thus know the sequence $\{Ax^{k+1} + By^{k+1} - b\}$ is also bounded because of the identity

$$Ax^{k+1} + By^{k+1} - b = \frac{1}{\beta}(\lambda^k - \lambda^{k+1}),$$

which is obvious from the update scheme of the scheme (1.3). Therefore, the boundedness of $\{v^k\}$ ensures that the sequence $\{Ax^k\}$ is bounded. Since matrix A is assumed

to be of full column rank, $\{x^k\}$ is bounded. Overall, we prove that the sequence $\{w^k\}$ is bounded. Let $\{w^{k_j}\}$ be a subsequence of $\{w^k\}$ converging to w^*. Then for any fixed $w \in \Omega$, considering the inequality (2.6) for the subsequence $\{w^{k_j}\}$ and taking $j \to \infty$, and using the fact $\|w^{k_j+1} - w^{k_j}\|_H \to 0$ implied by (2.8), we can conclude that $w^* \in S^*$. Now we need to prove that $w^k \to w^*$ as $k \to \infty$. It follows from (2.7) that $\|w^k - w^*\|_H \to 0$, which implies that $\|v^k - v^*\| \to 0$ because B is full column rank and hence $H_0 \succ 0$. We thus have $y^k \to y^*$ and $\lambda^k \to \lambda^*$. Notice that

$$A(x^k - x^*) + B(y^k - y^*) = Ax^k + By^k - b = \frac{1}{\beta}(\lambda^{k+1} - \lambda^k),$$

where the first equality follows from the optimality of (x^*, y^*), and the second equality is a direct consequence of the definition of λ^{k+1} in (1.3). Since $\|w^{k+1} - w^k\|_H \to 0$ implies $\lambda^{k+1} - \lambda^k \to 0$, we have $A(x^k - x^*) + B(y^k - y^*) \to 0$. Because $y^k \to y^*$ and A is full column rank, we immediately have $x^k \to x^*$, and hence $w^k \to w^*$ as $k \to \infty$.

Now, we consider Case (2) of Assumption 2.2. For this case, we have $H \succ 0$. Then, by (2.7), we know that the sequence $\{w^k\}$ is bounded and let $\{w^{k_j}\}$ be a subsequence of $\{w^k\}$ converging to w^*. Similar to the discussion above, for any fixed $w \in \Omega$, considering the inequality (2.6) for the subsequence $\{w^{k_j}\}$, taking the limit over j, and using the fact that $\|w^{k_j+1} - w^{k_j}\|_H \to 0$, we obtain that $w^* \in S^*$. Then, using (2.7), we have $\|w^k - w^*\|_H \to 0$. Since $H \succ 0$ for this case, we immediately have $w^k \to w^*$ as $k \to \infty$ and the proof is complete. □

Note that Assumptions 2.1 and 2.2, and strong CHIP are sufficient to ensure the convergence of the PADMM (1.3); but, they are not necessary. For Example 5.6 to be studied in Section 5.2, we shall show that Assumption 2.2 is not fulfilled but the convergence of the ADMM (1.2) is still ensured for this specific example.

Chapter 3

Algorithm-tailored error bound conditions

In this section, with the by-default given convergence of the sequence $\{w^k\}$ generated by the PADMM (1.3) to $w^* \in S^*$, we focus on the discussion of its linear convergence rate. Note that it is not necessary to assume Assumption 2.2 in the analysis.

As mentioned, in the literature, some generic error bound conditions depending only on the model have been studied for the linear convergence rate of the ADMM (1.2) and its variants; and in the literature, it is focused on how to sufficiently ensure these error bound conditions by posing more assumptions or requiring special structures in the model (1.1). These error bound conditions or related study are usually too restrictive; and they do not take into consideration the specific structures and properties of the algorithm under discussion. Meanwhile, it seems beneficial to estimate the error only for the specific iterative sequence, instead of arbitrary points within a region, when the convergence rate of a particular algorithm is studied. We hence prompt studying the linear convergence rate of the PADMM (1.3) under some PADMM-tailored error bound conditions, with specific consideration of the iterative scheme of (1.3). We shall show that this PADMM-tailored consideration can indeed weaken the mentioned generic error bound conditions.

We first make some notation clear. Recall the definition of H in (2.4). We shall use the notation

$$dist_H(w, \mathcal{C}) := \inf_{w' \in \mathcal{C}} \{\|w - w'\|_H\}, \tag{3.1}$$

for a given subset $\mathcal{C}$ and vector w in the same space. As mentioned, $H \succeq 0$ under Assumption 2.2. When $dist_H(\cdot, S^*)$ and $dist(\cdot, S^*)$ are considered, since

$$\|w\|_H^2 = w^T H w \leq \rho(H)\|w\|_2^2, \qquad \forall w \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m,$$

where $\rho(H)$ is the spectral radius of matrix H. Let $L_H = \sqrt{\rho(H)}$, it follows from (3.1) that

$$dist_H(w, S^*) \leq L_H \cdot dist(w, S^*), \qquad \forall w \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m. \tag{3.2}$$

Moreover, notice that the variable x is intermediate and it is not involved in the iteration of the original ADMM (1.2); see, e.g., Boyd et al. [2011]. When our analysis generally conducted for the PADMM (1.3) is specified for the original ADMM (1.2), i.e. $D = 0$, we also need the notation $v = (y, \lambda)$ to exclude the intermediate variable x and $S_v^* := \{(y^*, \lambda^*) \mid (x^*, y^*, \lambda^*) \in S^* \text{ for some } x^*\}$. Accordingly, H_0 is needed to present the analysis for (1.2) compactly; and instead of $dist_H(w, S^*)$, we just use

$$dist_{H_0}(v, S_v^*) := \inf_{v' \in S_v^*} \{\|v - v'\|_{H_0}\}, \tag{3.3}$$

when the original ADMM (1.2) is considered in our analysis. Also, we use the notation

$$S_\lambda^* := \{\lambda^* \mid (x^*, y^*, \lambda^*) \in S^* \text{ for some } (x^*, y^*)\} \tag{3.4}$$

when the convergence of the sequence of Lagrange multiplier $\{\lambda^k\}$ is highlighted.

3.1 PADMM-tailored error bound for the linear convergence rate

We first present a PADMM-tailored error bound condition associated with the sequence generated by PADMM (1.3); and show that it suffices to guarantee the linear convergence rate of the generated sequence. We refer to more literatures, e.g., Tao and Yuan [to appear]; Wang et al. [2017], for some preliminary study of algorithm-

tailored error bound conditions for other algorithms.

Definition 3.1 (PADMM-tailored error bound)**.** *Let $\{w^k\}$ be the sequence generated by the PADMM (1.3). If there exist $\kappa > 0$ and $\epsilon > 0$ such that*

$$dist_H(w^{k+1}, S^*) \leq \kappa \cdot \|w^{k+1} - w^k\|_H \qquad \text{provided} \quad w^{k+1} \in \mathcal{B}_\epsilon(w^*), \tag{3.5}$$

then $\{w^k\}$ is said to satisfy a PADMM-tailored error bound.

With (3.5), it is easy to prove the local linear convergence rate for the PADMM (1.3). We need one more theorem for preparation.

Theorem 3.2. *Let $\{w^k\}$ be the sequence generated by the PADMM (1.3) and it converge to w^*. If Assumptions 2.1 and strong CHIP are both satisfied, for any $\epsilon > 0$, there exists $\tilde{\epsilon} > 0$ such that*

$$\|w^{k+1} - w^k\|_H < \tilde{\epsilon} \Longrightarrow w^{k+1} \in \mathcal{B}_\epsilon(w^*).$$

Proof. It follows from the convergence of $\{w^k\}$ that, for any $\epsilon > 0$, there exists an integer $K > 0$ such that

$$w^{k+1} \in \mathcal{B}_\epsilon(w^*) \quad \forall\ k \geq K.$$

Taking $\tilde{\epsilon} := \min_{0 \leq k < K}\{\|w^{k+1} - w^k\|_H\} > 0$, we have

$$\|w^{k+1} - w^k\|_H < \tilde{\epsilon} \Longrightarrow k \geq K \Longrightarrow w^{k+1} \in \mathcal{B}_\epsilon(w^*),$$

and the proof is complete. □

We first prove a local property for the sequence $\{dist_H^2(w^{k+1}, S^*)\}$.

