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**LETTER TO THE EDITOR:
EXTENSION OF DELTHEIL'S STUDY ON RANDOM POINTS
IN A CONVEX QUADRILATERAL**

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Deltheil's 1926 treatise [4] is sometimes cited in the context of Sylvester's famous 4-point problem. The problem, finding the probability p_4 that four uniformly-distributed points within a planar convex body K have a triangular convex hull, has been solved, according to the research literature of the last 40 years, *only* for a few bodies – triangles, ellipses, parallelograms and regular polygons. Whilst Deltheil's name is sometimes linked with these solutions and also with certain extremal issues, it has apparently been forgotten that his work gives a simple expression for p_4 when K is a general convex quadrilateral. In this note we extend Deltheil's result and, as a pleasant side effect, draw attention to his forgotten study.

For n points, the probability p_n equals $\binom{n}{3}\mathbb{E}(A_3^{n-3})/|K|^{n-3}$, where A_n is the area of the convex hull formed by n points, uniformly and independently distributed within a convex quadrilateral (see Effron [5]). We report $\mathbb{E}(A_3^k)$ and $\mathbb{E}(A_n)$ for some small values of n and k , extending work of Deltheil. So we focus on the affine-invariant moments $\mathbb{E}(A_n^k)/|K|^k$. Let K be a convex quadrilateral $ABCD$, whose diagonal AC is cut by the other diagonal BD into two segments of ratio $a : 1$, with BD in turn being divided in the ratio $b : 1$. Using a straightforward analysis aided by symbolic calculations, we have derived the

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following formulae:

$$\begin{aligned}\mathbb{E}\left(\frac{A_3}{|K|}\right) &= \frac{1}{12} - \frac{ab}{9(1+a)^2(1+b)^2}; & \mathbb{E}\left(\frac{A_3^2}{|K|^2}\right) &= \frac{1}{72} - \frac{ab}{18(1+a)^2(1+b)^2}; \\ \mathbb{E}\left(\frac{A_3^3}{|K|^3}\right) &= \frac{31}{9000} - \frac{ab(132ab + 74(a+b)(1+ab) + 41(1+a^2)(1+b^2))}{1500(1+a)^4(1+b)^4}; \\ \mathbb{E}\left(\frac{A_3^4}{|K|^4}\right) &= \frac{1}{900} - \frac{ab(28ab + 20(a+b)(1+ab) + 13(1+a^2)(1+b^2))}{900(1+a)^4(1+b)^4}.\end{aligned}$$

Only $\mathbb{E}(A_3)/|K|$ was found by Deltheil (being expressed by him using a different parametrisation). Our analysis and further discussion can be found in [3].

Our (a, b) -quadrilateral collapses to a triangle when either a or b equals zero and our leading terms agree with known results for triangles, given by Reed [6]. Other special cases are $a = 1$ yielding a (possibly skewed) kite, $a = b$ creating a trapezium and $a = b = 1$, a parallelogram. Our results do not agree with Reed's parallelogram formula. Instead, we agree with a formula of Trott, recently reported by Weisstein [7]:

$$\mathbb{E}\left(\frac{A_3^k}{|K|^k}\right)_{\text{parallelogram}} = \frac{3\left(1 + (k+2)\sum_{r=1}^{k+1} r^{-1}\right)}{(1+k)(2+k)^3(3+k)^2 2^{k-3}}.$$

We have also found some results for $\mathbb{E}(A_n)$ when $n > 3$. Buchta [1] showed that, for general K , $\mathbb{E}(A_4) = 2\mathbb{E}(A_3)$. We supplement this with:

$$\begin{aligned}\mathbb{E}\left(\frac{A_5}{|K|}\right) &= \frac{43}{180} - \frac{ab(108ab + 56(a+b)(1+ab) + 29(1+a^2)(1+b^2))}{90(1+a)^4(1+b)^4}; \\ \mathbb{E}\left(\frac{A_6}{|K|}\right) &= \frac{3}{10} - \frac{ab(124ab + 68(a+b)(1+ab) + 37(1+a^2)(1+b^2))}{90(1+a)^4(1+b)^4}.\end{aligned}$$

Using Buchta's more recent theory [2], in combination with our results, we find that N_5 , the number of sides of the convex hull when $n = 5$, takes values 3, 4 or 5 with probabilities $\frac{5}{36} - \psi$, $\frac{5}{9}$ and $\frac{11}{36} + \psi$ respectively. Here $\psi := 5ab/[9(1+a)^2(1+b)^2]$. It is intriguing that the probability of this convex hull being 4-sided does not depend on a or b .

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References

- [1] BUCHTA, C. (1986). On a conjecture of R. E. Miles about the convex hull of random points. *Monatsh. Math.* **102**, 91–102.
- [2] BUCHTA, C. (2005). An identity relating moments of functions of convex hulls. *Discr. Comput. Geom.* **33**, 125–142.
- [3] COWAN, R. AND CHIU, S. N. (2005). Technical Report: A solution of Sylvester-like problems for convex quadrilaterals. <http://www.maths.usyd.edu.au/u/richardc>.
- [4] DELTHEIL, R. (1926). *Probabilités Géométriques*. In *Traité du calcul des probabilités et de ses applications*, **2**, ed E. Borel, Gauthier-Villars, Paris.
- [5] EFFRON, B. (1965). The convex hull of a random set of points. *Biometrika* **52**, 331–343.
- [6] REED, W. J. (1974). Random points in a simplex. *Pacific J. Math.* **54**, 183–198.
- [7] WEISSTEIN, E. W. (2005). <http://mathworld.wolfram.com/SquareTrianglePicking.html>