Theorem 3.3. *Assume that Assumptions 2.1 and strong CHIP are both satisfied. If the sequence $\{w^k\}$ generated by the PADMM (1.3) converges to w^* and it satisfies the PADMM-tailored error bound (3.5), then there exist $\kappa > 0$ and $\epsilon > 0$ such that*

$$dist_H^2(w^{k+1}, S^*) \leq (1 + \frac{1}{\kappa^2})^{-1} \cdot dist_H^2(w^k, S^*) \qquad \text{provided} \quad \|w^{k+1} - w^k\|_H < \epsilon.$$

Proof. First, it follows from (2.7) that

$$dist_H^2(w^{k+1}, S^*) \leq dist_H^2(w^k, S^*) - \|w^{k+1} - w^k\|_H^2, \qquad \forall k = 1, 2, \ldots.$$

By virtue of Theorem 3.2 and (3.5), there exist $\kappa > 0$ and $\epsilon > 0$ such that

$$dist_H(w^{k+1}, S^*) \leq \kappa \cdot \|w^{k+1} - w^k\|_H^2 \qquad \text{provided } \|w^{k+1} - w^k\|_H < \epsilon.$$

Subsequently, we have

$$dist_H^2(w^{k+1}, S^*) \leq dist_H^2(w^k, S^*) - \frac{1}{\kappa^2} dist_H^2(w^{k+1}, S^*) \quad \text{provided } \|w^{k+1} - w^k\|_H < \epsilon,$$

and the proof is complete. □

Moreover, we observe that when the convergence of sequence $\{w^k\}$ is guaranteed, the local property of the sequence $\{dist_H^2(w^{k+1}, S^*)\}$ established in Theorem 3.3 is essentially global. Hence, there is no difference in studying the local or global property for the sequence $\{dist_H^2(w^{k+1}, S^*)\}$ under the PADMM-tailored error bound condition (3.5). The following theorem is inspired by [Facchinei and Pang, 2007, Proposition 6.1.2].

Theorem 3.4. *Assume that Assumptions 2.1 and strong CHIP are both satisfied. If the sequence $\{w^k\}$ generated by the PADMM (1.3) converges to w^* and it satisfies the PADMM-tailored error bound condition (3.5), then there exists $\tilde{\kappa} > 0$ such that*

$$dist_H^2(w^{k+1}, S^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-1} \cdot dist_H^2(w^k, S^*), \qquad \forall\ k \geq 0. \tag{3.6}$$

Proof. Because of Theorem 3.3, there exist $\kappa > 0$ and $\epsilon > 0$ such that

$$dist_H(w^{k+1}, S^*) \leq \kappa \cdot \|w^{k+1} - w^k\|_H \qquad \text{provided } \|w^{k+1} - w^k\|_H < \epsilon.$$

Thus, we only need to consider indices k such that $\|w^{k+1} - w^k\|_H \geq \epsilon$. According to (2.7), there is a constant $M > 0$ such that $\|w^k - w^*\|_H \leq M$ for all $k \geq 0$. We

immediately have

$$dist_H(w^{k+1}, S^*) \leq \|w^{k+1} - w^*\|_H \leq M/\epsilon \cdot \|w^{k+1} - w^k\|_H \quad \text{provided } \|w^{k+1} - w^k\|_H \geq \epsilon.$$

Letting $\tilde{\kappa} := \max\{\kappa, M/\epsilon\}$, we obtain that

$$dist_H(w^{k+1}, S^*) \leq \tilde{\kappa} \cdot \|w^{k+1} - w^k\|_H, \qquad \forall\ k \geq 0.$$

Together with (2.7), we have

$$dist_H^2(w^{k+1}, S^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-1} \cdot dist_H^2(w^k, S^*), \qquad \forall\ k \geq 0,$$

and the proof is complete. □

Based on Theorem 3.4, the linear convergence rate of the sequence $\{\lambda^k\}$ generated by the PADMM (1.3) can be immediately derived. We summarize it in the following theorem.

Theorem 3.5. *Assume that Assumptions 2.1 and strong CHIP are both satisfied. If the sequence $\{w^k\}$ generated by the PADMM (1.3) converges to w^* and it satisfies the PADMM-tailored error bound condition (3.5), then there exists $\tilde{\kappa} > 0$ such that*

$$dist(\lambda^k, S_\lambda^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0,$$

where S_λ^* is defined in (3.4). That is, the sequence $\{\lambda^k\}$ generated by the PADMM (1.3) converges linearly.

If the convergence of PADMM (1.3) is guaranteed specifically by Assumption 2.2 as discussed in Section 2.3, then accordingly we can further specify the linear convergence rate of the PADMM (1.3) in the following two theorems. Note that the linear convergence results established below are both global, because of Theorem 3.4.

Theorem 3.6 (Globally Linear Convergence Rate of $\{v^k\}$)**.** *Let assumptions in Theorem 3.4 hold; and additionally if Case (1) of Assumption 2.2 holds, then it follows that*

$$dist_{H_0}(v^k, S_v^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0.$$

That is, the sequence $\{v^k\}$ generated by the PADMM (1.3) converges linearly.

Theorem 3.7 (Globally Linear Convergence Rate of $\{w^k\}$)**.** *Let assumptions in Theorem 3.4 hold; and additionally if Case (2) of Assumption 2.2 holds, then it follows that*

$$dist_H(w^k, S^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0.$$

That is, the sequence $\{w^k\}$ generated by the PADMM (1.3) converges linearly.

For the special case where $D = 0$, the PADMM (1.3) reduces to the original ADMM (1.2). Theorem 3.6 indicates the linear convergence rate of the ADMM (1.2) in sense of $\{v^k\}$ under Case (1) of Assumption 2.2, which is consistent with the analysis in the ADMM literature. Recall that the variable x is intermediate and it is not involved in the iteration performed by (1.2); hence convergence results of the ADMM (1.2) are measured only by the variables y and λ, and x does not appear.

3.2 FEB (1.8) is sufficient to ensure (3.5)

In the last section, we have proved the linear convergence rate of PADMM (1.3) under the PADMM-tailored error bound condition (3.5). Generally this condition cannot be checked directly. But we shall show that the FEB (1.8) suffices to ensure (3.5); hence (3.5) is theoretically weaker than (1.8).

Let us start with presenting a lemma which will be often used in the analysis later. The proof is trivial by using the characterization of an iterate of the PADMM (1.3) given in (2.5); it is thus omitted. We need one more matrix to simplify the notation in the analysis:

$$\hat{H} := \begin{pmatrix} D & -\beta A^T B & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} I \end{pmatrix}. \tag{3.7}$$

Lemma 3.8. *Let $\{w^k\}$ be the sequence generated by the PADMM (1.3); $\phi(\cdot)$ be*

defined in (1.6) and $\hat{H}$ in (3.7). Then, we have

$$\begin{pmatrix} D(x^k - x^{k+1}) - \beta A^T B(y^k - y^{k+1}) \\ 0 \\ \frac{1}{\beta}(\lambda^k - \lambda^{k+1}) \end{pmatrix} \in \phi(x^{k+1}, y^{k+1}, \lambda^{k+1}), \tag{3.8}$$

or equivalently,

$$\hat{H}(w^k - w^{k+1}) \in \phi(x^{k+1}, y^{k+1}, \lambda^{k+1}). \tag{3.9}$$

Based on (3.8), we immediately find that $dist(0, \phi(w^{k+1}))$ can be bounded by $\|w^{k+1} - w^k\|_H$. This is shown in the following lemma.

Lemma 3.9. *Let $\{w^k\}$ be the sequence generated by the PADMM (1.3); and $\phi(\cdot)$ be defined in (1.6). There exists $L_1 > 0$ such that*

$$dist(0, \phi(w^{k+1})) \leq L_1 \|w^{k+1} - w^k\|_H. \tag{3.10}$$

Proof. It follows from (3.8) that

$$\begin{aligned} dist(0, \phi(w^{k+1})) &= \left(\|D(x^k - x^{k+1}) - \beta A^T B(y^k - y^{k+1})\|^2 + \|\frac{1}{\beta}(\lambda^k - \lambda^{k+1})\|^2 \right)^{\frac{1}{2}} \\ &\leq \|D(x^k - x^{k+1}) - \beta A^T B(y^k - y^{k+1})\| + \|\frac{1}{\beta}(\lambda^k - \lambda^{k+1})\| \\ &\leq \|D(x^{k+1} - x^k)\| + \rho(A)\sqrt{\beta}\|\sqrt{\beta}B(y^{k+1} - y^k)\| + \frac{1}{\sqrt{\beta}}\|\frac{1}{\sqrt{\beta}}(\lambda^{k+1} - \lambda^k)\| \\ &\leq (\sqrt{\rho(D)} + \rho(A)\sqrt{\beta} + \frac{1}{\sqrt{\beta}})\|w^{k+1} - w^k\|_H, \end{aligned}$$

where $\rho(D) \geq 0$ and $\rho(A) \geq 0$ are the spectral radius of the matrices D and A, respectively. Therefore, the assertion (3.10) is proved with $L_1 := \sqrt{\rho(D)} + \rho(A)\sqrt{\beta} + \frac{1}{\sqrt{\beta}} > 0$. □

Now, it becomes clear that the FEB (1.8) gives the relationship between the terms $dist(w^{k+1}, S^*)$ and $dist(0, \phi(w^{k+1}))$, and thus effectively bridges the inequalities in (3.2) and (3.10), and eventually ensures the PADMM-tailored error bound condition (3.5). We give the full description in the following lemma. Our motivation of studying the FEB (1.8) for the linear convergence rate of the PADMM (1.3) is indeed justified; more details will be given in Chapter 4.

Lemma 3.10. *Let $\{w^k\}$ be the sequence generated by the PADMM (1.3) and it converges to w^*. Then the FEB* (1.8) *around w^* ensures the PADMM-tailored error bound condition (3.5).*

Proof. It follows from (3.10) in Lemma 3.9 and the FEB (1.8) that there exist $\kappa > 0$ and $\epsilon > 0$ such that

$$dist(w^{k+1}, S(0)) \leq \kappa dist(0, \phi(w^{k+1})) \leq L_1\kappa\|w^{k+1} - w^k\|_H, \qquad \text{provided } w^{k+1} \in \mathcal{B}_\epsilon(w^*).$$

According to (3.2), we know that $dist_H(\cdot, S^*) \leq L_H \cdot dist(\cdot, S^*)$ holds for $L_H > 0$. Thus, we have

$$dist_H(w^{k+1}, S^*) \leq L_H \cdot dist(w^{k+1}, S(0)) \leq L_H L_1 \kappa \|w^{k+1} - w^k\|_H, \quad w^{k+1} \in \mathcal{B}_\epsilon(w^*),$$

and the proof is complete. □

Remark 3.11. *Recall the definitions of $\phi(\cdot)$ in (1.6) and $S(p)$ in (1.7); also note that the sequence $\{w^k\}$ generated by the PADMM (1.3) ensures (3.8). Hence, the term $\hat{H}(w^k - w^{k+1})$ in (3.9) can be regarded as a perturbation p of $S(p)$. Moreover, it follows from (1.9) that the set-valued map $S(p)$ is calm around $(0, \bar{w})$ if and only if there exist $\kappa > 0, \sigma > 0$ and a neighborhood $\mathcal{B}_\epsilon(\bar{w})$ of $\bar{w}$ such that*

$$dist(w, S(0)) \leq k\|p\| \qquad \text{provided } \ w \in \mathcal{B}_\epsilon(\bar{w}) \cap S(p),\ \|p\| < \sigma.$$

Then, because of (3.10), it is clear that the calmness of $S(p)$, which is independent of the iterative sequence $\{w^k\}$ generated by the PADMM (1.3), suffices to ensure the algorithm-tailored error bound (3.5). Also, notice that the calmness of $S(p)$ at $(0, \bar{w})$ is equivalent to the FEB (1.8) *around $\bar{w}$. Hence, it is rationale to study the FEB (1.8) to ensure (3.5) for the PADMM (1.3). We refer to Wang et al. [2017] for a more general study, in which an unified framework is proposed to develop appropriate sufficient conditions for ensuring various error bound conditions that are tailored for some algorithms.*

Using Theorem 3.4 and Lemma 3.10, we immediately have the following theorem and its proof is omitted.

Theorem 3.12. *Let $\{w^k\}$ be the sequence generated by the PADMM (1.3) and it converge to w^*. If Assumptions 2.1 and strong CHIP are both satisfied, and the FEB (1.8) is fulfilled around w^*, then there exists $\tilde{\kappa}$ such that*

$$dist_H^2(w^{k+1}, S^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-1} \cdot dist_H^2(w^k, S^*), \qquad \forall\ k \geq 0.$$

Then, we can elaborate on the globally linear convergence rate of the sequence generated by the PADMM (1.3) under different scenarios. We summarize the results in following theorem and skip the proof.

Theorem 3.13 (Globally Linear Convergence Rate under (1.8))**.** *Let the assumptions of Theorem 3.12 hold. Then we have*

$$dist(\lambda^k, S_\lambda^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0,$$

where S_λ^* is defined in (3.4). That is, the sequence $\{\lambda^k\}$ generated by the PADMM (1.3) converges linearly. In addition, if Assumption 2.2 is assumed, then we have the following assertions.

(1) If Case (1) of Assumption 2.2 holds, it follows that

$$dist_{H_0}(v^k, S_v^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0.$$

That is, the sequence $\{v^k\}$ generated by the PADMM (1.3) converges linearly.

(2) If Case (2) of Assumption 2.2 holds, it follows that

$$dist_H(w^k, S^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0.$$

That is, the sequence $\{w^k\}$ generated by the PADMM (1.3) converges linearly.

In general, the FEB (1.8) may not hold (see the next chapter for such an example). The following corollary suggests some interesting cases with practical interests where the validation of the FEB (1.8) can be easily verified.

Corollary 3.14. *In the model (1.1), suppose that both ∂f and ∂g are polyhedral multifunctions, $\mathcal{X}$ and $\mathcal{Y}$ are polyhedral sets. Then, the FEB* (1.8) *is fulfilled around any point in S^*.*

Proof. Note first $\mathcal{F}$ defined in (2.1) is a polyhedra. Since the graph of $\mathcal{N}_{\mathcal{X}}$ is a finite union of polyhedral convex sets, $\mathcal{N}_{\mathcal{X}}$ is polyhedral. Hence, the sum of polyhedral maps $\partial f + \mathcal{N}_{\mathcal{X}}$ is polyhedral. Similarly, $\partial g + \mathcal{N}_{\mathcal{Y}}$ is polyhedral as well, and so is the inverse map

$$S(p) := \{(x, y, \lambda) : p \in \phi(x, y, \lambda)\}.$$

By [Robinson, 1980, Proposition 1], $S(\cdot)$ is upper-Lipschitz. Hence, FEB (1.8) is fulfilled around any KKT point. □

Chapter 4

More discussions on various error bound conditions

In this chapter, we show the equivalence among the FEB (1.8), the proximal EB-I (1.11) and the proximal EB-II (1.13). We also give more details of using the FEB (1.8) for studying the linear convergence rate of PADMM (1.3).

4.1 Equivalence of several error bound conditions

We show that the mentioned error bound conditions (1.8), (1.11) and (1.13) are all equivalent. First, we prove that (1.8) holds if (1.11) or (1.13) holds.

Proposition 4.1. *If the KKT system (1.4) admits either the proximal EB-I (1.11) or proximal EB-II (1.13) around a KKT point $\bar{w}$, it also admits the FEB (1.8) around $\bar{w}$.*

Proof. Given w, for any $u \in \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x)$, it holds that

$$x = \mathrm{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda + u),$$

and

$$x = \mathrm{Proj}_{\mathcal{X}}(x - \xi + A^T\lambda + u), \qquad for\ some\ \xi \in \partial f(x).$$

Since it holds that

$$0 \in \mathcal{N}_{\mathcal{X}}(x) + x - (x - \partial f(x) + A^T\lambda + u),$$

we have

$$\|x - \mathrm{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda)\| = \|\mathrm{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda + u) - \mathrm{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda)\| \leq \|u\|,$$

and thus

$$\begin{aligned} & dist(0, x - \mathrm{Proj}_{\mathcal{X}}(x - \partial f(x) + A^T\lambda)) \\ \leq & \|x - \mathrm{Proj}_{\mathcal{X}}(x - \xi + A^T\lambda)\| \\ = & \|\mathrm{Proj}_{\mathcal{X}}(x - \xi + A^T\lambda + u) - \mathrm{Proj}_{\mathcal{X}}(x - \xi + A^T\lambda)\| \leq \|u\|. \end{aligned}$$

Since u is arbitrarily chosen, we have the relations:

$$\|x - \mathrm{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda)\| \leq dist(0, \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x)),$$

and

$$dist(0, x - \mathrm{Proj}_{\mathcal{X}}(x - \partial f(x) + A^T\lambda)) \leq dist(0, \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x)).$$

Similarly, we can establish the same results for $dist(0, \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y))$. That is, we have

$$\|y - \mathrm{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda)\| \leq dist(0, \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y)),$$

and

$$dist(0, y - \mathrm{Proj}_{\mathcal{Y}}(y - \partial g(y) + B^T\lambda)) \leq dist(0, \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y)).$$

Using these inequalities, it is easy to see that

$$\|R_1(w)\| \leq dist(0, \phi(w)), \qquad dist(0, R_2(w)) \leq dist(0, \phi(w)),$$

and the proof is complete. □

Notice the equality $\text{Prox}_{th} = (I + t\partial h)^{-1}$. It is easy to show that (1.8) can imply either (1.11) or (1.13) as well. We summarize this result in the following proposition.

Proposition 4.2. *If the KKT system (1.4) admits the FEB (1.8) around a KKT point $\bar{w}$, it admits the proximal EB-I (1.11) and proximal EB-II (1.13) around $\bar{w}$ as well.*

Proof. First, by virtue of

$$\begin{aligned} x + A^T\lambda - \text{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda) &\in \big(\partial f + \mathcal{N}_{\mathcal{X}}\big)\big(\text{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda)\big), \\ y + B^T\lambda - \text{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda) &\in \big(\partial g + \mathcal{N}_{\mathcal{Y}}\big)\big(\text{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda)\big), \end{aligned}$$

we conclude that

$$dist\Big(0, \phi\big(\text{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda), \text{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda), \lambda\big)\Big) \leq \|R_1(w)\|. \tag{4.1}$$

Therefore, for any $w \in \mathcal{B}_\epsilon(\bar{w})$, we have

$$\begin{aligned} dist(w, S(0)) &\leq c_1(\|x - \text{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda)\| + \|y - \text{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda)\|) \\ &\quad + dist\Big(\big(\text{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda), \text{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda), \lambda\big), S(0)\Big) \\ &\leq c_1(\|x - \text{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda)\| + \|y - \text{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda)\|) \\ &\quad + \kappa \cdot dist\Big(0, \phi\big(\text{Prox}_{f+\delta_{\mathcal{X}}}(x + A^T\lambda), \text{Prox}_{g+\delta_{\mathcal{Y}}}(y + B^T\lambda), \lambda\big)\Big), \\ &\leq (2c_1 + \kappa) \cdot \|R_1(w)\|, \end{aligned}$$

where the second inequality follows from the FEB (1.8), and the third inequality is a direct consequence of (4.1). Thus we get the proximal EB-I (1.11) around $\bar{w}$. We can obtain the proximal EB-II (1.13) similarly. The proof is complete. □

With Propositions 4.1 and 4.2, the equivalence between the FEB (1.8) and the proximal EB-I (1.11) or proximal EB-II (1.13) is established. Technically one can employ anyone of (1.8), (1.11) and (1.13), however, we just focus on the error bound condition in form of (1.8). Together with the convergence rate analysis in the forth-

coming chapter, we will explain that (1.8) seems to be a better choice in the sense of a simplified analysis later.

4.2 Preference of (1.8)

In this section we provide more details of why we prefer the FEB (1.8) than the proximal EB-I (1.11) and proximal EB-II (1.13) for analyzing the linear convergence rate of PADMM (1.3), despite of their theoretical equivalence.

As briefly mentioned preceding Lemma 3.10, to meet the PADMM-tailored error bound condition (3.5), we need to bound the term $dist_H(w^{k+1}, S^*)$ by $\|w^{k+1} - w^k\|_H$. On the other hand, the inequalities in (3.2) and (3.10) give us

$$dist_H(w^{k+1}, S^*) \leq L_H \cdot dist(w^{k+1}, S^*), \quad dist(0, \phi(w^{k+1})) \leq L_1 \|w^{k+1} - w^k\|_H.$$

Hence, essentially we need to build up the link between the terms $dist(w^{k+1}, S^*)$ and $dist(0, \phi(w^{k+1}))$. This is perfectly achieved by the FEB (1.8).

For the proximal EB-I (1.11), however, it facilitates bridging the terms $dist(w^{k+1}, S^*)$ and $\|R_1(w^{k+1})\|$; or the terms $dist(w^{k+1}, S^*)$ and $dist(0, R_2(w^{k+1}))$ by the proximal EB-II (1.13). In other words, neither (1.11) nor (1.13) can be directly used for bridging the terms $dist_H(w^{k+1}, S^*)$ and $\|w^{k+1} - w^k\|_H$ and hence ensuring (3.5); additional and more complicated manipulations are needed if (1.11) or (1.13) is used.

Let us further explain the difference among these error bound conditions in studying the linear convergence rate of the particular PADMM (1.3) from the perturbation perspective. As mentioned, the FEB (1.8) around a reference point $\bar{w}$ is equivalent to the calmness of $S(p)$ at $(0, \bar{w})$. On the other hand, if we define $S_{prox-I} : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m \rightrightarrows \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m$ as

$$S_{prox-I}(p) := \left\{ (x, y, \lambda) \mid \begin{pmatrix} p_1 \in (\partial f + \mathcal{N}_{\mathcal{X}})(x - p_1) - A^T\lambda \\ p_2 \in (\partial g + \mathcal{N}_{\mathcal{Y}})(y - p_2) - B^T\lambda \\ p_3 = Ax + By - b \end{pmatrix} \right\}$$

with $p = (p_1, p_2, p_3) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m$, then we have $S_{prox-I}(0) = S^*$ and hence the proximal EB-I (1.11) around a reference point $\bar{w}$ is equivalent to the calmness of

$S_{prox-I}(p)$ at $(0, \bar{w})$. That is, there exist $\kappa > 0, \sigma > 0$ and a neighborhood $\mathcal{B}_\epsilon(\bar{w})$ of $\bar{w}$ such that

$$dist(w, S_{prox-I}(0)) \leq k\|p\| \qquad \text{provided} \quad w \in \mathcal{B}_\epsilon(\bar{w}) \cap S_{prox-I}(p),\ \|p\| < \sigma.$$

Let us further define $S_{prox-II} : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m \rightrightarrows \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m$ as

$$S_{prox-II}(p) := \left\{ (x, y, \lambda) \mid \begin{pmatrix} p_1 \in \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x - p_1) \\ p_2 \in \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y - p_2) \\ p_3 = Ax + By - b \end{pmatrix} \right\}$$

with $p = (p_1, p_2, p_3) \in \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m$. It is easy to see that $S_{prox-II}(0) = S^*$ and the proximal EB-II (1.13) around a reference point $\bar{w}$ is equivalent to the calmness of $S_{prox-II}(p)$ at $(0, \bar{w})$. That is, there exist $\kappa > 0, \sigma > 0$ and a neighborhood $\mathcal{B}_\epsilon(\bar{w})$ of $\bar{w}$ such that

$$dist(w, S_{prox-II}(0)) \leq k\|p\| \qquad \text{provided} \quad w \in \mathcal{B}_\epsilon(\bar{w}) \cap S_{prox-II}(p),\ \|p\| < \sigma.$$

According to Wang et al. [2017], for the sequence $\{z^k\}$ generated by an algorithm, if $\hat{H}(z^{k+1} - z^k)$ with an appropriate $\hat{H}$ is regarded as the perturbation of the corresponding optimality system, then the calmness of the induced set-valued mapping straightforwardly implies the desirable error bound that is tailored for the algorithm under investigation. When the PADMM (1.3) is considered, as shown by (3.8) in Lemma 3.8, $\hat{H}(w^k - w^{k+1})$ corresponds to the canonical perturbation of the KKT system $0 \in \phi(w)$. Hence, it is motivated to consider the perturbed multifunction $S(p)$ defined in (1.7), instead of $S_{prox-I}(p)$ or $S_{prox-II}(p)$. That is, we use the FEB (1.8), rather than the proximal EB-I (1.11) or proximal EB-II (1.13), for analyzing the linear convergence of the PADMM (1.3).

In addition to the superiority of yielding an easier analysis for the linear convergence rate of the problem (1.1), studying the calmness of $S(p)$, rather than $S_{prox-I}(p)$ or $S_{prox-II}(p)$, may lead to some interesting future work, as we shall mention in Chapter 7. Also, as we shall show soon in the next chapter, considering the perturbed mapping $S(p)$ in (1.7) enables us discern that the second part of the left-hand side

of (3.8) remains zero for each iteration of the PADMM (1.3). This insight inspires us to study a partial error bound condition to ensure the linear convergence of the PADMM (1.3), which seems to be novel in the literature.

Chapter 5

Partial error bound for the linear convergence of PADMM (1.3)

We have established the linear convergence rate for the PADMM (1.3) under the PADMM-tailored error bound condition (3.5) and shown that the FEB (1.8) sufficiently ensures (3.5). In this chapter, we show that the FEB (1.8) can be further weakened if the specific iterative scheme (1.3) is fully considered. As mentioned, this is accomplished by the observation that there is no perturbation to the second part of the perturbed mapping $S(p)$ in (1.7). Hence, taking into consideration the specific iterative scheme enables us to weaken the FEB (1.8) to guarantee (3.5) and hence the linear convergence rate of the PADMM (1.3).

5.1 Partial error bound conditions and linear convergence

Recall the KKT system (1.4) and the definition of the generic error bound condition (1.5). Using the terminology initiated in Liu et al. [2008], we can also define the so-called local partial error bound (PEB) for (1.4).

Definition 5.1. *Assume S is represented as the intersection of two closed sets, i.e., $S = \mathcal{C} \cap \mathcal{D}$. The KKT system (1.4) is said to admit a PEB on the set $\mathcal{C}$ around $w^* \in S$ if there exists a nonnegative function $\bar{r} : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m \to \boldsymbol{R}_+$ satisfying*

$\bar{r}(w) = 0$ *if* $w \in \mathcal{D}$, *a neighborhood* $\mathcal{B}_\epsilon(w^*)$ *of the point* w^*, *and a constant* $\kappa > 0$ *such that*

$$[\text{PEB}^{\bar{\text{r}}} \text{ on } \mathcal{C}] \quad dist(w, S) \leq \kappa \cdot \bar{r}(w) \quad \text{provided} \quad w \in \mathcal{B}_\epsilon(w^*) \cap \mathcal{C}.$$

Obviously, from the definition, for any given closed set $\mathcal{C}$, $\text{PEB}^{\bar{r}}$ on $\mathcal{C}$ is weaker than EB^r defined in (1.5). Taking a closer look at (2.5) and (3.8), we notice that the optimality condition with respect to y, i.e., $0 \in \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y)$, remains satisfied for all iterates of (1.3). Let us define

$$S_g := \{w \mid 0 \in \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y)\}.$$

Then, this observation motivates us to consider a partially perturbed KKT mapping $S_P : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^m \rightrightarrows \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m$ as

$$S_P(p) := \{w \in S_g \mid p \in \phi_P(x, y, \lambda)\},$$

where $\phi_P : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m \rightrightarrows \boldsymbol{R}^{n_1} \times \boldsymbol{R}^m$ is defined as

$$\phi_P(w) = \begin{pmatrix} \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x) \\ Ax + By - b \end{pmatrix}. \tag{5.1}$$

Hence, we define a partial error bound that is particularly tailored for the specific sequence $\{w^k\}$ generated by the PADMM (1.3). Note that this definition may not be extended to other algorithms evidently.

Definition 5.2 (PADMM-tailored PEB)**.** *Let* $\{w^k\}$ *be the sequence generated by the PADMM (1.3) and it converges to* w^*. *The KKT system (1.4) is said to admit a partial local error bound around* w^* *if there exists a neighborhood* $\mathcal{B}_\epsilon(w^*)$ *of* w^* *and some* $\kappa > 0$ *such that*

$$[\text{PEB}] \qquad dist(w, S_P(0)) \leq \kappa \cdot dist(0, \phi_P(w)) \qquad \text{provided} \quad w \in S_g \cap \mathcal{B}_\epsilon(w^*). \tag{5.2}$$

Apparently, it holds that $S_P(0) = S(0) = S^*$, and the following relationship is easy to obtain: the proximal EB-I (1.11) and proximal EB-II (1.13) $\Leftrightarrow$ the FEB (1.8) $\Rightarrow$ PADMM-tailored PEB (5.2). That is, the PADMM-tailored PEB (5.2) is

the weakest one.

We next show that the PADMM-tailored PEB (5.2) suffices to imply the (3.5) and hence to ensure the linear convergence for the PADMM (1.3).

Lemma 5.3. *Let $\{w^k\}$ be the sequence generated by the PADMM (1.3). If the PADMM-tailored PEB (5.2) holds, then the PADMM-tailored error bound (3.5) holds as well.*

Proof. First, by virtue of (3.8) in Lemma 3.8, there always holds

$$0 \in \partial g(y^{k+1}) - B^T\lambda^{k+1} + \mathcal{N}_{\mathcal{Y}}(y^{k+1}),$$

which indicates that $w^{k+1} \in S_g$. Then by Lemma 3.9 (3.10), there is $L_1 > 0$ such that

$$dist(0, \phi_P(w^{k+1})) \leq L_1\|w^{k+1} - w^k\|_H.$$

Furthermore, according to the PADMM-tailored PEB (5.2), it follows from $w^{k+1} \in S_g$ that there are $\kappa > 0$ and $\epsilon > 0$ such that

$$dist(w^{k+1}, S^*) = dist(w^{k+1}, S_P(0)) \leq L_1\kappa\|w^{k+1} - w^k\|_H, \quad \text{provided} \ \ w^{k+1} \in \mathcal{B}_\epsilon(w^*).$$

Then, it follows from (3.2) that $dist_H(\cdot, S^*) \leq L_H \cdot dist(\cdot, S^*)$ holds. Subsequently the desired estimate follows

$$dist_H(w^{k+1}, S^*) \leq L_2 \cdot dist(w^{k+1}, S^*) \leq L_2L_1\kappa\|w^{k+1} - w^k\|_H, \quad \text{provided} \ w^{k+1} \in \mathcal{B}_\epsilon(w^*),$$

and the proof is complete. □

Similar as the proof of Theorem 3.4, we can derive an important equality for the sequence $\{dist_H^2(w^{k+1}, S^*)\}$ under (5.2).

Theorem 5.4. *Let $\{w^k\}$ be the sequence generated by the PADMM* (1.3) *and it converges to w^*. If Assumptions 2.1 and strong CHIP are both satisfied, and the PADMM-tailored PEB (5.2) is fulfilled around w^*, then there exists $\tilde{\kappa}$ such that*

$$dist_H^2(w^{k+1}, S^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-1} \cdot dist_H^2(w^k, S^*), \qquad \forall \ k \geq 0.$$

Similar as Theorem 3.13, we can further specify Theorem 5.4 as the globally linear convergence rate of the PADMM (1.3) under various scenarios. We present them in the following theorem and skip the proof.

Theorem 5.5. *Let $\{w^k\}$ be the sequence generated by the PADMM* (1.3) *and it converges to w^*. If Assumptions 2.1 and strong CHIP are both satisfied, and the PADMM-tailored PEB (5.2) is fulfilled around w^*, then there exists $\tilde{\kappa}$ such that*

$$dist(\lambda^k, S^*_\lambda) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0,$$

where S^*_λ is defined in (3.4). That is, the sequence $\{\lambda^k\}$ generated by the PADMM (1.3) converges linearly. In addition, if Assumption 2.2 is assumed, then we have the following assertions.

(1) If Case (1) of Assumption 2.2 holds, it follows that

$$dist_{H_0}(v^k, S^*_v) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0. \tag{5.3}$$

That is, the sequence $\{v^k\}$ generated by the PADMM (1.3) converges linearly.

- *If Case (2) of Assumption 2.2 holds, it follows that*

$$dist_H(w^k, S^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0. \tag{5.4}$$

That is, the sequence $\{w^k\}$ generated by the PADMM (1.3) converges linearly.

5.2 Example

It is interesting to compare the FEB (1.8) and the PADMM-tailored PEB (5.2). We next present an example which ensures the PADMM-tailored PEB (5.2) while fails to guarantee the FEB (1.8) at its optimal solution point. Hence, together with the fact that the FEB (1.8) being sufficient to guarantee the PADMM-tailored PEB (5.2), we show that the PADMM-tailored PEB (5.2) is weaker than the FEB (1.8).

Example 5.6. *Consider a special case of the model (1.1) as*

$$\begin{aligned} \min_{x,y} \quad & \frac{1}{2}x^2 + \frac{1}{2}y_1^2 + \frac{1}{4}y_2^4 \\ s.t. \quad & Ax + By = 0, \end{aligned} \tag{5.5}$$

where

$$A = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.6}$$

with $x \in \boldsymbol{R}, y = (y_1, y_2, y_3, y_4) \in \boldsymbol{R}^4$. *Let* $w = (x, y, \lambda) \in \boldsymbol{R}^8$. *The strong CHIP follows trivially from the linearity. The KKT residual mapping in (1.6) can be specified as* $\phi : \boldsymbol{R} \times \boldsymbol{R}^4 \times \boldsymbol{R}^3 \to \boldsymbol{R} \times \boldsymbol{R}^4 \times \boldsymbol{R}^3$ *given by*

$$\phi(w) = \begin{pmatrix} x - \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ 0 \\ 0 \\ y_4^3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} y \end{pmatrix}.$$

It is easy to see that the optimal solution point is $w^* = (0, 0, 0, 0, 0, 0, 0, 0)$ *and the solution set* $S^* = \{w^*\}$. *Note that* $dist(w, S^*) = \|w\|$. *Without loss of generality, in this example we use the* l_1*-norm for the norm used in the involved error bound conditions.*

For simplicity, let us just take $D = 0$ *in (1.3) and consider the original ADMM scheme (1.2). It is easy to see that the exact expression of the iterative scheme (1.2)*

for this example is explicitly written as

$$w^{k+1} = (\frac{\lambda_3^k - \beta y_2^k}{1+2\beta}, 0, \frac{\beta y_2^k - \lambda_3^k}{1+2\beta}, 0, 0, 0, 0, \frac{\beta^2 y_2^k + (1+\beta)\lambda_3^k}{1+2\beta}), \qquad \forall k \geq 1.$$

Recall that the variable x is intermediate and we just need to focus on the non-zero y- and λ-variables of the iteration, i.e., $(y_2^{k+1}, \lambda_3^{k+1})$. Accordingly, we define the matrix T as

$$T = \begin{pmatrix} \frac{\beta}{1+2\beta} & \frac{-1}{1+2\beta} \\ \frac{\beta^2}{1+2\beta} & \frac{1+\beta}{1+2\beta} \end{pmatrix}, \tag{5.7}$$

and the iteration is essentially executed by the recursion:

$$(y_2^{k+1}, \lambda_3^{k+1}) = T(y_2^k, \lambda_3^k), \forall k \geq 1.$$

By straightforward calculation, the eigenvalues of the matrix T in (5.7) are $(1 + 2\beta \pm \sqrt{1-4\beta^2})/(2+4\beta)$. Therefore, $\rho(T)$, the spectral radius of T, is strictly smaller than 1 for any $\beta > 0$. Thus, we know that $\{w^k\}$ converges to $w^* = (0,0,0,0,0,0,0,0)$ linearly.

We next show by contradiction that the KKT system $\phi(x,y,\lambda) = 0$ fails to admit the FEB (1.8) around w^*. Let $\{\delta_k\}$ be a sequence such that $\delta_k \searrow 0$ and define the sequence $\{w^k\}$ by $w^k = (0,0,-\delta_k,-\delta_k,\delta_k,0,0,0), \forall k \geq 0$. It is clear from the construction that $w^k \to w^*$ and $dist(w^k, S^*) = 3\|\delta_k\|$. On the other hand, $dist(0, \phi(w^k)) = \|\phi(w^k)\| = \|\delta_k^3\|$, $\forall k \geq 0$. This leads to $dist(0, \phi(w^k)) = o(dist(w^k, S^*))$. Consequently, the KKT system $\phi(x,y,\lambda) = 0$ does not possess the FEB (1.8) around w^*.

For analyzing the PADMM-tailored PEB (5.2) around w^*, in this example, we specify the partial KKT residual mapping $\phi_P : \boldsymbol{R} \times \boldsymbol{R}^4 \times \boldsymbol{R}^3 \to \boldsymbol{R} \times \boldsymbol{R}^3$ defined in (5.1):

$$\phi_P(w) = \begin{pmatrix} x - \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} y \end{pmatrix}.$$

Let us further define

$$S_y := \left\{ (x,y,\lambda) \mid 0 = \begin{pmatrix} y_1 \\ 0 \\ 0 \\ y_4^3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \right\},$$

which can be simplified as:

$$S_y := \{(x,y,\lambda) \mid y_1 - \lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_1 = 0,\ \ y_4^3 - \lambda_1 - \lambda_2 = 0\}.$$

Therefore, for any sequence $\{w^k\} \subseteq S_y \cap \mathcal{B}_{\frac{1}{2}}(w^*)$ *and* $\|w^k\| \to 0$*, there holds that*

$$\begin{aligned} dist(0, \phi_P(w^k)) = \|\phi_P(w^k)\| =& |x^k - \lambda_2^k - \lambda_3^k| + |y_1^k + y_3^k + y_4^k| + |x^k + y_2^k + y_4^k| + |x^k| \\ =& |x^k| + |x^k - \lambda_3^k| + |y_3^k| + |x^k + y_2^k|. \end{aligned}$$

It is clear that,

$$|y_2| \le |x + y_2| + |x|,$$

and

$$|\lambda_3| \le |x - \lambda_3| + |x|.$$

Consequently, for the sequence $\{w^k\} \subseteq S_y \cap \mathcal{B}_{\frac{1}{2}}(w^*)$*, we have the following estimate:*

$$\begin{aligned} dist(w^k, S^*) = \|w^k\| =& |x^k| + |y_1^k| + |y_2^k| + |y_3^k| + |y_4^k| + |\lambda_1^k| + |\lambda_2^k| + |\lambda_3^k| \\ \le& |x^k| + |x^k + y_2^k| + |x^k| + |y_3^k| + |x^k - \lambda_3^k| + |x^k| \\ \le& 3(|x^k| + |x^k - \lambda_3^k| + |y_3^k| + |x^k + y_2^k|) \\ =& 3\, dist(0, \phi_P(w^k)). \end{aligned}$$

Therefore, the KKT system $\phi(x,y,\lambda) = 0$ *admits the PADMM-tailored PEB (5.2) around* w^**.*

Remark 5.7. *Example 5.6 with a few variables is sufficient to convince the advantage of considering the PADMM-tailored PEB (5.2) for studying the linear convergence rate of the ADMM (1.3). It is analogous to construct convex polynomial optimization*

problems in higher dimension so that only the PADMM-tailored PEB (5.2) holds while the FEB (1.8) does not.

Note that the matrix B given in (5.6) is not of full column rank; hence Assumption 2.2 is not satisfied and this reflects that Assumption 2.2 is sufficient, instead of necessary, to ensure the convergence of the PADMM (1.3). Moreover, it is verified that the FEB (1.8) fails and thus it is invalid to explain the linear convergence rate of the application of the ADMM (1.2) to this specific example. Instead, the PADMM-tailored PEB (5.2) is satisfied for this example and hence the linear convergence rate of the ADMM is theoretically explained.

Chapter 6

Difference of updating the primal variables in ADMM (1.2)

Despite of the main purpose of studying the linear convergence rate of the PADMM (1.3) under weaker error bound conditions, an interesting byproduct of this work is a theoretical explanation for the difference of updating the primal variables x and y in the iteration. For simplicity, let us just focus on the original ADMM (1.2) in this chapter.

6.1 Swap update order of ADMM

If we swap the order of the primal variables x and y in (1.2), another form of the ADMM is obtained:

$$\begin{cases} y^{k+1} = \arg\min\limits_{y\in\mathcal{Y}} \ \{g(y) - \langle\lambda^k, Ax^k + By - b\rangle + \dfrac{\beta}{2}\|Ax^k + By - b\|^2\}, \\ x^{k+1} = \arg\min\limits_{x\in\mathcal{X}} \ \{f(x) - \langle\lambda^k, Ax + By^{k+1} - b\rangle + \dfrac{\beta}{2}\|Ax + By^{k+1} - b\|^2\}, \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b). \end{cases} \tag{6.1}$$

Obviously, the iterative scheme (6.1) can be written as

$$\begin{cases} 0 \in \partial g(y^{k+1}) - B^T\lambda^{k+1} + \beta B^T A(x^k - x^{k+1}) + \mathcal{N}_{\mathcal{Y}}(y^{k+1}), \\ 0 \in \partial f(x^{k+1}) - A^T\lambda^{k+1} + \mathcal{N}_{\mathcal{X}}(x^{k+1}), \\ 0 = Ax^{k+1} + By^{k+1} - b + \dfrac{1}{\beta}(\lambda^{k+1} - \lambda^k). \end{cases} \tag{6.2}$$

The convergence of (6.1) certainly holds; given the proved convergence of (1.2). But these two schemes differ in the intermediate variables and the order of updating the primal variables: x and y. Numerically, it does make difference to place which of x and y as the first variable to be updated. An immediate explanation is that if the x-subproblem is significantly more complicated than the y-subproblem, it seems smarter to update y first so as to avoid the possible transmission of error caused by solving the x-subproblem inexactly. Such situations arise in the case where, e.g., one subproblem is in higher dimension or more complicated natures than the other one. Representative examples are the so-called sparse and low-rank optimization models which at each iteration require to solve a subproblem involving the singular value decomposition of a large matrix and thus inner iterations with accumulative errors are inevitable, and the other subproblem which usually has the closed-form solution and hence no inner iteration is needed. For such problems, it is highly suggested to update the easier subproblem first; and this makes significant difference in the eventual numerical performance , see, e.g., Lin et al. [2010]; Yang and Yuan [2013]. Meanwhile, it seems no theory is known to explain this difference caused by different orders of updating the primal variables. We next show that the two schemes may admit different convergence rates in sense of different partial error bound assumptions; and thus provide a theoretical explanation for this issue.

6.2 Partial error bound condition for (6.1)

We need the matrix to simplify the notation in the analysis:

$$\tilde{H} := \begin{pmatrix} 0 & 0 & 0 \\ -\beta B^T A & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} I \end{pmatrix}. \tag{6.3}$$

Similar as Lemma 3.8, we present the following lemma which follows directly from the characterization of an iterate of (6.1) given in (6.2).

Lemma 6.1. *Let $\{w^k\}$ be the sequence generated by (6.1); $\phi(\cdot)$ be defined in (1.6) and $\tilde{H}$ in (6.3). Then, we have*

$$\begin{pmatrix} 0 \\ -\beta B^T A(x^k - x^{k+1}) \\ \frac{1}{\beta}(\lambda^k - \lambda^{k+1}) \end{pmatrix} \in \phi(x^{k+1}, y^{k+1}, \lambda^{k+1}). \tag{6.4}$$

or equivalently,

$$\tilde{H}(w^k - w^{k+1}) \in \phi(x^{k+1}, y^{k+1}, \lambda^{k+1}).$$

Taking a closer look at (6.2) and (6.4), we notice that the optimality condition with respect to x, i.e., $0 \in \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x)$ remains satisfied for all iterates of (6.1). Following our discussion in the preceding chapter, by letting

$$S_f := \{(x, y, \lambda) \mid 0 \in \partial f(x) - A^T\lambda + \mathcal{N}_{\mathcal{X}}(x)\},$$

we can define a PEB tailored particularly for the ADMM scheme (6.1) as follows.

Definition 6.2 (ADMM-tailored Partial Error Bound-yx)**.** *Let $\{w^k\}$ be the sequence generated by (6.1) and it converges to w^*. The KKT system (1.4) is said to admit a local PEB-yx around w^* if there exists a neighborhood $\mathcal{B}_\epsilon(w^*)$ of w^* and $\kappa > 0$ such that*

$$[\text{PEB} - \text{yx}] \qquad dist(w, S^*) \leq \kappa \cdot dist(0, \bar{\phi}_P(w)) \quad \text{provided} \quad w \in S_f \cap \mathcal{B}_\epsilon(w^*), \tag{6.5}$$

where $\bar{\phi}_P : \boldsymbol{R}^{n_1} \times \boldsymbol{R}^{n_2} \times \boldsymbol{R}^m \rightrightarrows \boldsymbol{R}^{n_1} \times \boldsymbol{R}^m$ *is defined as:*

$$\bar{\phi}_P(w) = \begin{pmatrix} \partial g(y) - B^T\lambda + \mathcal{N}_{\mathcal{Y}}(y) \\ Ax + By - b \end{pmatrix}.$$

We define the matrix $H_{x\lambda}$ and its submatrix $H^0_{x\lambda}$ as follows to simplify the notation in our analysis:

$$H_{x\lambda} = \begin{pmatrix} \beta A^T A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\beta} I \end{pmatrix} \quad \text{and} \quad H^0_{x\lambda} = \begin{pmatrix} \beta A^T A & 0 \\ 0 & \frac{1}{\beta} I \end{pmatrix}.$$

Also, we use the notation

$$S^*_{x\lambda} := \{(x^*, \lambda^*) \mid (x^*, y^*, \lambda^*) \in S^* \text{ for some } y^*\} \tag{6.6}$$

when the convergence of the sequence of $\{x^k, \lambda^k\}$ is highlighted. Consequently, we can prove the globally linear convergence rate of the scheme (6.1) if the PEB-yx (6.5) is assumed. The details are omitted.

Proposition 6.3. *Let the sequence* $\{w^k\}$ *be generated by (6.1) and it converges to* w^**. If Assumptions 2.1 and strong CHIP are both satisfied; and the PEB-yx condition (6.5) is fulfilled around* w^**, then there exists* $\tilde{\kappa}$ *such that*

$$dist^2_{H_{x\lambda}}(w^{k+1}, S^*) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-1} \cdot dist^2_{H_{x\lambda}}(w^k, S^*), \qquad \forall\ k \geq 0.$$

Moreover, we have

$$dist(\lambda^k, S^*_\lambda) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0,$$

where S^*_λ *is defined in (3.4). That is, the sequence* $\{\lambda^k\}$ *generated by (6.1) converges linearly. In addition, if* A *is full column rank, then* $H^0_{x\lambda} \succ 0$ *and it follows that*

$$dist_{H^0_{x\lambda}}\left((x^k, \lambda^k), S^*_{x\lambda}\right) \leq (1 + \frac{1}{\tilde{\kappa}^2})^{-\frac{k}{2}} \cdot dist_H(w^0, S^*), \qquad \forall\ k \geq 0,$$

where $S^*_{x\lambda}$ is defined in (6.6). That is, the sequence $\{(x^k, \lambda^k)\}$ generated by (6.1) converges linearly.

6.3 Difference between PEB (5.2) and PEB-yx (6.5)

By comparing Corollary 5.4 and Proposition 6.3, a clear conclusion can be drawn from the difference in the PEB conditions (5.2) and (6.5). Let us reconsider Example 5.6 for an illustration of the difference. In particular, we will show that Example 5.6 does not meet the PEB-yx around the optimal solution. The PEB (5.2), on the other hand, is satisfied according to the analysis for Example 5.6 in Section 5.2.

As previously, we can easily write down the explicit recursion for the application of the ADMM scheme (6.1) to Example 5.6; and the convergence is clearly implied. We omit the details for succinctness. We further show the difference in the two PEB conditions (5.2) and (6.5) in this example. Therefore, the convergence rates of (1.2) and (6.1) may be different according to the proposed PEB theory. To this end, the associated partial KKT residual mapping $\bar{\phi}_P : \boldsymbol{R} \times \boldsymbol{R}^4 \times \boldsymbol{R}^3 \to \boldsymbol{R} \times \boldsymbol{R}^4$ reads as:

$$\bar{\phi}_P(x, y, \lambda) = \begin{pmatrix} \begin{pmatrix} y_1 \\ 0 \\ 0 \\ y_4^3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} y \end{pmatrix}.$$

Let $\{\delta_k\}$ be a sequence such that $\delta_k \searrow 0$ and define the sequence $\{w^k\}$ by $w^k = (0, 0, -\delta_k, -\delta_k, \delta_k, 0, 0, 0)$, where $k = 0, 1, \ldots$. It is clear from the construction that $w^k \to w^*$, $\{w^k\} \subseteq S_x$ and $dist(w^k, S) = 3\|\delta_k\|$, where

$$S_x := \left\{ (x, y, \lambda) \mid 0 = x - \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \right\}.$$

On the other hand, $dist(0, \bar{\phi}_P(w^k)) = \|\bar{\phi}_P(w^k)\| = \|\delta_k^3\|$ for $k = 0, 1, \ldots$. This leads to $dist(0, \bar{\phi}_P(w^k)) = o(dist(w^k, S^*))$. Consequently, the PEB-$yx$ is not fulfilled around w^*. That is, the ADMM (1.2) with the updating order of $x - y$ admits the PEB and it converges linearly, while the PEB-yx (6.5) is not satisfied and there is no guarantee to the linear convergence rate for the ADMM (6.1) with the updating order of $y - x$.

Chapter 7

Discussion

We study error bound conditions to ensure the linear convergence rate for the alternating direction method of multipliers (ADMM) in the convex programming context. Different from existing literatures that requires stronger assumptions or special structures on the model under discussion to sufficiently ensure certain error bound conditions, we prompt weakening these error bound conditions by taking into consideration the structures and properties of the specific algorithm under discussion. That is, algorithmic-tailored error bound conditions should be considered and they could be weaker than the generic-purpose error bound conditions. We give both full and partial error bound conditions in accordance with the ADMM's special iterative scheme to derive its linear convergence rate; the idea of partial error bound is inspired by an observation on the partially perturbed system (3.8). Furthermore, we construct an example to show that the partial error bound condition is weaker than the generic ones. The main analysis also inspires byproducts. First, a theoretical interpretation is given to explain the difference if the two primal variable are updated by different orders in ADMM's iteration. Second, the equivalence among various error bound conditions widely used in the literature is established. Our new philosophy of weakening existing error bound conditions in accordance with the specific structure of an algorithm under investigation may inspire other similar research in other contexts. Moreover, we use the concepts of calmness/metric subregularity in our analysis, and the main partial error bound result is inspired by a perturbation perspective. We believe that more deliberated draw on the experience of these well-developed tech-

niques in variational analysis and perturbation analysis will lead to more interesting and deeper results for the convergence analysis of other popular algorithms.

The linear convergence rates of ADMM schemes and other first-order methods via various error bound conditions have been studied in other contexts as well. For examples, in Hong and Luo [2017], an extended version of the ADMM scheme (1.2) with a sufficiently small step size for updating the dual variable λ is considered for a similar but more complicated case of (1.1) where there are more than two blocks of functions in the objective; a variant of the PADMM is studied in Han et al. [to appear] under calmness condition of S_{prox-I} defined in (4.2). In this paper, we concentrate on the relatively simpler convex scenario where only the two-block separable convex minimization model (1.1) and the proximal version of ADMM (1.3) are considered so that our idea can be exposed with simplier notation more clearly. Technically, we believe it is possible to extend our analysis to various more complicated scenarios such as models with nonconvex function components in their objectives, more sophisticated variants of the ADMM for two-block or even multiple-block models. Let us just mention one specific extension: it is trivial to extend our analysis to a more general version of the PADMM (1.3) considered in Eckstein and Bertsekas [1992]; Fang et al. [2015]:

$$\begin{cases} x^{k+1} = \arg\min\limits_{x\in\mathcal{X}} \ \{f(x) - (\lambda^k)^T(Ax + By^k - b) + \dfrac{\beta}{2}\|Ax + By^k - b\|^2 + \dfrac{1}{2}\|x - x^k\|_D^2\}, \\ y^{k+1} = \arg\min\limits_{y\in\mathcal{Y}} \ \{g(y) - (\lambda^k)^T(Ax^{k+1} + By - b) \\ \qquad\qquad\qquad\qquad + \dfrac{\beta}{2}\|\alpha Ax^{k+1} - (1-\alpha)(By^k - b) + By - b\|^2\}, \\ \lambda^{k+1} = \lambda^k - \beta(\alpha Ax^{k+1} - (1-\alpha)(By^k - b) + By^{k+1} - b), \end{cases}$$

with $\alpha \in (0, 2)$ is a relaxation factor. We skip the tedious analysis for more complicated extension and just present our analysis in the simplest context.

In the recent paper Drusvyatskiy and Lewis [2016], the authors investigate some unconstrained separable convex optimization problems and illustrate that subregularity of the gradient-like mapping is equivalent to the subregularity of its subdifferetial. Therefore, they employ the quadratic growth condition as the characterization of the error bound condition and establish the linear convergence rate for the proximal gra-

dient method. In the literature, there are many interesting works that provide characterizations and criteria for error bound properties in terms of various derivative-like objects in either the primal space (via directional derivatives, slopes, etc.) or the dual space (via subdifferentials, normal cones). Notice that for a given $\bar{w}$, the FEB (1.8) around $\bar{w}$ is equivalent to the the metric subregularity of the KKT mapping ϕ at $(\bar{w}, 0)$ and the calmness of $S(p)$ at $(0, \bar{w})$. The proximal EB-I (1.11) and proximal EB-II (1.13) around $\bar{w}$, on the other hand, are equivalent to the calmness of $S_{prox-I}(p)$ and $S_{prox-II}(p)$ at $(0, \bar{w})$, respectively. It is known that computing the exact formula of those derivative-like objects is much simpler in absence of any proximal operators. This perspective motivates us to call on existing extensive literature of the verifiable first-order and second-order sufficient conditions for the metric subregularity of ϕ or the calmness of $S(p)$, see, e.g. Gfrerer [2011, 2013]; Gfrerer and Klatte [2016]; Gfrerer and Mordukhovich [2015, 2017]; Gfrerer and Outrata [2016]; Gfrerer and Ye [2017]; Henrion et al. [2002]. Therefore, for the particular constrained model (1.1), we may consider investigating verifiable sufficient conditions for the metric subregularity/-calmness and hence the linear convergence rates for various ADMM-type algorithms and other schemes. In particular, it is interesting to notice that for the problem data with underlying polyhedral structures, the second-order sufficient condition is nearly necessary, see, e.g. Gfrerer [2013]. This is our future work.

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CURRICULUM VITAE

Academic qualifications of the thesis author, Mr. ZENG Shangzhi:

- Received the degree of Bachelor of Natural Sciences (Mathematics and Applied Mathematics) from Wuhan University, June 2015.

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